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EXISTENCE OF SOLUTIONS OF SEMILINEAR DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract. The aim of this paper is to prove the existence and uniquencess of local, strong and global solutions of a nonlocal Cauchy problem for a differential equation. The method of analytic semigroups and the contraction mapping principle are used to establish the results.

1. Introduction

The problem of existence of solutions of evolution equation with nonlocal conditions in Banach space has been studied first by Byszewski [5]. In that paper he has establised the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)), \qquad t \in (t_0, t_0 + a] \tag{1}$$

$$u(t_0) + g(t_1, t_2, \ldots, t_p, u(\cdot)) = u_0 \tag{2}$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, in a Banach space X, $0 \leq t_0 < t_1 < \cdots < t_p \leq t_0 + a$, $a > 0$, $u_0 \in X$ and $f : [t_0, t_0 + a] \times X \to X$, $g : [t_0, t_0 + a]^p \times X \to X$ are given functions. Subsequently he has investigated the same type of problem to a different class of evolution equations in Banach spaces [3-7]. Here the symbol $g(t_1, \ldots, t_p, u(\cdot))$ is used in the sense that in the place of '$\cdot$' we can substitute only elements of the set $\{t_1, \ldots, t_p\}$.

The purpose of this paper is to prove the existence and uniqueness of local, strong and global solutions for a semilinear differential equation with nonlocal conditions of the form:

$$\frac{du(t)}{dt} + Au(t) = f(u(t)), \qquad t \in (0, b] \tag{3}$$

$$u(0) + g(t_1, t_2, \ldots, t_p, u(t_1), \ldots, u(t_p)) = u_0 \tag{4}$$

where $0 \leq t_0 < t_1 < \cdots < t_p \leq b$. For example,

$$g(t_1, t_2, \ldots, t_p, u(t_1), \ldots, u(t_p)) = c_1 u(t_1) + \cdots + c_p u(t_p)$$

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where $c_i (i = 1, \ldots, p)$ are constants. In this case, equation (4) allows the measurements at $t = 0, t_1, \ldots, t_p$ rather than just at $t = 0$.

Here we assume that $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator $T(t)$, $t \geq 0$, in a Banach space X. The operator A^α can be defined for $0 \leq \alpha < 1$ and A^α is a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. The closedness of A^α implies that $D(A^\alpha)$ endowed with the graph norm of A^α, that is the norm $\|x\| = \|x\| + \|A^\alpha x\|$, is a Banach space. Sine A^α is invertible its graph norm $\| \cdot \|$ is equivalent to the norm $\|x\|_\alpha = \|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $\| \cdot \|_\alpha$, is a Banach space which we denote by X_α. From the definition it is clear that $0 < \alpha < \beta$ implies $X_\alpha \supset X_\beta$ and that the imbedding of X_β in X_α is continuous. Throughout the paper we shall use the symbol $J = [0, b]$. The nonlinear operators $f : X_\alpha \to X$, $g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) : J^p \times X_\alpha^p \to Y$ are given functions.

The motivation for an abstract theory such as this comes from the following partial differential equation:

$$\begin{aligned} &v_t(x,t) - v_{xx}(x,t) = \sigma(v(x,t))_x, && 0 < x < 1 \\ &v(0,t) = v(1,t) = 0 && t > 0 \\ &v(x,0) = v(x,1) + s(x) && 0 < x < 1 \end{aligned}$$

It is not true in general that $\frac{\partial}{\partial x} = A^{1/2}$, however [13] there exist a bounded linear operator B from X into itself such that $A^{1/2} B = \frac{\partial}{\partial x}$. Letting $G = B\sigma$ we can fit the above equations into the abstract theory developed in this paper.

The abstract theory one can find in the books [8, 9, 12] handles partial differential equations of the above forms, however the theory illustrated in these works does not distinguish between the problems of the form

$$v_t(x,t) - v_{xx}(x,t) = v(x,t)|v(x,t)|^{\beta-1}$$

with nonlocal conditions and the above equations but with right hand side $\frac{\partial}{\partial x}[v(x,t)$ $|v(x,t)|^{\beta-1}]$

As in [1-3, 7, 8, 10, 11] the nonlocal condition (4) can be applied in physics with better effect than the classical condition $u(0) = u_0$ since condition (4) is usually more precise for physical measurements than the classical condition.

