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# QUATERNION HOMOLOGY OF BANACH SPACE

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**Abstract.** In this article we are concerned with the quaternion Banach spaces and their homology. We obtain the relation between the quaternion and the dihedral homology for a unital involutive Banach algebra.

## Introduction

It is well known that is there are two types of homological algebra theory; the homology theory of discrete algebras, initiated and developed by Hochschild 1945-1947 [7], [8], [9] and the homology theory of operator algebras studied by Jonson, Kadison and Ringrose [4], [5], [6]. In 1983 Connes and Tsygan [2], [14] have introduced a new type, the cyclic (co)homology of a unital algebra. The analog Banach cyclic (co)homology has been studied by Helemskii [10], Christensen and Sinclair [1] and Wodziski [15]. In 1987 Loday [13], and Krassawkas, Lapin and Solovev [12] introduced and studied the reflexive and dihedral (co)homology of involutive unital algebras. The author studied the analog reflexive and dihedral (co)homology of $C^*$-algebras [11]. The dihedral homology of algebras and its relation with quaternion homology has been studied by Loday [13]. In this article we are concerned with the quternion Banach spaces and their homology. Firstly we recall the definition and some properties of generalized quternion group.

## 1. Generalized Quaternion Group $Q_m$

Let $\mathbb{H}$ be algebra of quaternions $\mathbb{R}\oplus\mathbb{R}_i\oplus\mathbb{R}_j\oplus\mathbb{R}_k$. For every natural number $m \geq 2$, the generalized quaternion group $Q_m$ is defined as a subgroup of the multiplicative group $\mathbb{H}^*$, generated by the elements $x = e^{\pi i/m}$ and $y = j$. It is clear that the element $x$ has order $2m$ and the relations $y^2 = x^m$ and $yxy^{-1} = x^{-1}$ are fulfilled. Hence $x^m y^{-1} = yxy^{-1}yxy^{-1}\cdots yxy^{-1} = x^{-m}$ and we deduce that

$$y \cdot y^2 \cdot y^{-1} = y^{-2}, \quad \text{i.e.} \quad x^{2m} = y^4 = 1.$$

Thus, the cyclic subgroup $C$ generated by the element $x$, is a normal subgroup and has index two in $Q_m$. It follows that the group $Q_m$ itself has order $4m$. Let us list the most important properties of the generalized quaternion group $Q_m$:

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(i) The group $Q_m$ is given by a co-representation

$$Q_m = \{x, y; x^m = y^2, yxy^{-1} = x^{-1}\}.$$

(ii) In the extension

$$0 \to C \to Q_m \to \mathbb{Z}/2 \to 0,$$

the generator of $\mathbb{Z}/2$ acts on $C$ as the multiplication by $-1$.

(iii) Every element in the set $Q_m/C$ has order 4.

(iv) An extension

$$0 \to C \to Q_m \to \mathbb{Z}/2 \to 0$$

is not splittable.

**Proposition 1.1.** *Let $\mathbb{R}$ be a commutative ring with unit. Then there exist 4-periodic resolution of the trivial $Q_m$-module $\mathbb{R}$:*

$$\cdots \longrightarrow R[Q_m] \xrightarrow{N} R[Q_m] \xrightarrow{w} R[Q_m]^2 \xrightarrow{v} R[Q_m]^2 \xrightarrow{u} R[Q_m] \xrightarrow{\varepsilon} R \longrightarrow 0.$$

*where $\varepsilon$ is the natural augmentation,*

$$u = (1-x, 1-y), v = \begin{bmatrix} T & 1+xy \\ -(1+y) & x-1 \end{bmatrix}, w = \begin{bmatrix} 1-x \\ yx-1 \end{bmatrix}, T = 1 + x + x^2 + \cdots + x^{m-1},$$

$$N = \sum_{g \in Q_m} g = (1 + y^2 + y^3 + y)T.$$

**Proof.** We use Fox's derivatives [1]. Let

$$G = \{g_1, \ldots, g_k / r_1, \ldots, r_\ell\}$$

be a group generated by the elements $g_1, \ldots, g_k$ with relations $r_1, \ldots, r_\ell$. The free differential $\partial r_i / \partial g_j$ of the group ring $\mathbb{Z}[G]$ is defined by:

$$\frac{\partial(ab)}{\partial g} = \frac{\partial a}{\partial g} + a\frac{\partial b}{\partial g}, \quad \frac{\partial g}{\partial g} = 1, \quad \frac{\partial h}{\partial g} = 0,$$

where $h$ is any generator of $G$ not equal to $g$. Then according to Fox [1], the sequence

$$\mathbb{Z}[G]^\ell \xrightarrow{v} \mathbb{Z}[G]^k \xrightarrow{u} \mathbb{Z}[g]^k \xrightarrow{\varepsilon} \mathbb{Z}, \longrightarrow 0,$$

where $\varepsilon(g) = 1$, $u = (1 - g_1, \ldots, 1 - g_k)$, $v = (\frac{\partial r_i}{\partial g_j})$, $1 \le i \le k$, $\le j \le \ell$ is the first part of the free resolution of the trivial G-module $\mathbb{Z}$.

By using the Fox's derivatives of the generalized quaternion group $Q_m$, when $k = 2$, $\ell = 2$, $g_1 = x$, $g_2 = y$, $r_1 = x^m y^{-2}$, $r_2 = xyxy^{-1}$ and $u = (1-x, 1-y)$,

$$v = \begin{bmatrix} \frac{\partial(x^m y^{-2})}{\partial x} & \frac{\partial(xyxy^{-1})}{\partial x} \\ \frac{\partial(x^m y^{-2})}{\partial y} & \frac{\partial(xyxy^{-1})}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 + x + \cdots + x^{m-1} & 1+xy \\ -(1+y) & x-1 \end{bmatrix},$$

We get the following exact sequence

$$R[Q_m]^2 \xrightarrow{v} R[Q_m]^2 \xrightarrow{u} R[Q_m] \xrightarrow{\varepsilon} R \longrightarrow 0. \tag{1.1}$$

Considering in (1.1), the factor $\mathcal{H}om_{R[Q_m]}(-, R[Q_m])$ and modifying the $Q_m$-module structure by means of the isomorphism $f : Q_m \to Q_m$, $f(x) = x^{-1}$, $f(y) = (by)^{-1}$ we get the exact sequence:

$$\mathcal{H}om_{R[Q_m]}(R[Q_m]^2, R[Q_m]) \xleftarrow{v^*} \mathcal{H}om_{R[Q_m]}(R[Q_m]^2, R[Q_m]) \xleftarrow{u^*}$$
$$\longleftarrow \mathcal{H}om_{R[Q_m]}(R[Q_m], R[Q_m]) \xleftarrow{\varepsilon} \mathcal{H}om_{R[Q_m]}(R, R[Q_m]) \longleftarrow 0.$$

It is easy to verify that:

$$u^* = (1-x, 1-y) = \begin{bmatrix} 1-x \\ yx-1 \end{bmatrix} = w,$$
$$v^* = \begin{bmatrix} T & 1+xy \\ -(1+y) & x-1 \end{bmatrix}^* = \begin{bmatrix} T & 1+xy \\ -(1+y) & x-1 \end{bmatrix} = v \quad \text{and we get}$$

the following exact sequence:

$$0 \longrightarrow R \xrightarrow{\varepsilon^*} R[Q_m] \xrightarrow{w} R[Q_m]^2 \xrightarrow{v} R[Q_m]^2. \tag{1.2}$$

Since the composition $\varepsilon^* \circ \varepsilon$ is a homomorphism $N$, from (1.1) and (1.2) we get the required 4-periodic resolution.

## 2. Quaternion Banach Spaces

Let $E = \otimes_{n \geq 0} E_n$ be a graded Banach space over the field of complex numbers $\mathbb{C}$. Consider the families of continuous linear maps on $E$:

$$d_n^1 : E_n \to E_{n-1}, \qquad s_n^j : E_n \to E_{n+1},$$
$$\tau_n, \omega_n : E_n \to E_n, \qquad 0 \leq i \leq n,\ 0 \leq j \leq n,$$

