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BEST COAPPROXIMATION IN METRIC LINEAR SPACES

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Abstract. In order to obtain some characterizations of real Hilbert spaces among real Banach spaces, a new kind of approximation, called best coapproximation, was introduced in normed linear spaces by *C.* Franchetti and M. Furi [3] in 1972. Subsequently, the study was pursued in normed linear spaces and Hilbert spaces by H. Berens, L. Hetzelt, T. D. Narang, P. L. Papini, Geetha *S.* Rao and her students, Ivan Singer and a few others (see, e.g., [1], [4], [7], [9], [13 to 15], and [17 to 20]). In this paper, we discuss best coapproximation in metric linear spaces thereby generalizing some of the results proved in [3], [7], [13], and [18]. The problems considered are those of existence of elements of best coapproximation and their characterization, characterizations of coproximinal, co-semi-Chebyshev and co-Chebyshev subspaces, and some properties of the best coapproximation map in metric linears spaces.

1. Introduction

The main object of the theory of best approximation is to seek a solution to the problem: Given a subset G of a metric space (X,d) and an element $x \in X$, find an element $g_0 \in G$ such that

$$d(x,g_0) \leq d(x,g) \text{ for all } g \in G \tag{1}$$

The set of all such $g_0 \in G$ (if any), called the set of best approximation of x by elememts of G, is denoted by $P_G(x)$. Clearly

$$P_G(x) = [\cap_{g\in G} B(x, d(x,g))] \cap G,$$

where $B(x,d(x,g))$ denotes the closed ball in X with centre x and radius $d(x,g)$.

As a counterpart to best approximation, another kind of approximation, called best coapproximation, was introduced by Franchetti and Furi [3], who considered those elements $g_0 \in G$ satisfying

$$d(g_0,g) \leq d(x,g) \text{ for all } g \in G \tag{2}$$

The set of all such $g_0 \in G$ (if any) is denoted by $R_G(x)$. Clearly

$$R_G(x) = [\cap_{g\in G} B(g, d(x,g))] \cap G.$$

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An element $g_0 \in G$ satisfying (1) is called a best approximation to x in G, and satisfying (2) is called a best coapproximation to x in G. The set G is said to be proximinal (respectively, coproximinal) if $P_G(x)$ (respectively, $R_G(x)$) is non empty for each x in X. It is said to be semi-Chebyshev (respectively, co-semi-Chebyshev) if $P_G(x)$ (respectively, $R_G(x)$) contains at most one element for each x in X and it is said to be Chebyshev (respectively, co-Chebyshev) if $P_G(x)$ (respectively, $R_G(x)$) contains exactly one element for each x in X. If $D(P_G) = \{x \in X : P_G(x) \neq \emptyset\}$ (respectively, $D(R_G) = \{x \in X : R_G(x) \neq \phi\}$), the mapping $P_G : D(P_G) \to G$ (respectively, the mapping $R_G : D(R_G) \to G$), defined by $x \to P_G(x)$ (respectively, $x \to R_G(x)$) is called the best approximation map or metric projection (respectively, best coapproximation map or metric coprojection). In general, $D(P_G)$ (respectively, $D(R_G)) \neq X$ and the mapping P_G (respectively, R_G) is multivalued on $D(P_G)|G$ (respectively, $D(R_G)|G$), but the restriction of the mapping P_G (respectively, R_G) to G is single-valued. We have $D(P_G)$ (respectively, $D(R_G)) = X$ if G is proximinal (respectively, coproximinal) and is single-valued on X if G is Chebyshev (respectively, co-Chebyshev).

As in the case of best approximation, the theory of best coapproximation has been developed to a large extent in normed linear spaces and in Hilbert spaces by H. Berens and U. Westphal [1], C. Franchetti and M. Furi [3], L. Hetzelt [4], T. D. Narang [7], [9], P. L. Papini and I. Singer [13], Geetha S. Rao and her students [4], [15], [17], [18], U. Westphal [20] and a few others. Geetha S. Rao was the first to develop the theory of best coapproximation after the appearance of the paper of Papini and Singer [13]. In a series of papers she and her students have proved many results on best coapproximation in normed linear spaces. The situation in the case of metric linear spaces is somewhat different. Whereas many successful attempts have been made to develop the theory of best approximation in metric linear spaces (although the theory is comparatively less developed than that in normed linear spaces due to the non-convexity of spheres, lack of duality theory in metric linear spaces, etc.), the theory of best coapproximation in such spaces is yet to make a beginning. The present paper is a step in this direction. We discuss in this paper some results on existence of elements of best coapproximation and their characterization, characterizations of coproximinal, co-semi-Chebyshev and co-Chebyshev subspaces and some properties of the best coapproximation map in metric linear spaces.