2. Preliminaries

It is known that equations (3) – (4) are related to the integral equation

$$u(t) = T(t)u_0 - T(t)g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) + \int_0^t T(t-s) f(u(s)) ds, \quad t \geq 0 \quad (5)$$

where $T(t)$ is the semigroup of operators generated by $-A$. The solution $u(t)$ of equation (5) is called a mild solution of equations (3) – (4) and is not necessarily a solution of equations (3) – (4).

Definition 1. A function $u(t)$ is a mild solution of equations (3) – (4) on $[0, b)$ if $u \in C([0, b); X_\alpha)$, $u(0) = u_0 - g$, and $u(t)$ satisfies the integral equation (5) on $[0, b)$.

Definition 2. A function $u(t)$ is a strong solution of equations (3) – (4) if $u \in C([0, b); X_\alpha)) \cap C^1([0, b); X)$, $u(0) = u_0 - g$, and $u(t)$ satisfies (3) – (4) on $[0, b)$.

3. Existence of Solutions

We shall make the following assumptions on the operator A and the nonlinear operators f and g:

(i) $-A$ is the infinitesimal generator of a bounded analytic semigroup of linear operator $T(t)$, $t > 0$, in X.
(ii) There exist real constants M and δ such that $||T(t)|| \leq Me^{\delta t}$ for $t > 0$.
(iii) For $0 \leq \alpha < 1$, the fractional power A^α satisfies $||A^\alpha T(t)|| \leq C_\alpha t^{-\alpha}$ for $t > 0$ where C_α is a real constant.

We shall assume $X_\alpha \subseteq Y \subseteq X$ so that $T(t) : X \to Y$ for all $t > 0$ is a bounded linear operater and

(iv) $A^\beta T(t) : X \to Y$ for $t > 0$ and $||A^\beta T(t)|| \in L^1(0, r)$ for $\beta \in [\alpha, \alpha + d]$ for some $d > 0$ and every $r > 0$.
(v) The function f maps X_α into X, and satisfies:
there exists $G : Y \to X$ such that $||G(u) - G(v)|| \leq K||u - v||$ where K is a constant, $G : X_\alpha \to X_\alpha$ and for each $u(0) \in X_\alpha$, $f(u(0)) = A^\alpha G(u(0))$
(vi) The function $g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p))$ maps $J^p \times X^p_\alpha$ into Y and satisfies:
there exist $h(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) : J^p \times X^p \to X$ and a constant $K_1 > 0$ such that

$$||h(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) - h(t_1, \ldots, t_p, v(t_1), \ldots, v(t_p))|| \leq K_1||u - v||,$$
$$h : J^p \times X^p_\alpha \to X_\alpha \text{ and } g = A^\alpha h.$$

Theorem 1. *If the assumptions (i) to (vi) hold, then for each $u(0) \in Y$ there exists a $b > 0$ and a unique continuous function $u : [0, b) \to Y$ such that*

$$u(t) = T(t)u_0 - T(t)A^\alpha h(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p)) + \int_0^t A^\alpha T(t - s)G(u(s))ds, \quad t \geq 0 \tag{6}$$

Proof. Define the set $S = \{u : [0, t] \to Y : u(t)$ and $g \in Y$ are continuous, $u(0) = u_0 - g$ and $||u(t) - u(0)|| \leq R\}$. Choose 'b' such that for

$$||(T(t) - I)||(||u_0|| + ||g(t_1, \ldots, t_p, u(t_1), \ldots, u(t_p))||)$$
$$+Cb^{(1-\alpha)}/(1 - \alpha)\{||G(u(0))|| + KR\} \leq R$$
$$\text{and} \qquad \{||A^\alpha T(t)||K_1 + Kb^{(1-\alpha)}/(1 - \alpha)\} < 1$$

Moreover, define the mapping P on S by

$$(Pu)(t) = T(t)u_0 - T(t)g(t_1,\dots,t_p,u(t_1),\dots,u(t_p)) + \int_0^t A^\alpha T(t-s)G(u(s))ds, \quad t \geq 0$$

First note that for $u \in S$, P is well defined since

$$\begin{aligned}\int_0^t \|A^\alpha T(t-s)G(u(s))\|ds &\leq \{\int_0^t A^\alpha T(t-s)\|ds\} \cdot \{\|G(u(0)) + G(u(t)) - G(u(0))\|\} \\ &\leq \{\int_0^t \|A^\alpha T(t-s)\|ds\} \cdot \{\|G(u(0))\| + KR\}\end{aligned}$$

