TAMKANG JOURNAL OF MATHEMATICS
Volume 30, Number 3, Autumn 1999

HÖLDER CONTINUOUS FUNCTIONS AND THEIR ABEL AND LOGARITHMIC MEANS

SUSHIL SHARMA AND S. K. VARMA

Abstract. Mohapatra and Chandra [8] have obtained the degree of approximation for $f \in H_\alpha (0 \leq \beta < \alpha \leq 1)$ using infinite matrix $A = (a_{n,k})$. Mohapatra and Chandra [7] used Euler, Boral and Taylor means. In the present paper we have obtained the analogous results using Abel (A_λ) and Logarithmic (L)-means.

1. Introduction

Let $C_{2\pi}$ be the Banach space of all 2π periodic functions defined on $[-\pi, \pi]$ under the sup norm. For $0 < \alpha \leq 1$ and some positive constant k.

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq k|x-y|^\alpha\}. \tag{1.1}$$

The space H_α is a Banach space with the norm $||\cdot||_\alpha$ defined by

$$||f||_\alpha = ||f||_c + \sup_{x,y}\{\Delta^\alpha f(x,y)\} \tag{1.2}$$

where

$$||f||_c = \sup_{-\pi \leq x \leq \pi} |f(x)|$$

and

$$\Delta^\alpha f(x,y) = \frac{|f(x) - f(y)|}{|x-y|^\alpha} \qquad (x \neq y) \tag{1.3}$$

we shall use the convention that $\Delta^0 f(x,y) = 0$. The elements of the space H_α are called Hölder continuous functions. If D is the collection of all differentiable functions defined on $[-\pi, \pi]$ then it is easy to see that

$$C_{2\pi} \supseteq H_\beta \supseteq H_\alpha \supset D \quad \text{for} \quad 0 \leq \beta \leq \alpha \leq 1.$$

For each $f \in H_\alpha$, $0 < \alpha \leq 1$, let the Fourier series be given by

$$f(x) \sim \frac{a_0}{2} + \sum_{v=1}^{\infty}(a_v \cos vx + b_v \sin vx) \tag{1.4}$$

Received June 20, 1997; revised March 3, 1999.
1991 *Mathematics Subject Classification.* 42A10, 41A25
Key words and phrases. Approximation, Abel mean, Logarithmic mean.

where a_v and b_v are Fourier coefficients.

Given $\lambda > -1$, we say that the series $\sum U_m$ is A_λ-summable to a finite sum S if the series

$$(1-r)^{\lambda+1}\sum_{n=0}^{\infty}\epsilon_n^\lambda r^n S_n$$

is convergent for all r in (0,1) and tends to S as $r \to 1$ in (0,1), where

$$e_n^\lambda = \binom{n+\lambda}{n} \quad \text{and} \quad S_n = \sum_{m=0}^{n} U_m.$$

If $\lambda = 0$, A_λ-method reduces to well known Abel method of summability (Hardy [3]).

Borwein [1] introduced Logarithmic (L)-method of summabililty. He defined a series $\sum U_m$ to be summable by L-method of summability to the sum S if, for r in the interval (0,1)

$$\lim_{r\to 1-0}\frac{1}{|\log(1-r)|}\sum_{n=1}^{\infty} S_n\frac{r^n}{n} = S.$$

We have following inclusion relation

$$(L) \supset (A,\lambda) \supset A \supset (C,\delta) \text{ for every } \delta > -1.$$

We shall use the following notations throughout this paper

$$2\phi_x(t) = f(x+t) + f(x-t) - 2f(x), \tag{1.5}$$

$$A_\lambda(r,t) = (1-r)^{\lambda+1}\sum_{n=0}^{\infty} e_n^\lambda r^n \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}} \tag{1.6}$$

$$L(r,t) = \frac{1}{t}\tan^{-1}\left(\frac{r\sin t}{1-r\cos t}\right). \tag{1.7}$$

2. Statement of Results

Khan [5] obtained the necessary and sufficient condition for A_λ -summability of Fourier series (1.4) and Hsiang [4] obtained the necessary and sufficient condition for L-summability of Fourier series (1.4).

