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# SCHATTEN-TYPE CLASSES ON SEQUENCE SPACES

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**Abstract.** Let $H$ be a Hilbert space and $L(H)$ be the bounded linear operators on H. For $T \in L(H)$, let $\|T\|_p = \sup[\sum_{n=1}^{\infty} | < Te_n, e_n > |^p]^{1/p}$, where the supremum is taken over all orthonormal sequences $(e_n)$. Set $C_p(H) = \{T \in L(H) : \|T\|_p < \infty\}$. The object of this paper is to define and study $C_p(X, Y)$ where $X$ and $Y$ are sequences spaces.

## 0. Introduction

Let $H$ be a Hilbert space and $L(H)$ be the space of bounded linear operators on H. For $T \in L(H)$ let

$$\|T\|_p = \sup\left[\sum_{n=1}^{\infty} | < Te_n, e_n > |^p\right]^{1/p}, 1 \leq p < \infty$$

where the supremum is taken over all orthonormal sequences $(e_n)$ in H. The Schatten class of index $p$ is defined to be

$$C_p(H) = \{T \in L(H) : \|T\|_p < \infty\}.$$

We refer to [4], [5] for more on Schatten classes. The set of compact operators in $L(H)$ are denoted by $C_\infty$. It is known that $C_p(H) \subset C_\infty$, for all $1 \leq p < \infty$, and that $C_p(H)$ is a two sided ideal in $L(H)$. Further, for $2 \leq p < \infty$ and $T \in C_p(H)$,

$$\|T\|_p = \sup_{(e_n)}\left[\sum_{n=1}^{\infty} \|Te_n\|^p\right]^{1/p}.$$

And for $1 \leq p \leq 2$,

$$\|T\|_p = \inf_{(e_n)}\left[\sum_{n=1}^{\infty} \|Te_n\|^p\right]^{1/p}.$$

Schatten classes can be defined on Banach spaces either via singular numbers of bounded operators or via $(p, 2)$-summing operators. We refer to [4] for both cases.

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The object of this paper is to define Schatten type classes on sequence spaces ($\ell^p -$ *sapces*) via $p$-orthogonal sequences.

Throughout this paper $L(E,F)$ is the space of all bounded linear operators from $E$ to $F$, where $E$ and $F$ are any two Banach spaces. the compact operators in $L(E,F)$ will be denoted by $K(E,F)$ while the finite rank operators will be denoted by $F(E,F)$. The class of $(p,q)$-summing operators in $L(E,F)$ is denoted by $\pi_{p,q}(E,F)$, [4], and the class of $p$-nuclear operators will be denoted by $N_p(E,F)$, [4]. The class of weakly $p$-summable sequences on $E$ is denoted by $\ell^p(E)$, [2]. The dual of $E$ is $E^*$, the unit sphere of $E$ is $S(E)$ and the conjugate of $p$ is $p^*[\frac{1}{p}+\frac{1}{p^*}=1]$.

## I. $C_p(\ell^{p^*}, E)$

Let $E$ be any Banach space and $1 \leq p < \infty$. To define our class of operators, we need first to introduce the concept of $p$-orthogonal elements in Banach spaces.

**Definiton 1.1.** A sequence $(x_n)$ in $E$ is called $p$-orthogonal if

$$\|\sum_{n=1}^{\infty}\lambda_n x_n\| = \Big[\sum_{n=1}^{\infty}|\lambda_n|^P\|x_n\|^p\Big]^{1/p}.$$

If $\|x_n\|=1$, we say $(x_n)$ is $p$-orthonormal. For $p=\infty$, $(x_n)$ is called $p$-orthogonal if

$$\|\sum_{n=1}^{\infty}\lambda_n x_n\| = \sup_n(|\lambda_n|\|x_n\|).$$

We refer to [1] for more on $p$-orthogonal sequences in Banach spaces. Some of the basic properties of $p$-orthogonal sequences in listed in:

**Lemma 1.2.** *Let $(x_n)$ be a sequence in the Banach space $E$. Then:*

(i) *For $E=\ell^p$, $(x_n)$ is p-orthogonal if and only if $supp(x_n)\cap supp(x_m)=\varphi$ for $n\neq m$, where $supp(x_n) =$ closure of $\{i : x_n(i)\neq 0\}$.*
(ii) *For $E=\ell^p$, $(x_n)$ is p-orthogonal if and only if $(|x_n|)$ is p-orthogonal.*
(iii) *If $(x_n)$ is p-orthonormal in $E$, then*

$$\Big[\sum_{n=1}^{\infty}|<x_n,x^*>|^p\Big]^{1/p}\leq\|x^*\|$$

*for all $x^*\in E^*$.*

The proof of (i) can be found in [1], (ii) follows from (i) and (iii) follows from the definition of $p$-orthogonal sequences.

