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SIMPLE RINGS WITH (R, R, R) IN LEFT NUCLEUS

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Abstract. It is shown that assuming associators in the left nucleus, simple rings with identity and characteristic$\neq$ 2 must be associative.

1. Introduction

In 1986 a paper appeared in the American Mathematical Monthly [2] which proved, using methods understandable by undergraduates, that a semi-prime ring of characteristic $\neq 2$, satisfying (R, R, R) in the nucleus must be associative, where the nucleus N is defined by

$$N = \{n \in R | (n, R, R) = 0 = (R, n, R) = (R, R, n)\}.$$

Since that time a number of results have appeared, particularly those by Chen-Te Yen and Irvin Roy Hentzel which are built on top of this result. Specifically Yen [3] has proved that one can weaken the hypothesis to (R, R, R) in the left and middle nucleus, provided one strengthens the hypothesis to simple rings. Here we shall then add a proof that the same result holds if one assumes (R, R, R) in the left and right nucleus together with the assumption of simplicity and characteristic$\neq$ 2. Hentzel and Yen have assumed both (R, R, R) and $[R, (R, R, R)]$ in the left nucleus and shown that semi-prime rings of this variety, of characteristic not 2 are associative [1]. In fact one can weaken the asumption on what is in the left nucleus even further. It turns out that under the assumption of associators in the left nucleus, simple rings with identity and characteristic$\neq$ 2 must be associative. Our proofs still remain simple and direct, so as to be on the level of an undergraduate mathematics student.

2. Main Section

We shall make use of the Teichmueller identity

$$(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z, \tag{1}$$

which holds in every ring R. Henceforth we assume that all associators $u = (a, b, c)$, with $a, b, c \in R$, are in the left nucleus L of R; in other words $(u, R, R) = 0$. Note that

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because of (1) this implies that $w(x,y,z)+(w,x,y)z \in L$. The associator ideal A of R can be characterized as the additive subgroup generated by all associators and all right multiples of associators. Symbolically $A = \Sigma(R,R,R) + (R,R,R)R$. Since we assume that R is simple, either $A = 0$, in which case R is associative, or $A = R$. Let us take up the case where R is not associative, so $A = R$. For every $n \in L$, and $x, y, z \in R$, equation (1) implies

$$(nx, y, z) = n(x, y, z). \tag{2}$$

Replace n by $(d,(a,b,c),e)$ in (2), where $a, b, c, d, e, \in R$ and $(d,(a,b,c),e) \in L$ by hypothesis. Thus

$$(d,(a,b,c),e)(x,y,z) = ((d,(a,b,c),e)x, y, z). \tag{3}$$

Applying (1) we obtain

$$(d,(a,b,c),e)x + d((a,b,c),e,x) = (d(a,b,c),e,x) - (d,(a,b,c)e,x) + (d,(a,b,c),ex). \tag{4}$$

The right hand side of (4) is in L by hypothesis. Then $((a,b,c),e,x) = 0$, also by hypothesis. Thus $d((a,b,c),e,x) = 0$. Consequently

$$(d,(a,b,c),e)x \in L. \tag{5}$$

This implies

$$(d,(a,b,c),e)(x,y,z) = 0. \tag{6}$$

Because $(d,(a,b,c),e) \in L$, it is now a one step proof to obtain $(R,(R,R,R),R)A = 0$. Since we are considering the case where $A = R$, this gives $(R,(R,R,R),R)R = 0$, which in the presence of $1 \in R$ gives $(R,(R,R,R),R) = 0$, so that associators (R,R,R) are now in the middle and left nucleus and so one may quote [3] to obtain a contradiction. At this point we observe however that $(a,b,(c,d,e))(x,y,z) = ((a,b,(c,d,e))x,y,z) = -(a(b,(c,d,e),x),y,z) = 0$, using (1) and the fact that associators are in the middle and left nucleus. From this it follows by the same proof that (R,R,R) must be in the right nucleus as well, thus [2] applies. We have shown:

Theorem 1. *Let R be a simple ring with 1, characteristic$\neq 2$, satisfying $(R,R,R) = 0$. Then R must be associative.*

Theorem 2. *Let R be a simple ring of characteristic$\neq 2$, satisfying $((R,R,R),R,R) = 0 = (R,R,(R,R,R))$. Then R must be associative.*

Proof. We get as far as $(R,(R,R,R),R) = 0$ just as in the proof of Theorem 1. By essentially passing to the anticommutative copy of R, that is by considering R under a new product $x * y = yx$, it becomes clear that $(R,R,(R,R,R)) = 0$ implies $R(R,(R,R,R),R) = 0$. Let $T = \{t \in R | tR = 0 = Rt\}$. Obviously $(R,(R,R,R),R) \subset T$. It is easy to prove that T is an ideal of R. Then simplicity yields $T = R$ or $T = 0$. However $T = R$ implies $RR = 0$, a contradiction. Thus $T = 0$, and so $(R,(R,R,R),R) = 0$, resulting in $(R,R,R) \subset N$. At this point use the result in [2] again.

Theorem 3. *If R satisfies the identity $((R,R,R),R,R)=0$, has characteristic $\neq 2$, and if for all $a\in R$, $a^2=0$ implies $a=0$, then R must be associative.*

Proof. As in earlier computation, we deduce that $(R,(R,R,R),R)(R,R,R)=0$. Thus $(v,(w,x,y),z)^2=0$. Hence $(v,(w,x,y),z)=0$, proving that (R,R,R) is in the middle nucleus of R. This leads to $(R,R,(R,R,R))(R,R,R)=0$. Then $(v,w,(x,y,z))^2=0$, so that (R,R,R) is in N, the nucleus of R. At this point one may quote [2] to see that R must be associative.

References

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