2. Existence and Characterizations of Elements of Best Coapproximation

In this section, we discuss some results on the existence and characterization of elements of best coapproximation in metric linear spaces. We start with listing a few elementary observations.

Observation 2.1. If G is a subset of a metric space (X, d), then $G \subset R_G(x)$ whenever the diameter of G is smaller than dist (x, G).

Observation 2.2. If G is a convex subset of a strongly locally convex metric linear space(a metric linear space in which all spheres are convex-see [6] (X, d) then $R_G(x)$ is a convex set.

Observation 2.3. If G is a linear subspace of a metric linear space (X, d) and $R_G^{-1}(0) = \{x \in X : 0 \in R_G(x)\}$, then

(i) $R_G^{-1}(0)$ is a closed set containing 0,
(ii) $g_0 \in R_G(x) \Leftrightarrow 0 \in R_G(x - g_0)$ i.e., $x - g_0 \in R_G^{-1}(0)$, and
(iii) for $g \in G$, we have $z \in R_G^{-1}(0) \Leftrightarrow g \in R_G(g + z)$, i.e., $g + z \in R_G^{-1}(g)$.

The following theorem gives a necessary and sufficient condition under which $R_G(x) \neq \emptyset$:

Theorem 1. *If G is a linear subspace of a metric linear space (X, d), then $R_G(x) \neq \emptyset$ for some $x \in X \backslash G$ if and only if $R_G^{-1}(0)$ is not a singleton.*

Proof. By Observation 2.3 (i), $0 \in R_G^{-1}(0)$. Suppose, $g_0 \in R_G(x)$ for some $x \in X \backslash G$. Then by Observation 2.3 (ii), $0 \neq x - g_0 \in R_G^{-1}(0)$ and so $R_G^{-1}(0)$ is not a singleton.

Conversely, suppose $R_G^{-1}(0)$ is not a singleton. Then there exists an $x \neq \emptyset \in R_G^{-1}(0)$ and so $0 \in R_G(x)$ i.e., $R_G(x) \neq \emptyset$ for some $x \in X \backslash G$.

It may be remarked that a similar result holds for $P_G(x)$.

It was shown by Johnson [5] that if (X, d) is a metric space and x_0 is a fixed point of X then the set

$$X_0^{\#} = \{f : X \to \mathbb{R} \sup_{x \neq y_x, y \in x} \frac{|f(x) - f(y)|}{d(x, y)} < \infty, \quad f(x_0) = 0\},$$

with the usual operations of addition, and multiplication by real scalars, normed by

$$||f||_X = \sup_{x \neq y_x, y \in x} \frac{|f(x) - f(y)|}{d(x, y)}, \quad f \in X_0^{\#}$$

is a Banach space (even a conjugate Banach space). Using this idea of Johnson, we prove:

Theorem 2. *If G is a linear subspace of a metric linear space $(X, d), x \in X|G$ and $g_0 \in G$ then $g_0 \in R_G(x)$ if for every $g \in G$ there exists an $f^g \in X_0^{\#}$ with the following properties:*

(i) $|f^g(x) - f^g(y)| \leq d(x, y)$ *for all* $x, y \in X$,
(ii) $f^g(x - g_0) = 0$, *and*
(iii) $f^g(g_0 - g) = d(g_0, g)$.

Proof. Suppose for every $g \in G$ there exists an $f^g \in X_0^{\#}$ satisfying (i), (ii), and (iii). Consider

$$\begin{aligned} d(x,g) &\geq |f^g(x) - f^g(g)|, \text{ by (i)}, \\ &= |f^g(x - g_0 + g_0 - g)| \\ &= |f^g(g_0 - g)|, \text{ by (ii)} \\ &= d(g_0, g), \text{ by (iii)} \end{aligned}$$

i.e., $d(g_0, g) \leq d(x, g)$ for all $g \in G$ and so $g_0 \in R_G(x)$.