Now we introduce our basic definition.

**Definition 1.3.** For $1 \leq p < \infty$ and $E$ any Banach space, set:

$$C_p(\ell^{p^*}, E) = \Big\{T \in L(\ell^{p^*}, E) : \sup \Big[\sum_{n=1}^{\infty} ||T\theta_n||^p\Big]^{1/p} < \infty\Big\},$$

where the supremum is taken over all $p^*$-orthonormal sets, $(\theta_n)$, in $\ell^{p^*}$. for $T \in C_p(\ell^{p^*}, E)$, set:

$$||T||_p = \sup\Big\{\Big[\sum_{n=1}^{\infty} ||T\theta_n||^p\Big]^{1/p} : (\theta_n) \text{ is } p^*\text{-orthonormal set in } \ell^{p^*}\Big\}.$$

For $p = \infty$, we let

$$C_\infty(\ell^1, E) = \{T \in L(\ell^1, E) : \sup_n(||T\theta_n||) < \infty\},$$

where the supremum is taken over all 1-orthonormal sets, $(\theta_n)$, in $\ell^1$. For $T \in C_\infty(\ell^1, E)$, set:

$$||T||_\infty = \sup\{||(||T\theta_n||)||_\infty, (\theta_n) \text{ is 1-orthonormal set in } \ell^1\}.$$

Throughout this paper, we write $\sup\limits_{(\theta_n)_p}$ to denote that the supremum is taken over all $p$-orthonormal sets $(\theta_n)$ in $\ell^p$ where $1 \leq p \leq \infty$.

**Lemma 1.4.** *Let $E$ and $F$ be any Banach spaces, and $1 \leq p \leq \infty$. Then:*

(i) $||T|| \leq ||T||_p$ *for all* $T \in C_p(\ell^{p^*}, E)$
(ii) *For any* $A \in L(E, F)$ *and all* $T \in C_p(\ell^{p^*}, E) AT \in c_p(\ell^{p^*}, F)$ *and* $||AT||_p \leq ||A||||T||_p$.
(iii) $C_p(\ell^{p^*}, E)$ *is a Banach space*
(iv) $C_\infty(\ell^1, E) = L(\ell^1, E)$.

**Proof.** The proof of (i), (ii) and (iv) follows from the definition of $|| \ ||_p$. For (iii) we only prove that $C_p(\ell^{p^*}, E)$ is complete. for that, by proposition 4[6,p.116] it is enough to prove that if $T_n \in C_p(\ell^{p^*}, E)$ such that $\sum_{n=1}^{\infty} ||T_n||_p < \infty$, then $\sum_{n=1}^{\infty} T_n \in C_p(\ell^{p^*}, E)$.

Consider $T = \sum_{n=1}^{\infty} T_n$, where $Tx = \sum_{n=1}^{\infty} T_n x$ for all $x \in \ell^{p^*}$. Then:

$$||Tx|| \leq ||x|| \sum_{n=1}^{\infty} ||T_n|| \leq ||x|| \sum_{n=1}^{\infty} ||T_n||_p < \infty.$$

Hence, $T \in L(\ell^{p^*}, E)$.

Now, let $(\theta_k)$ be any $p^*$-orthonormal set in $\ell^{p^*}$. Then

$$\begin{aligned}\Big[\sum_{k=1}^{\infty} ||\sum_{n=1}^{\infty} T_n\theta_k||^p\Big]^{1/p} &\leq \sum_{n=1}^{\infty} \Big[\sum_{k=1}^{\infty} ||T_n\theta_k||^p\Big]^{1/p} \\ &\leq \sum_{n=1}^{\infty} \sup_{(\theta_k)_{p^*}} \Big[\sum_{k=1}^{\infty} ||T_n\theta_k||^p\Big]^{1/p} \\ &\leq \sum_{n=1}^{\infty} ||T_n||_p.\end{aligned}$$

**Lemma 1.5.** *$F(\ell^{p^*}, E) \subset C_p(\ell^{p^*}, E)$. Further for any $x \in \ell^p$ and $y \in E$ we have $\|x \otimes y\|_p = \|x\|\|y\|$.*

**Proof.** Follows from definition 1.3 and the basic properties of $\| \ \|_p$. We now give a nice characterization of $C_p(\ell^{p^*}, E)$.