which satisfy the following conditions

$$d_n^i d_{n+1}^j = d_n^{j-1} d_{n+1}^i, \qquad i < j,$$
$$s_{n+1}^i s_n^j = s_{n+1}^{j+1} s_n^i, \qquad i \leq j,$$
$$d_n^i s_{n-1}^j = \begin{cases} s_{n-2}^{j-1} s_{n-1}^j, & i < j, \\ Id(E_{n-1}), i = j, & j+1, \\ s_{n-2}^j d_{n-1}^{i-1}, & i > j, \end{cases}$$
$$d_n^i \tau_n = \tau_{n-1} d_n^{i-1}, \quad s_n^i \tau_n = \tau_{n+1} s_n^{i+1}, \quad 1 \leq i \leq n,$$
$$d_n^j \omega_n = \omega_{n-1} d_n^{n-j}, \quad s_n^j \omega_n = \omega_{n+1} s_n^{n-j}, \quad 0 \leq j \leq n,$$
$$\tau_n^n = \omega_n^2, \omega_n \tau_n \omega_n^{-1} = \tau_n^{-1}.$$

A graded Banach space $E = \oplus_{n\geq 0} E_n$ considered together with these families of continuous linear maps is called a quaternion Banach space. An arbitrary unital Banach algebra $A$ gernerates the quaternion Banach space.

Indeed, put

$$E_n = A\hat{\otimes}A\hat{\otimes}A\cdots\hat{\otimes}A \ (n+1 \text{ times}),$$

where $\hat{\otimes}$ is the continuous tensor product in the sense of Grothendiec. Define operators:

$$d_n^i : E_n \to E_{n-1}, \quad s_n^j : E_n \to E_{n+1},$$

by means of the formulas

$$\begin{aligned}
d_n^i(a_0\otimes,\ldots\otimes a_n) &= a_0\otimes\cdots a_i a_{i+1}\otimes\cdots\otimes a_n, \quad 0\leq i<n,\\
d_n^n(a_0\otimes,\ldots\otimes a_n) &= a_n a_0\otimes\cdots\otimes a_{n-1},\\
s_n^j(a_0\otimes,\ldots\otimes a_n) &= a_0\otimes\cdots\otimes a_j\otimes e\otimes a_{j+1}\otimes\cdots\otimes a_n, \ 0\leq j<n,\\
s_n^n(a_0\otimes,\ldots\otimes a_n) &= e\otimes a_0\otimes\cdots\otimes a_n.
\end{aligned}$$

Moreover, define the operators $\tau_n : E_n \to E_n$, $\omega_n : E_n \to E_n$, putting

$$\begin{aligned}
\tau_n(a_0\otimes,\ldots\otimes a_n) &= (-1)^n a_n\otimes a_0\otimes\ldots\otimes a_{n-1}\\
\omega_n(a_0\otimes,\ldots\otimes a_n) &= \alpha(-1)^{\frac{n(n+1)}{2}} a_0^*\otimes a_n^*\otimes\cdots\otimes a_1^*,
\end{aligned}$$

where $\alpha$ is a root of the $4^{\underline{\text{th}}}$ degree of $1$, $a_\ell^*$ is the image of elements $a_\ell \in A$ under involution $* : A \to A$. It is easy to verify that the family so defined of Banach spaces and continuous linear maps is a quternion Banach space. In what follows we denote the quaternion Banach space by:

$$Q(A) : Q(A)_n = A\hat{\otimes}\cdots\hat{\otimes}A(n+1-\text{times}).$$

## 3. Continuous Quaternion Homology

**Proposition 3.1.** *Let $E = \oplus_{n\geq 0} E_n$ be a quaternion Banach space. Put*

$$t_n = (-1)^n \tau_n, \ r_n = (-1)^{\frac{n(n+1)}{2}} \alpha\omega_n,$$

*where $\alpha = 1, -1, i, -i$. Then there exists a bicomplex ${}^\alpha\mathcal{E}(E)$ with 4-periodic rows:*

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
E_n & \xleftarrow{u} & E_n\oplus E_n & \xleftarrow{v} & E_n\oplus E_n & \xleftarrow{\omega} & E_n & \xleftarrow{N} & E_n & \longleftarrow & \\
b\downarrow & & -(b'\oplus b)\downarrow & & (b'\oplus b)\downarrow & & -b'\downarrow & & b\downarrow & & \\
E_{n-1} & \xleftarrow{u} & E_{n-1}\oplus E_{n-1} & \xleftarrow{v} & E_{n-1}\oplus E_{n-1} & \xleftarrow{\omega} & E_{n-1} & \xleftarrow{N} & E_{n-1} & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