For $u \in S$, we have

$$\begin{aligned}\|(Pu)(t) - u(0)\| &= \|T(t)u_0 - T(t)g(t_1,\dots,t_p,u(t_1),\dots,u(t_p)) \\ &\quad + \int_0^t A^\alpha T(t-s)G(u(s))ds - (u_0 - g)\| \\ &\leq \|(T(t)-I)\|(\|u_0\| + \|g(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\|) \\ &\quad + Cb^{(1-\alpha)}/(1-\alpha)\{\|G(u(0))\| + KR\} \\ &\leq R\end{aligned}$$

This implies that $P(S) \subseteq S$. Therefore, P maps S into itself. Let $u, v \in S$, then we have

$$\begin{aligned}&\|(Pu)(t) - (Pv)(t)\| \\ &\leq \|T(t)A^\alpha(h(t_1,\dots,t_p,u(t_1),\dots,u(t_p)) - h(t_1,\dots,t_p,v(t_1),\dots,v(t_p)))\| \\ &\quad + \|\int_0^t A^\alpha T(t-s)[G(u(s)) - G(v(s))]ds\| \\ &\leq \|A^\alpha T(t)\|K_1\|u-v\| + \{K\int_0^t \|A^\alpha T(s)\|ds\}\|u-v\| \\ &\leq \{\|A^\alpha T(t)\|K_1 + K\int_0^t \|A^\alpha T(s)ds\|\}\|u-v\| \\ &\leq \{\|A^\alpha T(t)\|K_1 + Kb^{1-\alpha}/1-\alpha)\}\|u-v\|\end{aligned}$$

By the contraction mapping theorem P has a unique fixed point $u \in S$.

Lemma 1. *Let assumptions (i) – (vi) be satisfied, then all $\theta > 0$ and t, $t+h \in [\theta, a]$ there exist σ such that $\|u(t+h) - u(t)\| \leq C(\theta)h^\sigma$, $0 < \sigma < 1$.*

Proof. Now

$$\begin{aligned}\|u(t+h) - u(t)\| &\leq \|(T(h)-I)T(t)u_0)\| + \|(T(h)-I)A^\alpha T(t)h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\| \\ &\quad + \|\int_0^t A^\alpha\{(T(h)-I)T(t-s)\}G(u(s))ds\|\end{aligned}$$

$$
\begin{aligned}
&+\|\int_t^{t+h} A^\alpha T(t+h-s)G(u(s))ds\| \\
\le &\|A^\alpha T(t)(T(h)-I)A^{-\alpha}u_0\| \\
&+\|A^\alpha T(t)(T(h)-I)h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\| \\
&+\|(T(h)-I)\int_0^t A^\alpha T(t-s)G(u(s))ds\| \\
&+\|\int_t^{t+h} A^\alpha T(t+h-s)G(u(s))ds\| \\
\le &C\theta^{-\tau}\|(T(h)-I)A^{-\alpha}u_0\| \\
&+C\theta^{-\tau}\|T(h)-I)A^{-\alpha}A^\alpha T(t)h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\| \\
&+\|\int_0^t A^{\alpha+\epsilon}T(t-s)(T(h)-I)A^{-\epsilon}G(u(s))ds\| \\
&+\int_t^{t+h} A^\alpha T(t+h-s)ds\cdot\{\|G(u(0))\|+KR\} \\
\le &C\theta^{-\tau}\|u_0\|h^\alpha + C\theta^{-\tau}\|A^\alpha T(t)h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\|h^\alpha \\
&+[C(\|G(u(0))\|+KR)\int_0^t \|A^{\alpha+\epsilon}T(s)\|ds\}h^\epsilon + C\{\|G(u(0))\|+KR\}h^{1-\tau}
\end{aligned}
$$

Taking $\sigma=\min\{\alpha,1-\tau,\varepsilon\}$, where $0<\varepsilon<\sigma$, hence the Lemma.

This establishes that a solution of equation (6) is locally Holder continuous on $(0,b]$. If the solution $u(t)$ of equation (6) is in X_α and if it is also Holder continuous in the X_α norm we can show that $u(t)$ is a solution of $(3)-(4)$ if f and g are locally Lipschitz continuous from X_α into X and $J^p\times X_\alpha^p$ into Y respectively.