**Theorem 1.6.** *Let $1 < p < \infty$ and $p \neq 2$. Then the following are equivalent:*

(i) $T \in C_p(\ell^{p^*}, E)$.
(ii) $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$, *where $g_n \in E$ with $\|g_n\| = 1$. In this case, $\|T\|_p = \|(\lambda_n)\|_p$.*

**Proof.** (i) $\to$ (ii).

Let $T \in c_p(\ell^{p^*}, E)$ and $x \in \ell^{p^*}$. Since $\delta_n$ is a basis for $\ell^{p^*}, 1 < p < \infty$, then $x = \sum_{n=1}^{\infty} < x, \delta_n > \delta_n$. This implies that

$$Tx = \sum_{n=1}^{\infty} < x, \delta_n > T\delta_n.$$

Thus,

$$T = \sum_{n=1}^{\infty} \delta_n \otimes T\delta_n,$$

where the series converges strongly. Hence, $T = \sum_{n=1}^{\infty} \lambda_n \delta \otimes g_n$ where $\lambda_n = \|T\delta_n\|$ and $g_n = \frac{T\delta_n}{\|T\delta_n\|}$. But since $T \in C_p(\ell^{p^*,E})$, then $[\sum_{n=1}^{\infty} |\lambda_n|^p]^{1/p} = [\sum_{n=1}^{\infty} \|T\delta_n\|^p]^{1/p} \leq \|T\|_p$. Thus

$$T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n,$$

where $(\lambda_n) \in \ell^p$ and $\|g_n\| = 1$.

(ii) $\to$(i).

Let $T = \sum_{n=1}^{\infty} \lambda_n \delta_n \otimes g_n$, where $(\lambda_n) \in \ell^p$ and $g_n \in E$ with $\|g_n\| = 1$. We claim that $T \in C_p(\ell^{p^*}, E)$. First, we prove that $T \in L(\ell^{p^*}, E)$. For this, let $x \in \ell^{p^*}$. Then

$$\begin{aligned} \|Tx\| &\leq \Big[\sum_{n=1}^{\infty} |\lambda_n|^p\Big]^{1/p} \cdot \Big[\sum_{n=1}^{\infty} < \delta_n, x > |^{p^*}\Big]^{1/p^*} \\ &\leq \Big[\sum_{n=1}^{\infty} |\lambda_n|^p\Big]^{1/p} \cdot \|x\|. \end{aligned}$$

Thus, $T \in L(\ell^{p^*}, E)$ and $\|T\| \leq [\sum_{n=1}^{\infty} |\lambda_n|^p]^{1/p}$.

Now, let $(\theta_k)$ be any $p^*$-orthonormal set in $\ell^{p^*}$. Then

$$\Big[\sum_{k=1}^{\infty} \|T\theta_k\|^p\Big]^{1/p} \leq \Big[\sum_{k=1}^{\infty} \Big[\sum_{n=1}^{\infty} |\lambda_n|| < \delta_n, \theta_k > |\Big]^p\Big]^{1/p}$$

$$\leq |\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \eta_k|\lambda_n|| < \delta_n, \theta_k > | \qquad (||(\eta_k)||_{p^*} = 1)$$

$$\leq \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} |\eta_k||\lambda_n| < |\delta_n|, |\theta_k| > .$$

Since $|\delta_n|$ is $p$-orthonormal, then $\|\sum_{n=1}^{\infty}|\lambda_n||\delta_n|\|_p = [\sum_{n=1}^{\infty}|\lambda_n|^p]^{1/p} < \infty$. Thus $\sum_{n=1}^{\infty}|\lambda_n||\delta_n| \in \ell^p$. Consequently, since $|\theta_k| \in \ell^{p^*}$, then:

$$\Big[\sum_{k=1}^{\infty} ||T\theta_k||^p\Big]^{1/p} \leq \sum_{k=1}^{\infty} |\eta_k| < \sum_{n=1}^{\infty} |\lambda_n||\delta_n|, |\theta_k| > .$$

Again, since $|\theta_k|$ is $p$-orthonormal, we get

$$||\sum_{k=1}^{\infty} |\eta_k||\theta_k|||_{p^*} = \Big[\sum_{k=1}^{\infty} |\eta_k|^{p^*}\Big]^{1/p^*} < \infty.$$

Thus $\sum_{k=1}^{\infty}|\eta_k||\theta_k| \in \ell^{p^*}$. Hence,

$$\begin{aligned}\Big[\sum_{k=1}^{\infty} ||T\theta_k||^p\Big]^{1/p} &\leq | < \sum_{n=1}^{\infty} |\lambda_n||\delta_n|, \sum_{k=1}^{\infty} |\eta_k||\theta_k| > |(\text{since } \sum_{n=1}^{\infty} |\lambda_n||\delta_n| \in \ell^p) \\ &\leq || \sum_{n=1}^{\infty} |\lambda_n||\delta|||_p \cdot || \sum_{k=1}^{\infty} |\eta_k||\theta_k|||_{p^*} \\ &\leq \Big[\sum_{n=1}^{\infty} |\lambda_n|^p\Big]^{1/p} \cdot \Big[\sum_{k=1}^{\infty} |\eta_k|^{p^*}\Big]^{1/p^*} \\ &= \Big[\sum_{n=1}^{\infty} |\lambda_n|^p\Big]^{1/p}.\end{aligned}$$