Problem 2.1. If $g_0 \in R_G(x)$ then can we find an $f^g \in X_0^{\#}$ satisfying (i), (ii), and (iii)?

Remark 2.1. A result similar to Theorem 2 and its converse were given by G. Pantelidis [12] for $P_G(x)$ in metric linear spaces and for $R_G(x)$ by Papini and Singer [13] in normed linear spaces.

An element x of a metric linear space (X, d) is said to be *orthogonal* to another element $y \in X$ (see [6]), and we write $x \perp y$, if $d(x, 0) \leq d(x, \alpha y)$ for every scalar α. x is said to be orthogonal to a subset G of $X (x \perp G)$ if $x \perp y$ for all $y \in G$. This definition of orthogonality is similar to that given by G. Birkhoff [2]. It is known (see [6]) that if G is a linear subspace of a metric linear space (X, d), $x \in X | \overline{G}$ and $g_0 \in G$ then $g_0 \in P_G(x)$ if and only if $x - g_0 \perp G$. It was proved in [3] that if G is a linear subspace of a normed linear space X and $g_0 \in G$ then $g_0 \in R_G(x)$ if and only if $G \perp x - g_0$. In metric linear spaces, the following is easy to prove:

If G is a linear subspace of a metric linear space (X, d) and $g_0 \in G$, then $g_0 \in R_G(x)$ if $G \perp (x - g_0)$ or if $G - g_0 \perp (x - g_0)$.

Problem 2.2. Is the converse also true, i.e., if $g_0 \in R_G(x)$, then can we prove that $G \perp (x - g_0)$ or $G - g_0 \perp (x - g_0)$?

Using orthogonality in metric linear space, we have:

Theorem 3. *A linear subspace G of a metric linear space (X, d) has the property $R_G(x) = \emptyset$ for every $x \in X \backslash G$ if there exists no $z \in X \backslash \{0\}$ such that $G \perp z$.*

Proof. Suppose there exists some $z \in X \backslash \{\emptyset\}$ such that $G \perp z$, i.e., $g \perp z$ for every $g \in G$. Then

$$d(g, \alpha z) \geq d(g, 0)$$

for all $g \in G$ and for all scalars α. This gives

$$d(0, g) \leq d(z, g)$$

for all $g \in G$, i.e., $0 \in R_G(z)$. Thus, $R_G(z) \neq \emptyset$ for $z \in X | G$, a contradiction.

Remark 2.2. In normed linear spaces, Theorem 3 and its converse were proved in [7]. It is not known whether the converse of Theorem 3 holds in metric linear spaces.

The following theorem on existence also connects elements of best approximation and elements of best coapproximation:

Theorem 4. *If G is a linear subspace of a metric linear space (X, d) and $x \in X \backslash G$ then*

(i) $A = \{g_0 \in G : g_0 \in \cap_{g \in G} P(g)\} \subset R_G(x)$, *where* $(g_0, x) = \{\alpha x + (1-\alpha) g_0 : \alpha \text{ scalar}\}$ *is the linear manifold spanned by* g_0 *and* x,

(ii) *for an element* $g_0 \in G$, *we have* $g_0 \in R_G(x)$ *if* $G \subset P^{-1}_{(x-g_0)}(0) \equiv \{z \in X : 0 \in P_{(x-g_0)}(z)\}$.

Proof.

(i)

$$\begin{aligned} g_0 \in A \Rightarrow\ & g_0 \in G \text{ and } g_0 \in P_{(g_0,x)}(g) \text{ for all } g \in G \\ \Rightarrow\ & g_0 \in G \text{ and } d(g_0, g) \le d(\alpha x + (1-\alpha, g_0, g) \text{ for all } g \in G \text{ and all scalars } \alpha \\ \Rightarrow\ & g_0 \in G \text{ and } d(g_0, g) \le d(x, g) \text{ for all } g \in G \end{aligned}$$

i.e., $g_0 \in R_G(x)$.