Prossdorf [10] proved the following theorem A with a view to obtain the degree of convergence of the Fejér means of the Fourier series of $f \in H_\alpha$.

Theorem A. *Let $f \in H_\alpha$, $0 < \alpha \leq 1$ and $0 \leq \beta < \alpha$, then*

$$\|\sigma_n(f:x) - f\|_\beta = \begin{cases} O(n^{\beta-\alpha}), & 0 < \alpha < 1, \\ O\{n^{\beta-1}(1+\log n)^{1-\beta}\}, & \alpha = 1. \end{cases} \tag{2.1}$$

Mohapatra and Chandra [7] replaced the Fejér means of the series (1.4) by well known Nörlund and $(\overline{N}, p_n)$ means. Mohapatra and Chandra [8] replaced Nörlund and $(\overline{N}, p_n)$ by a more general infinite matrix $T = (a_{n,k})$ with some conditions on $(a_{n,k})$. Mohapatra [9] proved results on degree of approximation of Hölder continuous functions. Das, Ojha and Ray [2] used Borel means to obtain the degree of approximation. Mohapatra and Chandra [7] used Euler, Boral and Taylor means to obtain the degree of approximation for $f \in H_\alpha$. They proved the following theorem:

Theorem B. *Let $0 \leq \beta < \alpha \leq 1$. Then, for $f \in H_\alpha$,*

$$\|B^r(f : x) - f\|_\beta = O\{r^{-(1/2)(\alpha-\beta)}(\log r)^{\beta/\alpha}\} \tag{2.2}$$

where $B^r(f : x)$ is the Boral mean of the series (1.4).

In the present paper we have used A_λ-mean and L-mean. In fact we shall prove the following theorems:

Theorem 1. *Let $0 \leq \beta < \alpha \leq 1$. Then, for $f \in H_\alpha$*

$$\|A_\lambda^r(f) - f\|_\beta = O[(1-r)^{\alpha-\beta}] \tag{2.3}$$

where $A_\lambda^r(f)$ is the A_λ-mean of the series (1.4).

Theorem 2. *Let $0 \leq \beta < \alpha \leq 1$. Then for $f \in H_\alpha$*

$$\|L^r(f) - f\|_\beta = O[\log(1-r)^{(\beta/\alpha)-1}(1-r)^{\alpha-\beta}r^\beta] \tag{2.4}$$

where $L^r(f)$ is the L-mean of the series (1.4).

3. Preliminary Lemmas

We shall use the following lemmas for the proof of our theorems.

Lemma 1. *If $f \in H_\alpha (0 < \alpha \leq 1)$ then*

$$|\phi_x(t) - \phi_y(t)| \leq 4k|x-y|^\alpha \tag{3.1}$$

and

$$|\phi_x(t) - \phi_y(t)| \leq 4k|t|^\alpha \tag{3.2}$$

where k is a positive constant. The proof of the lemma is obvious from the definition of $\phi_x(t)$ and the function space H_α.

Lemma 2. Khan [6]

$$A_\lambda(r,t) = \begin{cases} O[(\frac{1}{1-r})], & \text{when } t \leq 1-r, \\ & 0 < t \leq \pi, \\ & 0 \leq r < 1. \\ O[\frac{(1-r)^{\lambda+1}}{t^{\lambda+2}}], & \text{when } t \geq 1-r, \\ & 0 < t \leq \pi, \\ & 0 \leq r < 1. \end{cases} \tag{3.3}$$

Lemma 3. Hsiang [3] *For* $0 < r < 1$,

$$\frac{1}{t}\tan^{-1}\left(\frac{r\sin t}{1-r\cos t}\right) = \begin{cases} O[\frac{r}{1-r}], & 0 < t \leq 1-r \\ O[\frac{1}{t}], & 1-r < t \leq \pi. \end{cases} \tag{3.4}$$