Since $(\theta_k)$ was arbitrary $p^*$-orthonormal sequence, it follows that $T \in C_p(\ell^{p^*}, E)$ and $||T||_p \leq [\sum_{n=1}^{\infty}|\lambda_n|^p]^{1/p}$. But

$$\Big[\sum_{n=1}^{\infty} |\lambda_n|^p\Big]^{1/p} = \Big[\sum_{n=1}^{\infty} ||T\delta_n||^p\Big]^{1/p} \leq ||T||_p.$$

Hence

$$||T||_p = \Big[\sum_{n=1}^{\infty} |\lambda_n|^p\Big]^{1/p}.$$

If $p = 1$, then we have:

**Lemma 1.7.** *If $T = \sum_{n=1}^{\infty} \lambda_n\delta_n \otimes g_n$, where $(\lambda_n) \in \ell^1$ and $g_n \in E$ with $|g_n|| = 1$, then $T \in C_1(\ell^{\infty}, E)$. In this case, $||T||_1 = ||(\lambda_n)||_1$.*

**Proof.** This is just (ii) $\to$ (i) in Theorem 1.6.

Let $N_{p,q,r}(X,Y)$ denote the space of $(p,q,r)$ uclear operators, [4,Definition 18.1.1], from $X$ into $Y$. Using $N_{p,q,r}(X,Y)$ we give orther characterization of $C_p(\ell^{p^*},E)$.

**Theorem 1.8.** *Let* $1<p<\infty$, $p\neq 2$, *and* $E$ *be any Banach space. Then the following are equivalent:*

(i) $T\in C_p(\ell^{p^*},E)$

(ii) $T\in N_{p,1,p}(\ell^{p^*},E)$ and $T=\sum_{n=1}^{\infty}\lambda_n x_n\otimes y_n$, where $(\lambda_n)\in\ell^p$, $(x_n)=(|x_n|)\in\ell^{p^*}(\ell^p)$ and $\sup_n\|y_n|<\infty$.

**Proof.** (i)$\to$(ii):

Let $T\in C_p(\ell^{p^*},E)$. Then by Theorem 1.6, $T=\sum_{n=1}^{\infty}\lambda_n\delta_n\otimes g_n$ where $(\lambda_n)\in\ell^p$, $\delta_n=|\delta_n|\in\ell^p$ and $\|g_n\|=1$. Since $(\lambda_n)\in\ell^p$, $\sup_{\|x^*\|\le 1}[\sum_{n=1}^{\infty}|<\delta_n,x^*|^{p^*}]^{1/p^*}\le 1$ and $\sup_n\|g_n\|=1$, then it follows from Definition 18.1.1, [4], that $T\in N_{p,1,p}(\ell^{p^*},E)$.Further,

$$\begin{aligned}\|T\|_{p,1,p} &\le \Big[\sum_{n=1}^{\infty}|\lambda_n|^p\Big]^{1/p}\cdot\sup_{\|x^*\|\le 1}\Big[\sum_{n=1}^{\infty}|<\delta_n,x^*>|^{p^*}\Big]^{1/p^*}\cdot\sup_n\|g_n\|\\ &\le \Big[\sum_{n=1}^{\infty}|\lambda_n|^p\Big]^{1/p}\\ &=\|T\|_p.\end{aligned}$$

(ii) $\to$ (i).

Let $T\in N_{p,1,p}(\ell^{p^*},E)$ such that $T=\sum_{i=1}^{\infty}\lambda_i x_i\otimes y_i$ where $(\lambda_i)\in\ell^p$, $(x_i)=(|x_i|)\in\ell^{p^*}(\ell^p)$ and $\sup_i\|y_i\|<\infty$. Let $(\delta_k)$ be an $p^*$-orthonormal set in $\ell^{p^*}$. Then