(ii)

$$\begin{aligned} G \subset P^{-1}_{(x-g_0)}(0) \Rightarrow\ & 0 \in P_{(x-g_0)}(g) \text{ for all } g \in G \\ \Rightarrow\ & d(g, 0) \le d(\alpha(x - g_0), g) \text{ for all } g \in G \end{aligned}$$

Let $g' \in G$. Take $g = g' - g_0$ and $\alpha = 1$. We get $d(g' - g_0, 0) \le d(x - g_0, g' - g_0)$, i.e., $d(g_0, g') \le d(x, g')$. Therefore, $g_0 \in R_G(x)$.

Remark 2.3. Theorem 4 (i) clearly implies that $A = \{g_0 \in G : 0 \in \cap_{g \in G} P_{(x-g_0)}(g - g_0)\} \subset R_G(x)$.

It was proved by Papini and Singer [13] that in normed linear spaces $R_G(x) = A$ in (i) and the converse part of (ii) also holds. For locally convex spaces with a family of seminorms the equality of the sets $R_G(x)$ and A was proved by Geetha S. Rao and S. Elumalai [16].

3. Characterizations of Coproximinal, Co-Semi-Chebyshev and Co-Chebyshev Subspaces

In this section, we shall characterize coproximinal, co-semi-Chebyshev, and co-Chebyshev subspaces of metric linear spaces.

It was proved in [8] that a linear subspace G of a metric linear space (X, d) is proximinal if and only if $X = G + P_G^{-1}(0)$, where $P_G^{-1}(0) = \{x \in X : 0 \in P_G(x)\}$. Analogously, we have the following characterization of coproximinal linear subspaces of metric linear spaces.

Theorem 5. *For a linear subspace G of a metric linear space (X,d), the following statements are equivalent:*

(a) *G is coproximinal.*
(b) $X = G + R_G^{-1}(0)$.

Proof. $(a) \Rightarrow (b)$. Let $x \in X$. Since G is coproximinal, there exists $g_0 \in G$ such that $g_0 \in R_G(x)$ and so by Observation 2.3 (ii), $x - g_0 \in R_G^{-1}(0)$. Since $x = g_0 + (x - g_0) \in G + R_G^{-1}(0)$, we get $X \subset G + R_G^{-1}(0) \subset X$ and so $X = G + R_G^{-1}(0)$.

$(b) \Rightarrow (a)$. Let $x \in X = G + R_G^{-1}(0)$. Then $x = g_0 + y, g_0 \in G, y \in R_G^{-1}(0)$ and so $0 \in R_G(y) = R_G(x - g_0)$. Therefore, by Observation 2.3 (ii), $g_0 \in R_G(y)$ implying that G is coproximinal.

It was proved in [8] that a linear subspace G of a metric linear space (X,d) is proximinal if and only if G is closed and for the canonical mapping $W_G; X \to \frac{X}{G}$, we have $W_G[P_G^{-1}(0)] = \frac{X}{G}$. Analogously, we have the following characterization of coproximinal linear subspaces of metric linear spaces:

Theorem 6. *For a linear subspace G of a metric linear space (X,d), the following statements are equivalent:*

(a) *G is coproximinal.*
(b) *G is closed and for the canonical mapping $W_G : X \to \frac{X}{G}$, we have $W_G[R_G^{-1}(0)] = \frac{X}{G}$, i.e., W_G maps $R_G^{-1}(0)$ onto $\frac{X}{G}$.*

Proof. $(a) \Rightarrow (b)$. Firstly, we show that G is closed. Let $p \in \overline{G} \backslash G$ and $g_0 \in R_G(p)$. Then there exists a sequence $\langle g_n \rangle$ in G such that $\langle g_n \rangle \to p$ and $d(g_0, g) \leq d(p, g)$ for all $g \in G$ and so $d(g_0, g_n) \leq d(p, g_n$ for all n. This in the limiting case implies that $\langle g_n \rangle \to g_0$ and so $p = g_0 \in G$. Hence G is closed. Now suppose $x + G \in \frac{X}{G}$ and $g_0 \in R_G(x)$. Then by Observation 2.3 (ii), $x - g_0 \in R_G^{-1}(0)$ and $W_G(x - g_0) = (x - g_0) + G = x + G$.

