TAMKANG JOURNAL OF MATHEMATICS
Volume 30, Number 1, Spring 1999

AN OSCILLATION THEOREM FOR A NEUTRAL DIFFERENCE EQUATION WITH POSITIVE AND NEGATIVE COEFFICIENTS

WAN-TONG LI* AND SUI SUN CHENG

Abstract. An oscillation criterion is derived which supplements the oscillation theorems dervied in [1].

In [1], comparison and oscillation theorems are derived for a class of neutral type difference equations with positive and negative coefficients

$$\Delta(x_n - r_n x_{n-\xi}) + p_n x_{n-\tau} - q_n x_{n-\sigma} = 0, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where ξ is a positive integer, τ and σ are positive integers such that $\tau > \sigma$, $\{r_n\}_{n=0}^{\infty}$ is a real sequence, and $\{p_n\}_{n=0}^{\infty}$ as well as $\{q_n\}_{=0}^{\infty}$ are nonnegative sequences.

In this note, we will assume in addition, and also throughtout the sequel, that $\{r_n\}$ and $\{p_n - q_{n-\tau+\sigma}\}$ are eventually nonnegative and the latter sequence has a positive subsequence, and derive another oscillation theorem which supplements those in [1]. For the sake of brevity, preparatory definitions and material in [1] will not be repeated here. Lemma 1 in [1] will be assumed: In a ddition to the assumptions on (1), assume further that

$$r_n + \sum_{i=n-\tau+\sigma}^{n-1} q_i \leq 1 \tag{2}$$

holds for all large n, then for any eventually positive solution $\{x_n\}$ of (1), the sequence $\{z_n\}$ defined by

$$z_n = x_n - r_n x_{n-\xi} - \sum_{i=n-\tau+\sigma}^{n-1} q_i x_{i-\sigma}, \quad n \geq 0 \tag{3}$$

will satisfy $z_n > 0$ and $\Delta z_n \leq 0$ for large n. Here and in the sequel, we adopt the convention that empty sums are equal to zero. We will also make use of the following [1, Corollary 1] in the later two corollaries: in addition to the assumptions imposed on (1), assume further that $r_n > 0$ for all large n. Then every solution of (1) oscillates if, and only if, every solution of the following functional inequality

$$\Delta(x_n - r_n x_{n-\xi}) + p_n x_{n-\tau} - q_n x_{n-\sigma} \leq 0, \quad n = 0, 1, 2, \ldots$$

Received May 28, 1998.
1991 *Mathematics Subject Classification.* 39A10.
Key words and phrases. Neutral difference equation, oscillatory theroem.
* Supported by the Science Foundation of Gansu University of Technology.

oscillates.

Lemma 1. *Suppose*

$$r_n + \sum_{i=n-\tau+\sigma}^{n-1} q_i \geq 1 \tag{4}$$

for all large n. Suppose further that

$$\sum_{n=0}^{\infty} [p_n - q_{n-\tau+\sigma}] \exp\left\{\frac{1}{\mu}\sum_{j=0}^{n} j(p_j - q_{j-\tau+\sigma})\right\} = \infty, \tag{5}$$

where $\mu = \max\{\xi, \tau\} > 0$. Then for any eventually positive solution $\{x_n\}$ of (1), the sequence $\{z_n\}$ defined by (3) satisfies $z_n < 0$ and $\Delta z_n \leq 0$ for all large n.

Proof. Suppose $\{x_n\}$ is an eventually positive solution of (1). In view of (1),

$$\Delta z_n = -(p_n - q_{n-\tau+\sigma}) x_{n-\tau}$$

for all large n. Since $\{p_n - q_{n-\tau+\sigma}\}$ is eventually nonnegative and has a positive subsequence, we see further that $\{z_n\}$ is either eventually nonpositive or eventually negative. Suppose to the contrary that $\{z_n\}$ is eventually positive, then there is some integer T such that $x_n >$, $z_n > 0$ and $\Delta z_n \leq 0$ for $n \geq T - \max\{\xi, \tau, \sigma\}$. Let $\mu = \max\{\xi, \tau\}$ and $\kappa = \min\{\xi, \sigma\}$. Then in view of (3), for $T \leq n \leq T + \mu$,

$$x_n \geq M\left\{r_n + \sum_{i=n-\tau+\sigma}^{n-1} q_i\right\} \geq M,$$

where

$$M = \min\{x_{T-\mu}, x_{T-\mu+1}, \ldots, x_T\} > 0,$$

and by induction,

$$x_n \geq M, T + (k-1)\mu \leq n \leq T + k\mu,$$

for each $k = 1, 2, \ldots$. In other words, $x_n \geq M$ for $n \geq T - \mu$.