$$\begin{aligned}\Big[\sum_{k=1}^{\infty}\|T\delta_k\|^p\Big]^{1/p} &\le \Big[\sum_{k=1}^{\infty}|\sum_{n=1}^{\infty}|\lambda_n||<x_n,\delta_k>|||y_n|||^p\Big]^{1/p}\\ &\le|\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\eta-k|\lambda_n||<x_n,\delta_k>|||y_n|||(\text{For some }(\eta_k)\in S_1(\ell^{p^*}))\\ &\le|\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\eta_k|\lambda_n|<|x_n|,|\delta_k|>\|y_n\||.\\ &\le|<\sum_{n=1}^{\infty}|\lambda_n||x_n|\|y_n\|,\sum_{k=1}^{\infty}\eta_k|\delta_k|>|\\ &\le\|\sum_{n=1}^{\infty}|\lambda_n||x_n|\|y_n\|\|_p\|\sum_{k=1}^{\infty}\eta_k|\delta_k|\|_{p^*}\\ &\le\sup_n\|y_n\|\cdot\Big[\sum_{n=1}^{\infty}|\lambda_n|^p\Big]^{1/p}\Big[\sum_{n=1}^{\infty}|<x_n,x^*>|^{p^*}\Big]^{1/p^*}.\end{aligned}$$

Thus, by Theorem 1.6,

$$\|T\|_p \leq \Big[\sum_{n=1}^{\infty} |\lambda_n|^p\Big]^{1/p} \cdot \sup_{\|x^*\|\leq 1} \Big[\sum_{n=1}^{\infty} |< x_n, x^* >|^{p^*}\Big]^{1/p^*} \cdot \sup_n \|y_n\|.$$

Hence $T \in C_p(\ell^{p^*}, E)$.

## II. Ideal Realation of $C_p(\ell^{p^*}, E)$

The proof of the following is immediate and will be omitted:

**Theorem 2.1.**

(i) $C_p(\ell^{p^*}, E) \subseteq K(\ell^{p^*}, E)$
(ii) $\pi_p(\ell^{p^*}, E) \subseteq C_p(\ell^{p^*}, E)$
(iii) $N_p(\ell^{p^*}, E) \subseteq C_p(\ell^{p^*}, E)$.

For $p = 2$, we have the following nice result:

**Theorem 2.2.** *Let $E$ be any Banach space. Then $C_2(\ell^2, E) = \pi_2(\ell^2, E)$.*

**Proof.** By Theorem 2.1(ii), we have $\pi_2(\ell^2, E) \subseteq C_2(\ell^2, E)$.

To prove the other inclusion, let $T \in C_2(\ell^2, E)$ and $(x_n)$ be any sequence in $\ell^2$. If $\sup_{\|x^*\|\leq 1} \Big[\sum_{n=1}^{\infty} |< x_n, x^* >|^2]^{1/2} = \infty$, then we have:

$$\Big[\sum_{n=1}^{\infty} \|Tx_n\|^2\Big]^{1/2} \leq \infty = \sup_{\|x^*\|\leq 1} \Big[\sum_{n=1}^{\infty} < x_n, x^* > |^2\Big]^{1/2},$$

and $T \in \pi_2(\ell^2, E)$.

Assume that $\sup_{\|x^*\|\leq 1}[\sum_{n=1}^{\infty} |< x_n, x^* >|^2]^{1/2} < \infty$. Define

$$A : \ell^2 \to \ell^2$$
$$A = \sum_{n=1}^{\infty} \delta_n \otimes x_n.$$

The for each $x \in \ell^2$,

$$\begin{aligned} \|Ax\| &= |\sum_{n=1}^{\infty} < \delta_n, x >< x_n, x^* >| \qquad \text{(for some } x^* \in S_1(\ell^2)) \\ &\leq \|x\| \cdot \Big[\sum_{n=1}^{\infty} |< x_n, x^* >|^2\Big]^{1/2}. \end{aligned}$$

Consequently, $A \in L(\ell^2, \ell^2)$ and $\|A\| \leq [\sum_{n=1}^{\infty} |< x_n, x^* >|^2]^{1/2}$. Hence

$$\Big[\sum_{n=1}^{\infty} \|Tx_n\|^2\Big]^{1/2} = \Big[\sum_{n=1}^{\infty} \|TA\delta_n\|^2\Big]^{1/2} = \|A\|\Big[\sum_{n=1}^{\infty} \|T\frac{A}{\|A\|}\delta_n\|^2\Big]^{1/2}.$$