4. Proof of The Theorem 1

Let

$$\begin{aligned} I_n(x) &= A_\lambda^r(f,x) - f(x) \\ &= (1-r)^{\lambda+1}\sum_{n=0}^{\infty}[S_n(x)-f(x)]r^n\epsilon_n^\lambda \\ &\doteq \frac{(1-r)^{\lambda+1}}{\pi}\sum_{n=0}^{\infty} r^n\epsilon_n^\lambda \int_0^\pi \frac{\phi_x(t)\sin(n+1/2)t}{\sin t/2}dt. \end{aligned}$$

Therefore

$$\begin{aligned} I_n(x) - I_n(y) &= \frac{1}{\pi}\int_0^\pi |\phi_x(t)-\phi_y(t)|A_\lambda(r,t)dt \\ &= \frac{1}{\pi}\Big[\int_0^{1-r} + \int_{1-r}^\pi\Big]|\phi_x(t)-\phi_y(t)|A_\lambda(r,t)dt \\ &= I_1 + I_2, \text{ say }. \end{aligned} \tag{4.1}$$

So

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_0^{1-r} t^\alpha(\frac{1}{1-r})dt, \text{ by (3.2) and (3.3) }. \\ &= O(1-r)^\alpha. \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} I_2 &= \frac{1}{\pi}\int_{1-r}^\pi (1-r)^{\lambda+1}t^{\alpha-\lambda-2}dt, \text{ by (3.2) and (3.3)} \\ &= O(1-r)^\alpha. \end{aligned} \tag{4.3}$$

Similarly if we use (3.1) in place of (3.2), we get

$$\begin{aligned} I_1 &= O(1)\int_0^{1-r}\frac{|x-y|^\alpha}{(1-r)}dt \\ &= O|x-y|^\alpha. \end{aligned} \tag{4.4}$$

and

$$I_2 = O(1) \int_{1-r}^{\pi} \frac{|x-y|^\alpha}{t^{\lambda+2}} (1-r)^{\lambda+1} dt$$
$$= O|x-y|^\alpha. \tag{4.5}$$

Now for $k = 1, 2$, we observe that

$$I_k = I_k^{1-\beta/\alpha} I_k^{\beta/\alpha}$$

So by (4.2), (4.3), (4.4) and (4.5), we have

$$I_1 = O\{|x-y|^\beta (1-r)^{\alpha-\beta}\}$$

and

$$I_2 = O\{|x-y|^\beta (1-r)^{\alpha-\beta}\}$$

So

$$\sup_{\substack{x,y \\ x \neq y}} |\Delta^\beta I_n(x,y)| = O\{(1-r)^{\alpha-\beta}\}.$$

This completes the proof of the Theorem 1.

Proof of theorem 2.

$$I_n(x) = \frac{1}{|\log(1-r)|} \frac{1}{\pi} \int_0^\pi \frac{\phi_x(t)}{\sin t/2} \sum_{n=1}^\infty \frac{r^n}{n} \sin(n+1/2)tdt.$$

So

$$I_n(x) - I_n(y) = \frac{1}{|\log(1-r)|} \frac{1}{\pi} \int_0^\pi \frac{\phi_x(t) - \phi_y(t)}{\sin t/2} \sum_{n=1}^\infty r^n \sin(n+1/2)tdt$$
$$= \frac{1}{|\log(1-r)|} \frac{1}{\pi} \Big[\int_0^{1-r} + \int_{1-r}^\pi \Big] \frac{|\phi_x(t) - \phi_y(t)|}{\sin t/2} \sum_{n=1}^\infty r^n . \sin(n+1/2)tdt.$$
$$= I_1 + I_2, \text{ say }. \tag{4.6}$$

$$I_1 = \frac{1}{|\log(1-r)|} O(1) \int_0^{1-r} \frac{r}{(1-r)} t^\alpha dt, \text{ by using (3.2) and (3.4)}$$
$$= O\Big(\frac{r}{|\log(1-r)|} (1-r)^\alpha\Big) \tag{4.7}$$
$$I_2 = \frac{1}{|\log(1-r)|} \frac{1}{\pi} \int_{1-r}^\pi \frac{|\phi_x(t) - \phi_y(t)|}{\sin t/2} \sum_{n=1}^\infty r^n \sin(n+1/2)tdt.$$
$$= O(1) \frac{1}{|\log(1-r)|} \int_{1-r}^\pi t^{\alpha-1} dt, \text{ by using (3.2) and (3.4) }.$$
$$= O\Big\{\frac{(1-r)^\alpha}{|\log(1-r)|}\Big\} \tag{4.8}$$