Lemma 2. *Let assumptions of Lemma 1 be hold. Then the solution $u(t)$ of equation (6) is in $X_{1-\alpha}$ for $t\in(0,b]$.*

Proof. Let $\mu>0$, then the solution $u(t)$ of equation (6) satisfies

$$
\begin{aligned}
u(t)=&T(t-\mu)u_0-T(t-\mu)A^\alpha h(t_1,\dots,t_p,u(t_1),\dots,u(t_p)) \\
&+\int_\mu^t A^\alpha T(t-s)G(u(s))ds
\end{aligned}
$$

and

$$
\begin{aligned}
u(t)=&T(t-\mu)u_0-T(t-\mu)A^\alpha h(t_1,\dots,t_p,u(t_1),\dots,u(t_p)) \\
&+\int_\mu^t A^\alpha T(t-s)G(u(t))ds+\int_\mu^t A^\alpha T(t-s)[G(u(s))-G(u(t))]ds
\end{aligned}
$$

Since $T(t-\mu)u_0$, $T(t-\mu)A^\alpha h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\in D(A)$ for all $t>\mu$ and $\int_\mu^t A^\alpha T(t-s)G(u(t))ds=A^{\alpha-1}[G(u(t))-T(t-\mu)G(u(t))]$ for all $t\ge\mu$. We have only to show that

$$\int_{\mu}^{t} A^{\alpha}T(t-s)[G(u(s))-G(u((t))]ds$$

is in $X_{1-\alpha}$. Note that

$$\begin{aligned}\|A^{1-\alpha}\int_{\mu}^{t} A^{\alpha}T(t-s)[G(u(s))-G(u(t))]ds\| &= \|\int_{\mu}^{t} AT(t-s)[G(u(s))-u(t)]ds\| \\ &\leq CK\int_{\mu}^{t}(t-s)^{-1}(t-s)^{\sigma}ds \\ &= CK(t-\mu)^{\sigma}/\sigma.\end{aligned}$$

By Lemma 1, the last inequality is true.

Lemma 3. *Let the assumptions of the previous Lemma holds and that $X_{1-\alpha} \subseteq X_{\alpha}$, the imbedding being continuous, then the solution $u(t)$ of equation (6) is a mild solution of equations (3) – (4) and is locally Holder continuous into X_{α}.*

Proof. From Lemma 2 and assuption $X_{1-\alpha} \subseteq X_{\alpha}$ implies that $u(t) \in X_{\alpha}$ for all $t > 0$. Thus for $\mu > 0$ and $t > \mu/2$ we have

$$\begin{aligned}u(t) &= T(t-\mu/2)u_0 - T(t-\mu/2)g(t_1,\ldots,t_p,u(t_1),\ldots,u(t_p)) \\ &\quad + \int_{\mu/2}^{t} T(t-s)f(u(s))ds \qquad (7)\end{aligned}$$

Since $u(t)$ is continuous into $X_{1-\alpha}$ and $X_{1-\alpha}$ is continuously imbedded into X_{α}, $u(t)$ is continuous into X_{α}. Now we show that $u(t)$ is locally Holder continuous into X_{α}. Let $\mu > 0$ and $t+h$, $t \in [\mu, b]$, then

$$\begin{aligned}\|u(t+h)-u(t)\| &\leq \|A^{\alpha}(T(h)-I)T(t-\mu/2)u_0\| \\ &\quad + \|A^{\alpha}(T(h)-I)T(t-\mu/2)g(t_1,\ldots,t_p,u(t_1),\ldots,u(t_p))\| \\ &\quad + \|\int_{\mu/2}^{t} A^{\alpha}T(t-s)(T(h)-I)f(u(s))ds\| \\ &\quad + \|\int_{t}^{t+h} A^{\alpha}T(t+h-s)f(u(s))ds\| \\ &\leq C\|u_0\|h^{\alpha} + C\|g(t_1,\ldots,t_p,u(t_1),\ldots,u(t_p))\|h^{\alpha} \\ &\quad + \{C\int_{\mu/2}^{t}\|A^{\alpha+\varepsilon}T(t-s)\|ds\}\sup\|f(u(s))\|h^{\varepsilon} \\ &\quad + C/(1-\alpha)\sup\|f(u(t))\|h^{1-\alpha}\end{aligned}$$

where ε is chosen so that $\alpha+\varepsilon < 1$. Thus there exist a $C > 0$ and a $0 < \theta < 1$ such that $\|u(t+h)-u(t)\| \leq Ch^{\theta}$ for $t, t+h, \in [\mu, b]$.