Set $\tilde{A} = \frac{A}{\|A\|}$. Then $\|\tilde{A}\| = 1$. Lemma 2.2.3, [5], implies that $\tilde{A} = \sum_{i=1}^{4} \alpha_i u_i$, where $u_i$'s are unitary operators and $\sum_{i=1}^{4} |\alpha_i| = 1$. Thus

$$\begin{aligned}\Big[\sum_{n=1}^{\infty} \|Tx_n\|^2\Big]^{1/2} &\leq \|A\|\Big[\sum_{n=1}^{\infty}\Big[\sum_{i=1}^{4} |\alpha_i|\|Tu_i\delta_n\|\Big]^2\Big]^{1/2} \\ &\leq \|A\|\Big[\sum_{i=1}^{4} |\alpha_i|\Big[\sum_{n=1}^{\infty} \|Tu_i\delta_n\|^2\Big]^{1/2}\Big] \\ &\leq \|A\|\|T\|_2,\end{aligned}$$

noting that a unitary operator maps orthonormal sets to orthonormal sets. Consequently,

$$\Big[\sum_{n=1}^{\infty} \|Tx_n\|^2\Big]^{1/2} \leq \|T\|_2 \cdot \sup_{\|x^*\|\leq 1} \Big[\sum_{n=1}^{\infty} |< x_n, x^* >|^2\Big]^{1/2}.$$

Thus, $T \in \pi_2(\ell^2, E)$ and $\|T\|_{\pi_2} \leq \|T\|_2$.

## III. Duality in $C_p(\ell^{p^*}, E)$

**Theorem 3.1.** *let $E$ be a reflexive Banach space. Then*

(i) $[C_2(\ell^2, E)]^*$ *is isometrically isomorphic to* $\pi_2(E, \ell^2)$.
(ii) $[C_p(\ell^{p^*}, E)]^*$ *is isometrically isomorphic to* $C_{p^*}(\ell^p, E^*)$, $1 < p < \infty$ *and* $p \neq 2$.

**Proof.**

(i) Follows from Theorem 2.2 and the fact that $[\pi_2(\ell^2, E)]^* = \pi_2(E, \ell^2)$, [4, p.296].
(ii) For $A \in C_{p^*}(\ell^p, E^*)$, define

$$F_A : C_p(\ell^{p^*}, E) \to \mathbb{C},$$
$$F_A(T) = \sum_{n=1}^{\infty} < A\delta_n, T\delta_n > .$$

Then

$$\begin{aligned}|F_A(T)| &\leq \Big[\sum_{n=1}^{\infty} \|A\delta_n\|^{p^*}\Big]^{1/p^*} \cdot \Big[\sum_{n=1}^{\infty} \|T\delta_n\|^{p}\Big]^{1/p} \\ &\leq \|A\|_{p^*} \cdot \|T\|_p.\end{aligned}$$

Hence, $\|F_A\| \leq \|A\|_{p^*}$. This implies that $F_A$ is a bounded linear functional on $C_p(\ell^{p^*}, E)$.
Further

$$\begin{aligned}\|A\|_{p^*} &= \Big[\sum_{n=1}^{\infty} \|A\delta_n\|^{p^*}\Big]^{1/p^*} \\ &= |\sum_{n=1}^{\infty} < A\delta_n, y_n > | \qquad (\|(\|y_n\|)\|_p = 1).\end{aligned}$$

Now, define

$$T_0 : \ell^{p^*} \to E,$$
$$T_0 = \sum_{n=1}^{\infty} \delta_n \otimes y_n.$$

Then

$$\|T_0 x\| \le \Big[\sum_{n=1}^{\infty} | < \delta_n, x > |^{p^*}\Big]^{1/p^*} \cdot \Big[\sum_{n=1}^{\infty} \|y_n\|^p\Big]^{1/p} \le \|x\|.$$

Hence, $T_0 \in L(\ell^{p^*}, E)$. But

$$\Big[\sum_{n=1}^{\infty} \|T_0 \delta_n\|^p\Big]^{1/p} = \Big[\sum_{n=1}^{\infty} \|y_n\|^p\Big]^{1/p} = 1.$$

Then by Theorem 1.6, $T_0 \in C_p(\ell^{p^*}, E)$ and $\|T_0\|_p = 1$. Thus

$$\|A\|_{p^*} = |\sum_{n=1}^{\infty} < A\delta_n, T_0\delta_n > | = |F_A(T_0)| \le \|F_A\| \cdot \|T_0\|_p = \|F_A\|.$$

This implies that $\|F_A\| = \|A\|_{p^*}$.

Now, define:

$$J : C_{p^*}(\ell^p, E^*) \to [c_p(\ell^{p^*}, E)]^*,$$
$$J(A) = F_A.$$

Since $\|F_A\| = \|A\|_{p^*}$, it follows that $J$ is an isometry.