$(b) \Rightarrow (a)$. Let $x \in X$. Then $x + G \in \frac{X}{G} = W_G[R_G^{-1}(0)]$, i.e., $x + G = W_G(y)$ where $y \in R_G^{-1}(0)$, i.e., $x + G = y + G$ where $0 \in R_G(x)$, i.e., $x - y = g_0 \in G$ and $0 \in R_G(x - g_0)$. So, by Observation 2.3 (ii), $g_0 \in R_G(x)$. Hence G is coproximinal.

The following characterization of Chebyshev subspace of metric linear spaces was given in [8].

For a closed linear subspace G of a metric linear space (X,d), the following statements are equivalent:

(i) G is a Chebyshev subspace.
(ii) $X = G \oplus P_G^{-1}(0)$, where $\oplus$ means that the sum decomposition of each $x \in X$ is unique.

(iii) G is proximinal and $[P_G^{-1}(0) - P_G^{-1}(0) \cap G = \{0\}$.
(iv) G is proximinal and the restriction map $W_G|P_G^{-1}(0)$ is one to one.

Analogously, we have the following characterization of co-Chebyshev subspaces of metric linear spaces:

Theroem 7. *For a closed linear subspace G of a metric linear space (X, d), the following statements are equivalent:*

(a) *G is a co-Chebyshev subspace.*
(b) *$X = G \oplus R_G^{-1}(0)$, where $\oplus$ means that the sum decomposition of each $x \in X$ is unique.*
(c) *G is coproximinal and $[R_G^{-1}(0) - R_G^{-1}(0)] \cap G = \{0\}$.*
(d) *G is coproximinal and the restriction map $W_G|R_G^{-1}(0)$ is one to one.*

Proof. $(a) \Rightarrow (b)$. Since G is co-Chebyshev, it is coproximinal and so by Theorem 5, $X = G + R_G^{-1}(0)$. Now we show that the sum decomposition of each $x \in X$ is unique. Suppose $x \in X$ and $x = g_1 + y_1$ and $x = g_2 + y_2$ where $g_1, g_2 \in G$, $y_1, y_2 \in R_G^{-1}(0)$. This gives $g_1 - g_2 = y_2 - y_1$. Now, $y_1 \in R_G^{-1}(0) \Rightarrow 0 \in R_G(y_1) \Rightarrow g_1 \in R_G(y_1 + g_1)$ by Observation 2.3 (iii). i.e., $g_1 \in R_G(x)$. Similarly, $g_2 \in R_G(x)$. Since G is co-Chebyshev, $g_1 = g_2$ and consequently, $y_1 = y_2$. Hence, $X = G \oplus R_G^{-1}(0)$.

$(b) \Rightarrow (c)$. $X = G \oplus R_G^{-1}(0) \Rightarrow G$ is coproximinal by Theorem 6. Suppose $0 \neq y \in [R_G^{-1}(0) - R_G^{-1}(0)] \cap G$. Then $y = y_1 - y_2$, $y_1 \in R_G^{-1}(0)$, $y_2 \in R_G^{-1}(0)$, $y_1 \neq y_2$, So $0 \in R_G(y_1)$, $0 \in R_G(y_2)$. Now $y_1, y_2 \in R_G^{-1}(0)$, $y_1 - y_2 \in G \backslash \{0\}$ and $y_1 = 0 + y_1 = (y_1 + y_2) + y_2$, a contradiction to the uniqueness of the sum decomposition. Hence, $[R_G^{-1}(0) - R_G^{-1}(0)] \cap G = \{0\}$.

$(c) \Rightarrow (d)$. Suppose $W_G|R_G^{-1}(0)$ is not one to one, i.e., there exists $y_1, y_2 \in R_G^{-1}(0)$, $y_1 \neq y_2$ and $W_G(y_1) = W_G(y_2)$. Then $0 \neq y_1 - y_2 \in [R_G^{-1}(0) - R_G^{-1}(0)] \cap G$, a contradiction.

$(d) \Rightarrow (a)$. Suppose $x \in X$ has two distinct best coapproximation in G, say g_1 and g_2. Then by Observation 2.3 (ii), $x - g_1$ and $x - g_2 \in R_G^{-1}(0)$, $x - g_1 \neq x - g_2$ but $W_G(x - g_1) = W_G(x - g_2)$ as $(x - g_1) - (x - g_2) = g_2 - g_1 \in G$ which is a contradiction.