Theorem 2. *If the conditions* $(i)-(vi)$ *and the assumptions of Lemma 3 are satisfied, and* f *and* g *are locally Lipshitz from* X_α *into* X *and* $J^p \times X_\alpha^p$ *into* Y *respectively, then any solution of equation* (6) *is also a strong solution of equations* (3) $-$ (4).

Proof. Since f and g are locally Lipshitz and $u(t)$ is locally Holder continuous into X_α, the fuctions f and g are locally Holder continuous on $[\mu, b]$ for any $\mu > 0$. Thus the theory of analytic semigroups of linear operators [12] gives the desired result.

Theorem 3. *Let the assumptions of Lemma 1 be satisfied, then* $u(t)$ *may be extended to a maximum interval of existence* $I = [0, c)$, *where* $c = \max b$. *If* $b < \infty$ *then*

$$\lim_{t\to c}\int_0^t \|A^\alpha T(t-s)\|\ \|G(u(s))\| ds = \infty \tag{8}$$

$$\lim_{t\to c}\|u(t)\| = \infty \tag{9}$$

Proof. Suppose $c < \infty$. For every $t \in I$, $u(t)$ satisfies the integral equation (6). Claim $\lim_{t\to c}\sup\|u(t)\| \le C$ for all $t\in I$ and some $C > 0$. This gives

$$\begin{aligned}\|u(t)\| \le \|T(t)\mu_0\| &+ \|T(t)A^\alpha h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\|\\ &+\{\int_0^c \|A^\alpha T(t-s)\|ds\}.\{\sup_{t\in I}\|G(u(t))\|\}\end{aligned}$$

For $0 < \tau < t < c$, we have

$$\begin{aligned}&\|u(t)-u(\tau)\|\\ \le\ &\|A^\alpha T(\tau)[T(t-\tau)-I]A^{-\alpha}u_0\|\\ &+\|A^\alpha T(\tau)[T(t-r)-I]A^{-\alpha}A^\alpha h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\|\\ &+\|\int_0^\tau A^{\alpha+\epsilon}T(\tau-s)[T(t-\tau)-I]A^{-\epsilon}G(u(s))\|ds+\|\int_\tau^t A^\alpha T(t-s)G(u(s))\|ds\\ \le\ &C\tau^{-\varepsilon}\|u_0\|(t-\tau)^\alpha + C\tau^{-\varepsilon}\|A^\alpha h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\|(t-\tau)^\alpha\\ &+C(t-\tau)^\varepsilon\{\int_0^\tau \|A^{\alpha+\varepsilon}T(s)ds\|\}\cdot\{\sup_{t\in I}\|G(u(t))\|\}\\ &+\{\int_\tau^t \|A^\alpha T(s)ds\|\}\cdot\{\sup_{t\in I}\|G(u(t))\|\end{aligned}$$

Thus we have $\|u(t)-u(\tau)\| = 0$ as $t,\tau\to c$, contradicting the maximality of c.

If $\lim_{t\to c}\|u(t)\| \ne \infty$ then there exist numbers $r > 0$ and $d > 0$ with d arbitrarily large and sequences $\tau_n \to c$, $t_n \to c$ as $n\to\infty$ and such that $\tau_n < t_n < c$. $\|u(\tau_n)\| = r$, $\|u(t_n)\| = r+d$ and $\|u(t)\| \le r+d$ for $t\in[\tau_n, t_n]$. We have

$$\begin{aligned}\|u(t_n)-u(\tau_n)\| \le\ &\|[T(t_n-\tau_n)-I]u(\tau_n)\|\\ &+\|A^\alpha[T(t_n-\tau_n)-I]h(t_1,\dots,t_p,u(t_1),\dots,u(t_p))\|\\ &+\|\int_{\tau_n}^{t_n} A^\alpha T(t_n-s)G(u(s))\|\end{aligned}\tag{10}$$

The right hand side of (10) approaches zero as τ_n and t_n approach c while the left hand side of (10) is bounded below by $d > 0$. This contradiction gives the result and (8) follows from (9).

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