We claim that $J$ is onto. To see, let $F \in [C_p(\ell^{p^*}, E)]^*$. Define a map

$$A : \ell^p \to E^*$$
$$< Ax, y >= F(x \otimes y).$$

As $x \otimes y \in C_p(\ell^{p^*}, E)$, it follows that

$$| < Ax, y > | \le \|F\| \cdot \|x \otimes y\|_p = \|F\| \cdot \|x\|\|y\|.$$

Hence $\|A\| \le \|F\|$ and $A \in L(\ell^p, E^*)$. If $(\delta_n)$ is the $p$-orthonormal basis in $\ell^p$, then

$$\Big[\sum_{n=1}^{\infty} \|A\delta_n\|^{p^*}\Big]^{1/p^*} = |\sum_{n=1}^{\infty} < A\delta_n, g_n > | \qquad (\|(\|g_n\|)\|_p = 1).$$
$$= |\sum_{n=1}^{\infty} F(\delta_n \otimes g_n)|.$$

Theorem 1.6 implies that $\sum_{n=1}^{\infty}\delta_n\otimes g_n \in C_p(\ell^{p^*}, E)$ and $\|\sum_{n=1}^{\infty}\delta_n\otimes g_n\|_p = [\sum_{n=1}^{\infty}\|g_n\|^p]^{1/p}$. Thus

$$\Big[\sum_{n=1}^{\infty}\|A\delta_n\|^{p^*}\Big]^{1/p^*} = |F\Big[\sum_{n=1}^{\infty}\delta_n\otimes g_n\Big]|$$
$$\leq \|F\|\cdot\|\sum_{n=1}^{\infty}\delta_n\otimes g_n\|_p$$
$$= \|F\|.$$

Theorem 1.6 now implies that $A \in C_{p^*}(\ell^p, E^*)$.

Now, let $T \in C_p(\ell^{p^*}, E)$. Then, by Theorem 2.1, there exists $T_N = \sum_{n=1}^{N}\delta_n\otimes\delta_n \in F(\ell^{p^*}, E)$ such that $\lim_{N\to\infty}\|T_N - T\|_p = 0$. Hence

$$F(T) = \lim_{N\to\infty} F(T_N) = F_A(T).$$

Consequently, $F_A = F$. This implies that $J$ is an isometric onto operator.

**Corollary 3.2.** *Let $1 < p < \infty$, $p \neq 2$ and $E$ be a reflexive Banach space. Then $C_p(\ell^{p^*}, E)$ is reflexive.*

## IV. Ideal Properties of $C_p[\ell^{p^*}]$

Let $Q$ be an operator ideal [4]. The following definitions are taken from Pietsch [4].

(i) $Q$ is called small if whenever $Q(X,Y) = L(X,Y)$, then $X$ or $Y$ is a finite dimensional space.

(ii) $Q$ is called closed if the closure of $Q(X,Y)$ in $L(X,Y)$ is $Q(X,Y)$ for all Banach spaces $X$ and $Y$.

(iii) $Q$ is called regular if for all Banach spaces $X$ and $Y$, $T \in Q(X,Y)$ if and only if $k_YT \in Q(X,Y^{**})$, where $k_Y$ is the natural embedding of $Y$ into $Y^{**}$.

(iv) $Q$ is called injective if whever $J_YT \in Q(X, \ell^{\infty}(B_1(Y^*)))$, then $T \in Q(X,Y)$ for all Banach spaces $X$ and $Y$. Here $J_Y$ is the natural embedding of $Y$ into $\ell^{\infty}(B_1(Y^*))$.

**Theorem 4.1.** *Let $2 \leq p < \infty$. Then $C_p[\ell^{p^*}]$ is a small left operator ideal.*

**Proof.** Suppose $C_p(\ell^{p^*}, E) = L(\ell^{p^*}, E)$ for some Banach space $E$. Since $\|T\| \leq \|T\|_p$, then the identity map I from $C_p(\ell^{p^*}, E)$ into $L(\ell^{p^*}, E)$ is a bounded linear operator which is onto. The opern mapping theorem now implies that there exists $\gamma > 0$ such that for each $T \in C_p(\ell^{p^*}, E)$, $\|T\|_p \leq \gamma\|T\|$.

Now, assume if possible that $E$ is infinite dimensional Banach space. Chosse $N$ and $\epsilon > 0$ such that $N^{1/p} > \gamma(1+\epsilon)$. Then by Dvoretzky's Lemma [4, p.39], there exists $(x_i)_{n=1}^{N} \in E$ with $\|x_i\| = 1$ such that

$$\sup_{\|x^*\|\leq 1}\Big[\sum_{i=1}^{N}| < x_i, x^* > |^2\Big]^{1/2} \leq 1+\epsilon \tag{1}$$