Remark 3.1. By requiring instead of (b) that each element $x \in X$ has at most one sum decomposition and by omitting in (c) and (d) the condition of coproximinality of G, we obtain the following characterizations of co-semi-Chebyshev subspaces of metric linear spaces:

Theorem 8. *For a closed linear subspace G of a metric linear space (X, d), the following statements are equivalent:*

(a) *G is co-semi-Chebyshev subspace.*
(b) *Each element $x \in X$ has atmost one sum decomposition as $G + R_G^{-1}(0)$.*
(c) $[R_G^{-1}(0) - R_G^{-1}(0)] \cap G = \{0\}$.

(d) $W_G | R_G^{-1}(0)$ *is one to one.*

4. The Best Coapproximation Map R_G

In this section, we shall discuss some properties of the best coapproximation map R_G and conditions under which the mapping R_G is upper semi-continuous or continuous.

Theorem 9. *If G is a subset of a metric linear space (X, d) and $x \in X$, the set-valued mapping R_G has the following properties:*

(a) *$D(R_G) \supset G$ and $R_G(x) = \{x\}$ for all $x \in G$.*
(b) *If $x \in D(R_G)$, then $R_G(x) \in D(R_G)$ and $R_G^2(x) = R_G(x)$, i.e., the mapping R_G is idempotent on $D(R_G)$.*
(c) *If $x \in D(R_G)$ and $P_G(x) \neq \emptyset$ then $d(x, R_G^0(x)) \leq 2d(x, P_G^0(x))$, $R_G^0(x) \in R^G(x)$ and $P_G^0(x) \in P_G(x)$.*
(d) *If $0 \in G$ then $d(R_G^0(x), 0) \leq d(x, 0)$ for all $x \in D(R_G)$ and $R_G^0(x) \in R_G(x)$. So R_G is continuous at the origin and is a bounded mapping, in fact $R_G(x) \subset B(0, d(x, 0))$.*
(e) *If G is a linear subspace and R_G is a single-valued on $D(R_G)$ then for $x \in D(R_G)$ and $g \in G$, we have $x + g \in D(R_G)$ and*

$$R_G(x + g) = R_G(x) + R_G(g) = R_G(x) + g$$

i.e., R_G is quasi-additive.
(f) *If G is a linear subspace and $R_G^{-1}(0)$ is a closed linear subspace of X, then R_G is single-valued and additive on $D(R_G)$.*

Proof.

(a) Let $g_0 \in G$ then $g_0 \in R_G(g_0)$ as $d(g_0, g) \leq d(g_0, g)$ for all $g \in G$ and so $g_0 \in D(R_G)$. Thus, $G \subset D(R_G)$. Further, $g_0 \in R_G(g_0) \Rightarrow \{g_0\} \subset R_G(g_0)$. Now, suppose $y \in R_G(g_0)$. Then $d(y, g) \leq d(g_0, g)$ for all $g \in G$ and so in particular, $d(y, g_0) \leq d(g_0, g_0) = 0$ and so $y = g_0$. Therefore, $R_G(g_0) \subset \{g_0\}$. Hence, $R_G(x) = \{x\}$ for all $x \in G$.
(b) Let $x \in D(R_G)$ then $R_G(x) \subset G \subset D(R_G)$ by Part (a). Further, $R_G(x) \in G \Rightarrow R_G[R_G(x)] = R_G(x)$, i.e., $R_G^2(x) = R_G(x)$.
(c) $x \in D(R_G) \Rightarrow R_G(x) \neq \emptyset$. Let $R_G^0(x) \in R_G(x)$ and $P_G^0(x) \in P_G(x)$. Now, $R_G^0(x) \in R_G(x) \Rightarrow d(R_G^0(x), g) \leq d(x, g)$ for all $g \in G \Rightarrow d(R_G^0(x), P_G^0(x)) \leq d(x, P_G^0(x))$ as $P_G^0(x) \in G$.
Consider,

$$\begin{aligned} d(x, R_G^0(x)) &\leq d(x, P_G^0(x)) + d(P_G^0(x), R_G^0(x)) \\ &\leq 2d(x, P_G^0(x)) \end{aligned}$$

(d) $x \in D(R_G) \Rightarrow R_G(x) \neq \emptyset$. Let $R_G^0(x) \in R_G(x)$. Then

$$d(R_G^0(x), g) \leq d(x, g)$$

for all $g \in G$ implies

$$d(R_G^0(x), 0) =\leq d(x, 0).$$

The continuity of R_G at origin is now immediate. Also, $R_G(x) \subset B(0, d(x, 0))$ and so R_G is a bounded mapping.