*where* $u = (1-t, 1-r)$, $v = \begin{bmatrix} T & 1-tr \\ -1+r & t-1 \end{bmatrix}$, $w = \begin{bmatrix} 1-t \\ -rt-1 \end{bmatrix}$, $T = 1+t+\cdots+t^{n-1}$, $N = (1+r+r^2+r^3)T$, $b = \sum_{i=0}^{n}(-1)^i d^i$, $b' = \sum_{i=0}^{n-1}(-1)^i d^i$.

**Proof.** This assertion follows immediately from the following:

$$b(1-t) = (1-t)b', \quad br = rb, \quad b'dr = drb', \quad b'T = Tb, \quad b'N = Nb.$$

**Definition 3.2.** [3]. Let $E = \oplus_{n\geq 0} E_n$ be a quaternion Banach space. Define the quaternion homology of $E$ by the formula:

$$^{\alpha}\mathcal{H}Q_n(E) = \mathcal{H}_n(Tot^{\alpha}\mathcal{E}(E)).$$

Let $E = Q_n(A)$, then

$$^{\alpha}\mathcal{H}Q_n(E) = \mathcal{H}_n(Tot^{\alpha}\mathcal{E}(A)).$$

Consider now the bicomplex consisting the first four columns of the bicomplex $^{\alpha}\mathcal{E}(E)$, we shall denote it by $^{\alpha}P(E)$, and suppose the following exact sequence

$$0 \longrightarrow^{\alpha} P(E) \longrightarrow Tot^{\alpha}\mathcal{E}(EA) \xrightarrow{q} Tot^{\alpha}\mathcal{E}(A)) \longrightarrow 0.$$

Following [13] the homology of the complex $^{\alpha}P(E)$ is a periodic and given by:

$$^{\alpha}\mathcal{H}P_n(E) = \mathcal{H}_n(Tot^{\alpha}P(E)).$$

Since the bicomplex $^{\alpha}\mathcal{E}(E)$ has 4-periodic rows, we get the following exact sequence relating the periodic homology with quternion homology.

**Theorem 3.3.** *There exists the exact sequence*

$$\begin{aligned} &\longrightarrow^{\alpha} \mathcal{H}P_n(E) \longrightarrow^{\alpha} \mathcal{H}Q_n(E) \longrightarrow^{\alpha} \mathcal{H}Q_{n-4}(E) \longrightarrow \\ &\longrightarrow^{\alpha} \mathcal{H}P_{n-1}(E) \longrightarrow^{\alpha} \mathcal{H}Q_{n-1}(E) \longrightarrow^{\alpha} \mathcal{H}Q_{n-5}(E) \longrightarrow \end{aligned} \tag{2.1}$$

Following [12], the relation between $^{\alpha}\mathcal{H}P_n(E)$ and the dihedral homology is given by:

$$\begin{aligned} &\longrightarrow^{\alpha} \mathcal{H}P_n(E) \longrightarrow^{\alpha} \mathcal{H}D_n(E) \longrightarrow^{\alpha} \mathcal{H}D_{n-4}(E) \longrightarrow \\ &\longrightarrow^{\alpha} \mathcal{H}P_{n-1}(E) \longrightarrow^{\alpha} \mathcal{H}D_{n-1}(E) \longrightarrow^{\alpha} \mathcal{H}D_{n-5}(E) \longrightarrow \end{aligned} \tag{2.2}$$

Comparing the relations (3.1) and (3.2) we get the relation between dihedral and quternion homology in the following.

**Proposition 3.4.** *There exist the following natural isomorphism:*

$$\begin{aligned} ^{1}\mathcal{H}Q_n(A) &\cong {}^{1}\mathcal{H}d_n(A), \\ ^{-1}\mathcal{H}Q_n(A) &\cong {}^{-1}\mathcal{H}D_n(A), \end{aligned}$$

Similarly, one can define a quaternion cohomology for an unital Banach algebra with an involution and get the results.

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