Similarly if we use (3.1) in place of (3.2), we get

$$I_1 = \frac{O(1)}{|\log(1-r)|}\int_0^{1-r} |x-y|^\alpha(\frac{r}{1-r})dt$$
$$= O(\frac{|x-y|^\alpha r}{|\log(1-r)|}) \tag{4.9}$$

and

$$I_2 = \frac{O(1)}{|\log(1-r)|}\int_{1-r}^{\pi} \frac{|x-y|^\alpha}{t}dt$$
$$= O\{|x-y|^\alpha\} \tag{4.10}$$

Now for $k = 1, 2$ we observe that

$$I_k = I_k^{1-(\beta/\alpha)} I_k^{\beta/\alpha}$$

So by (4.7), (4.8), (4.9) and (4.10), we have

$$I_1 = O[\frac{|x-y|^\beta r^{\beta/\alpha} r^{(\alpha-\beta)/\alpha}(1-r)^{\alpha-\beta}}{\log(1-r)}]$$
$$= O[\frac{|x-y|^\beta r^\beta (1-r)^{\alpha-\beta}}{\log(1-r)}] \tag{4.11}$$
$$I_2 = O[\frac{|x-y|^\beta \{\log(1-r)\}^{\beta/\alpha}(1-r)^{\alpha-\beta}}{|\log(1-r)|}] \tag{4.12}$$

Since $\beta < 1$ and $r \le 1$, (4.11) and (4.12) gives

$$\sup_{\substack{x,y \\ x\neq y}} |\Delta^\beta I_n(x,y)| = O[\{\log(1-r)\}^{(\beta/\alpha)-1}(1-r)^{\alpha-\beta} r^\beta]$$

This completes the proof of the Theorem 2.

Acknowledgement

Authors extend thanks to Professor Prem Chandra for this paper.

References

[1] D. Borwein, *A logarithmic method of summability*, J. Lond. Math. Soc. **33**(1958), 212-220.

[2] G. Das, A. K. Ojha and B. K. Ray, *Degree of approximation of functions associated with Hardy-Littlewood series in the Hölder metric by Boral means*, J. Math. Anal. Appl. **219**(1998), 279-293.

[3] G. H. Hardy, *Divergent Series*, Oxford Publication, 1949.

[4] F. C. Hsiang, *Summability of the Fourier series*, Bull. Amer. Math. Soc. **67**(1961), 150-153.

[5] H. Khan, *On some aspects of summability*, Ind. J. pure appl. Math. **6**(1975), 1468-1472.
[6] H. Khan, *On the degree of approximation*, Math. Chronicle **10**(1981), 63-72.
[7] R. N. Mohapatra and P. Chandra, *Hölder continuous functions and their Euler*, Boral and Taylor means, Math. Chronicle **11**(1982), 81-96.
[8] R. N. Mohapatra and P. Chandra, *Degree of approximation of functions in the Hölder matrix*, Acta Math. Hung. **41**(1983), 67-76.
[9] R. N. Mohapatra, *Degree of approximation of Hölder continuous functions*, Math. Nachr. **140**(1988), 91-96.
[10] S. Prossdorf, *Zur Konvergenz der Fourierreihen Hölderstetiger Funktionen*, Math. Nachr., **69**(1975), 7-14.

Department of Mathematics, Madhav Science College, Ujjain, M. P., India.

Department of Mathematics, Govt. Autonomous Science Post-Graduate College, Bilaspur, M. P., India.