(e) Suppose R_G is single valued on $D(R_G)$, $x \in D(R_G)$ and $g \in G$.

$$\begin{aligned} x \in D(R_G) &\Rightarrow d(R_G(x), g') \leq d(x, g') \text{ for all } g' \in G \\ &\Rightarrow d(R_G(x) + g, g' + g) \leq d(x + g, g' + g) \text{ for all } g' \in G \\ &\Rightarrow d(R_G(x) + g, g^*) \leq d(x + g, g^*) \text{ for all } g^* \in G \\ &\Rightarrow R_G(x) + g \in R_G(x + g). \end{aligned}$$

Consequently, $x+g \in D(R_G)$ and since R_G is single valued, $R_G(x+g) = R_G(x)+g = R_G(x) + R_G(g)$, by (a). Thus, R_G is quasiadditive.

(f) Let $x \in D(R_G)$ and $g_1, g_2 \in R_G(x)$. Then by Observation 2.3 (ii), $x - g_1, x - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is a linear subspace, $(x - g_1) - (x - g_2) \in R_G^{-1}(0)$, i.e., $g_2 - g_1 \in R_G^{-1}(0) \cap G = \{0\}$ and so $g_2 = g_1$, i.e., R_G is single valued on $D(R_G)$.

Now we show that R_G is additive on $D(R_G)$. Let $x, t \in D(R_G)$ and $R_G(x) = g_1$, $R_G(y) = g_2$. Then by Observation 2.3 (ii), $x - g_1, y - g_2 \in R_G^{-1}(0)$. Since $R_G^{-1}(0)$ is a linear subspace, $(x - g_1) + (y - g_2) \in R_G^{-1}(0)$. So, $0 = R_G(x + y - g_1 - g_2)$. Consider,

$$\begin{aligned} R_G(x + y) - (g_1 + g_2) &= R_G(x + y) - R_G(g_1 + g_2) \\ &= R_G(x + y - g_1 - g_2), \text{ by (e)} \\ &= 0 \\ &= R_G(x) - g_1 + R_G(x) - g_2. \end{aligned}$$

This gives $R_G(x + y) = R_G(x) + R_G(y)$, i.e., R_G is additive on $D(R_G)$.

Remark 4.1. In normed linear spaces, Property (a) was observed in [3], Properties (b) and (e) in [13], and Property (f) in [7]. Properties (a), (b) and (e) were proved in locally convex spaces in [11] and [16]. Properties (a) to (e) are also true for metric projections in metric linear spaces (see Pantelidis [12]).

We may recall that a mapping $T : X \to 2^Y$ where X and Y are metric spaces and 2^Y denotes the collection of all subsets of Y, is said to be upper semi-continuous if the set

$$\{x \in X : T(x) \cap N \neq \emptyset\}$$

is closed for every clsoed $N \subset Y$.

Now, we discuss some conditions under which the mapping R_G is upper semi-continuous or continuous.

Theorem 10. *If G is a closed linear subspace of a metric linear space (X, d) then R_G is upper semi-continuous on $D(R_G)$ if and only if for each closed subset N of G, $N + R_G^{-1}(0)$ is closed.*

Proof. Suppose R_G is upper semi-continuous and N is a closed subset of G. Let x be a limit point of $N + R_G^{-1}(0)$. Then there exists a sequence $\langle x_n \rangle$ in $N + R_G^{-1}(0)$ such that $x_n \to x$. Suppose $x_n = g_n + y_n$ where $g_n \in N$ and $y_n \in R_G^{-1}(0)$. Since $y_n = x_n - g_n \in R_G^{-1}(0)$, by Observation 2.3 (ii) $g_n \in R_G(x_n) \cap N$. The upper semi-continuity of R_G implies $R_G(x) \cap N \neq \emptyset$ and so there exists some $g \in R_G(x) \cap N$. This gives $x - g \in R_G^{-1}(0)$, i.e., $x \in N + R_G^{-1}(0)$ and so $N + R_G^{-1}(0)$ is closed.