Next, in view of (3), for $n \geq t + \mu$,

$$\begin{aligned} x_n &= z_n + r_n x_{n-\xi} + \sum_{i=n-\tau+\sigma}^{n-1} q_i x_{i-\sigma} \\ &\geq z_n + \left(r_n + \sum_{i=n-r+\sigma}^{n-1} q_i\right) \min_{n-\mu \leq t \leq n-\kappa} x_t \\ &\geq z_n + \min_{n-\mu \leq t \leq n-\kappa} x_t \geq z_n + \min_{n-\mu \leq t \leq n} x_t. \end{aligned}$$

Let $[x]$ be the greatest integral part of the number x and let $N(n) = [n/\mu]$. Then by applying the same arguments, we see further that

$$\begin{aligned} x_n &\geq z_n + \min_{n-\mu\leq t\leq n} x_t \\ &\geq z_n + \min_{n-\mu\leq t_1\leq n} \left\{ z_t + \min_{t-\mu\leq t_1\leq t} x_s \right\} \\ &\geq z_n + \min_{n-\mu\leq t_1\leq n} z_{t_1} + \min_{n-\mu\leq t_1\leq n} \left\{ \min_{t_1-\mu\leq t_2\leq t_1} z_{t_2} \right\} + \cdots \\ &\quad + \min_{n-\mu\leq t_1\leq n} \left\{ \cdots \left\{ \min_{t_{N(n-T)-2}-\mu\leq t_{N(n-T)-1}\leq t_{N(n-T)-2}} z_{t_{N(n-T)-1}} \right\}\right\} \\ &\quad + \min_{n-\mu\leq t_1\leq n} \left\{ \cdots \left\{ \min_{t_{N(n-T)-1}\leq t_{N(n-T)}\leq t_{N(n-T)-1}} x_{t_{N(n-T)}} \right\}\right\}. \end{aligned}$$

Hence when $t_{N(n-T)} \geq T$, from the monotonicity of the sequence $\{z_n\}$, we see that

$$x_n \geq N(n-T)z_n + M, \quad n \geq T + \mu.$$

But then

$$\begin{aligned} 0 &= \Delta z_n + (p_n - q_{n-\tau+\sigma})x_{n-\tau} \\ &\geq \Delta z_n + (p_n - q_{n-\tau+\sigma})(N(n-\tau-T)z_{n-\tau} + M) \\ &\geq \Delta z_n + (1 - \exp(-(p_n - q_{n-\tau+\sigma})N(n-\tau-T)))z_{n-\tau} + (p_n - q_{n-\tau+\sigma})M \end{aligned}$$

for n, say, greater than or equal to $T + \mu$, where we have used the fact that $e^x \geq 1 + x$ in deriving the last inequality. If we multiply the above inequality by the "integrating factor" (cf. [5, Theorem 1])

$$\exp\Big(\sum_{i=T+\tau}^{n} (p_i - q_{i-\tau+\sigma})N(i-\tau-T)\Big),$$

we obtain

$$\begin{aligned} &\Delta\Big\{ z_n \exp\Big(\sum_{i=T+\tau}^{n-1} (p_i - q_{i-\tau+\sigma})N(i-\tau-t)\Big)\Big\} \\ &+M(p_n - q_{n-\tau-\sigma}) \exp\Big(\sum_{i=T+\tau}^{n} (p_i - q_{i-\tau+\sigma})N(i-\tau-T)\Big) \leq 0 \end{aligned}$$

for $n \geq T + \tau$. Summing the above functional inequality from $T + \tau$ to n, we obtain

$$\begin{aligned} z_{T+\tau} &\geq z_{n+1} \exp\Big(\sum_{i=T+\tau}^{n} (p_i - q_{i-\tau+\sigma})N(i-\tau-T)\Big) \\ &\quad +M \sum_{j=T+\tau}^{n} (p_j - q_{j-\tau+\sigma}) \exp\Big(\sum_{i=T+\tau}^{j} (p_i - q_{i-\tau+\sigma})N(i-\tau-T)\Big) \geq 0. \end{aligned}$$