Define,

$$J : \ell^{p^*} \to E,$$
$$J = \sum_{n=1}^{N} \delta_i \otimes x_i.$$

Then for each $x \in \ell^{p^*}$, we have:

$$\begin{aligned}
||Jx|| &= |\sum_{i=1}^{N} < \delta_i, x >< x_i, x^* > | \qquad \text{(For some } x^* \in S_1(E^*)) \\
&\leq \Big[\sum_{i=1}^{N} | < \delta_i, x > |^{p^*}\Big]^{1/p^*} \cdot \Big[\sum_{i=1}^{N} | < x_i, x^* > |p\Big]^{1/p} \\
&\leq ||x|| \cdot \Big[\sum_{i=1}^{N} | < x_i, x^* > |^2\Big]^{1/2} \\
&\leq (1+\epsilon)||x|| \qquad \text{(By(1))}.
\end{aligned}$$

Thus, $||J|| \leq (1+\epsilon)$. Further

$$||J||_p \geq \Big[\sum_{i=1}^{N} ||J\delta_i||^p\Big]^{1/p} = \Big[\sum_{i=1}^{N} ||x_i||^p\Big]^{1/p} = N^{1/p}.$$

Hence, $||J||_p \geq N^{1/p}$. Consequently, $N^{1/p} \leq ||J||_p \leq \gamma||J|| \leq \gamma(1+\epsilon) < N^{1/p}$. This is a contradiction. Hence $E$ must be finite dimensional.

**Theorem 4.2. Let $2 \leq p < \infty$. Then $C_p[\ell^{p^*}]$ is not closed.**

**Proof.** Suppose $C_p[\ell^{p^*}]$ is closed. Then $C_p(\ell^{p^*}, E)$ is closed in $L(\ell^{p^*}, E)$ for all Banach spaces $E$. Let $E = \ell^q$ where $1 < q < p^* \leq 2$. lemma 1.4 implies that $F(\ell^{p^*}, \ell^q) \subseteq C_p(\ell^{p^*}, \ell^q)$. Then $\overline{F(\ell^{p^*}, \ell^q)} \subseteq C_p(\ell^{p^*}, \ell^q)$ where the closure is in $L(\ell^{p^*}, \ell^q)$. Thus, $K(\ell^{p^*}, \ell^q) \subseteq C_p(\ell^{p^*}, \ell^q)$, [2, p.242]. Hence, by Theorem 2.1, $C_p(\ell^{p^*}, \ell^q) = K(\ell^{p^*}, \ell^q)$. Corollary 4.2, [3], now implies that $C_p(\ell^{p^*}), \ell^q = L(\ell^{p^*}, \ell^q)$. This contradicts Theorem 4.2. Hence $C_p[\ell^{p^*}]$ is not closed.

**Theorem 4.3.** *Let $1 < p < \infty$. Then $C_p[\ell^{p^*}]$ is regular.*

**Proof.** Let $E$ be any Banach space and $K_E$ be the natural embedding of $E$ into $E^{**}$. We want to prove that $T \in C_p(\ell^{p^*}, E)$ if and only if $K_E T \in C_p(\ell^{p^*}, E^{**})$.

Let $p = 2$ and $T \in C_2(\ell^2, E)$. Then by lemma 1.4, we have $K_E T \in C_2(\ell^2, E^{**})$.

Conversely, suppose $K_E T \in C_2(\ell^2, E^{**})$. Theorem 2.2 implies that $K_E T \in \pi_2(\ell^2, E^{**})$. AS $\pi_2$ is a regular operator ideal, [4, p.109], it follows that $T \in \pi_2(\ell^2, E)$. Consequently, Theorem 2.2 implies that $T \in C_2(\ell^2, E)$. Hence $C_2[\ell^2]$ is regular.

For $1 < p < \infty$ and $p \neq 2$, let $T \in C_p(\ell^{p^*}, E)$. Then Lemma 1.4 implies that $K_E T \in C_p(\ell^{p^*}, E^{**})$.

Now, suppose $K_E T \in C_p(\ell^{p^*}, E^{**})$. Then, Theorem 1.6 implies that:

$$K_E T = \sum_{n=1}^{\infty} \delta_n \otimes K_E T \delta_n,$$

where $[\sum_{n=1}^{\infty} \|K_E T \delta_n\|^p]^{1/p} < \infty$. Since $K_E$ is an isometric operator, then,

$$\left[\sum_{n=1}^{\infty} \|T\delta_n\|^p\right]^{1/p} < \infty.$$

Thus, by Theorem 1.6, we have $T \in C_p(\ell^{p^*}, E)$. Hence, $C_p[\ell^{p^*}]$ is regular.

In a similar way one can prove:

**Theorem 4.4.** *Let* $1 < p < \infty$. *Then* $C_p[\ell^{p^*}]$ *is injective.*

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