Conversely, suppose $N + R_G^{-1}(0)$ is closed for each closed subset N of G. Suppose R_G is not upper semi-continuous on $D(R_G)$. Then there exists an $x \in D(R_G)$ and a sequence $\langle x_n \rangle$ in $D(R_G)$ such that $x_n \to x$, $R_G(x_n) \cap N \neq \emptyset$ but $R_G(x) \cap N = \emptyset$. So, there exists $g_0 \in R_G(x)$ such that $g_0 \notin N$ and so $x \notin N + R_G^{-1}(0)$, contradicting that $N + R_G^{-1}(0)$ is closed. Hence R_G is upper semi-continuous.

Remark 4.2. For normed linear spaces, Theorem 10 was proved in [18]. Some more results on the upper semicontinuity of the mapping R_G given in normed linear spaces in [18], were proved in metric spaces in [10].

Next theorem proves the upper semi-continuity of R_G when $R_G^{-1}(0)$ is boundedly compact (i.e., when every bounded sequence in $R_G^{-1}(0)$ has a subsequence converging to an element of X). For normed linear spaces this result was stated in [18].

Theorem 11. *If G is a closed linear subspace of a metric linear space (X, d) such that $R_G^{-1}(0)$ is boundedly compact, then $R_G(x)$ is compact and R_G is upper semi-continuous on $D(R_G)$.*

Proof. Let $\langle g_n \rangle$ be an arbitrary sequence in $R_G(x)$, i.e.,

$$d(g_n, g) \leq d(x, g)$$

for all $g \in G$. Then $\langle x - g_n \rangle$ is a bounded sequence in $R_G^{-1}(0)$ and so, it has a subsequence $\langle x - g_{n_i} \rangle \to x - g_0 \in R_G^{-1}(0)$ as $R_G^{-1}(0)$ is also closed by Observation 2.3 (i). Consequently, $\langle g_n \rangle$ has a subsequence $\langle g_{n_i} \rangle \to g_o \in R_G(x)$ and hence $R_G(x)$ is compact.

Now suppose N is a closed subset of G and

$$B = \{x \in D(R_G) : R_G(x) \cap N \neq \emptyset\}.$$

To show B is closed, let x be a limit point of B. Then there exists a sequence $\langle x_n \rangle$ in B such that $x_n \to x$. Now $x_n \in B \Rightarrow$ there exists $g_n \in R_G(x_n) \cap N$, $n = 1, 2, \cdots$ So

$$d(g_n, g) \leq d(x_n, g)$$

for all $g \in G$. This gives $x_n - g_n \in R_G^{-1}(0)$ and is a bounded sequence as $\langle x_n \rangle$ and $\langle g_n \rangle$ are both bounded. Since $R_G^{-1}(0)$ is boundedly compact, there is a subsequence $\langle x_{n_i} - g_{n_i} \rangle \to x - g_0 \in R_G^{-1}(0)$ as $R_G^{-1}(0)$ is also closed by Observation 2.3 (i). This gives $g_0 \in R_G(x) \cap N$, i.e., $x \in B$. Hence R_G is upper semi-continuous.

Remark 4.3. In case R_G is single-valued (this is so if G is co-semi-Chebyshev), Theorem 10 and 11 give the continuity of R_G on $D(R_G)$ and on X if G is also coproximinal.

Remark 4.4. If $R_{S,G}(x)$ is the set of all those elements of G which belong to $R_G(x)$ strongly (we say that '$g_0 \in R_G(x)$ strongly' if $x \notin G$ and there exists an $r (0 < r \leq 1)$ such that $d(g_0, g) + rd(g_0, x) \leq d(x, g)$ for all $g \in G$), the mapping $R_{S,G} : x \to R_{S,G}(x)$ defined on $D(R_{S,G}) = \{x \in X : R_{S,G}(x) \neq \emptyset\}$, is called 'strong best coapproximation map.' For linear subspace G of normed linear spaces, Theorem 10, 11, and some other results have been proved for $R_{S,G}$ in [17].

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