By letting n tend to infinity, we see that

$$\sum_{j=T+\tau}^{\infty}(p_j - q_{i-\tau+\sigma})\exp\Big(\sum_{i=T+\tau}^{j}(p_i - q_{i-\tau+\sigma})N(i-\tau-T)\Big) < \infty.$$

we see finally that

$$\sum_{j=T+\tau}^{\infty}[p_j - q_{j-\tau+\sigma}]\exp\Big\{\frac{1}{\mu}\sum_{i=T+\tau}^{j} i(p_i - q_{i-\tau+\sigma})\Big\} < \infty.$$

This is contrary to (5). The proof is complete.

Theorem 1. *Suppose*

$$r_n + \sum_{i=n-\tau+\sigma}^{n-1} q_i = 1 \tag{6}$$

for all large n. Suppose further that (5) *holds. Then every solution of* (1) *oscillates.*

Indeed, recall that under the condition that (2) holds for all large n, for every eventually positive solution $\{x_n\}$ of (1), the sequence $\{z_n\}$ defined by (3) is eventually positive. But this is contrary to the conclusion of Lemma 1 here. Thus (1) cannot have any eventually positive, nor any eventually negative, solutions.

As an example, consider the equation

$$\Delta(x_n - (1-\alpha)x_{n-1}) + \Big(\alpha + \frac{1}{n^\beta}\Big)x_{n-2} - \alpha x_{n-1} = 0,$$

where $\mu = \max\{\xi, \tau\} = 2$, $0 < \alpha < 1$, and $3/2 < \beta < 2$. Take $r_n = 1-\alpha$, $p_n = \alpha + 1/n^\beta$ and $q_n = \alpha$, then (6) is satisfied for all large n. Furthermore, since

$$\sum_{j=1}^{k}\frac{1}{j^{\beta-1}} \geq \int_2^{k+1}\frac{dx}{x^{\beta-1}} = \frac{1}{(2-\beta)(k+1)^{\beta-2}} - \frac{1}{(2-\beta)2^{\beta-2}},$$

and

$$\exp\Big(\sum_{j=1}^{k}\frac{1}{j^{\beta-1}}\Big) \geq \exp\Big(\frac{-1}{(2-\beta)2^{\beta-2}}\Big)\exp\Big(\frac{1}{(2-\beta)(n+1)^{\beta-2}}\Big),$$

as well as

$$\int_1^\infty \frac{1}{x^\beta}\exp\Big(\frac{1}{(2-\beta)x^{\beta-2}}\Big)dx = \infty,$$

by means of the integral test, we see that

$$\sum_{k=1}^{\infty}[p_k - q_{k-\tau+\sigma}]\exp\Big\{\frac{1}{\mu}\sum_{j=1}^{k} j(p_j - q_{j-\tau+\sigma})\Big\}$$
$$= \sum_{k=1}^{\infty}\frac{1}{k^\beta}\exp\Big(\frac{1}{2}\sum_{j=1}^{k}\frac{1}{j^{\beta-1}}\Big) = \infty.$$

This shows that condition (5) is satisfied. All the assumptions in Theorem 1 are satisfied and hence all solutions oscillate. But he results in [1,4] are not applicable when $3/2 < \beta < 2$. This is because for such a β,

$$\sum_{n=2}^{\infty} n(p_n - q_{n-\tau+\sigma}) \sum_{k=n}^{\infty}(p_k - q_{k-\tau+\sigma}) = \sum_{n=2}^{\infty} \frac{1}{n^{\beta-1}} \sum_{k=n}^{\infty} \frac{1}{k^\beta}$$
$$\leq \sum_{n=2}^{\infty} \frac{1}{n^{\beta-1}} \int_{n-1}^{\infty} \frac{dt}{t^\beta} = \sum_{n=2}^{\infty} \frac{1}{(\beta-1)(n(n-1))^\beta - 1} < \infty.$$

In case (4) is not satisfied for all large n, we may try to apply the following two results.

Corollary 1. *Suppose* (2) *holds for all large* n. *Suppose further that* (5) *holds and that*

$$r_{n-\tau}(p_n - q_{n-\tau+\sigma}) \geq (p_{n-\xi} - q_{n-\xi-\tau+\sigma}) \tag{7}$$

for all large n. *Then equation* (1) *is oscillatory.*

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1). Then by means of Lemma 1 in [1], the sequence $\{z_n\}$ defined by (3) will satisfy $z_n > 0$ for all large n. In view of (1), we have

$$\Delta z_n = -(p_n - q_{n-\tau+\sigma})x_{n-\tau}$$
$$= -(p_n - q_{n-\tau+\sigma})\Big\{z_{n-\tau} + r_{n-\tau}x_{n-\xi-\tau} + \sum_{i=n-\tau+\sigma}^{n-1} q_{i-\tau}x_{i-\tau-\sigma}\Big\}$$

so that

$$\Delta z_n + (p_n - q_{n-\tau+\sigma})z_{n-\tau} + r_{n-\tau}(p_n - q_{n-\tau+\sigma})x_{n-\xi-\tau} \leq 0.$$

In view of (1) again, we also have

$$\Delta z_{n-\xi} + (p_{n-\xi} - q_{n-\xi-\tau+\sigma})x_{n-\tau-\xi} = 0.$$

Subtracting the latter equation from the former, we obtain

$$\Delta(z_n - z_\xi) + (p_n - q_{n-\tau+\sigma})z_{n-\tau}$$
$$\leq \{(p_{n-\xi} - q_{n-\xi-\tau+\sigma}) - r_{n-\tau}(p_n - q_{n-\tau+\sigma})\}x_{n-\tau-\xi} \leq 0,$$

which implies that $\{z_n\}$ is an eventually positive solution of the recurrence relation

$$\Delta(z_n - z_{n-\xi}) + (p_n - q_{n-\tau+\sigma})z_{n-\tau} \leq 0.$$

By Corollary 1 in [1] mentioned above, we see that the equation

$$\Delta(z_n - z_{n-\xi}) + (p_n - q_{n-\tau+\sigma})z_{n-\tau} = 0$$

has an evntually positive solution. This is contrary to Theorem 3.

Corollary 2. *Suppose that the conditons* (4) *and* (5) *in Lemma* 1 *hold, that* $\{q_n/(p_n - q_{n-\tau+\sigma})\}$ *is eventually nondecreasing, that*

$$h_1(p_{n-\xi} - q_{n-\tau-\xi+\sigma}) \geq r_{n-\tau}(p_n - q_{n-\tau+\xi}), \quad h_1 > 0, \tag{8}$$

and that

$$q_{n-\tau}(p_n - q_{n-\tau+\sigma}) \leq h_2(p_{n-\sigma} - q_{n-\tau}) \tag{9}$$

for all large n, *where* $h_1 + h_2(\tau - \sigma) = 1$. *Then every solution of* (1) *oscillates.*

Indeed, suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1), then by Lemma 1, we see that the sequence $\{z_n\}$ defined by (3) will satisfy $z_n < 0$ for all large n. Furthermore, in view of (8) and (9), we get

$$\begin{aligned}
\Delta z_n &= -(p_n - q_{n-\tau+\sigma})x_{n-\tau} \\
&= -(p_n - q_{n-\tau+\sigma})\Big[z_{n-\tau} + r_{n-\tau}x_{n-\xi-\tau} + \sum_{i=n-\tau+\sigma}^{n-1} q_{i-\tau}x_{i-\sigma-\tau}\Big] \\
&\geq -(p_n - q_{n-\tau+\sigma})z_{n-\tau} - h_1(p_{n-\xi} - q_{n-\tau+\sigma-\xi})x_{n-\xi-\tau} \\
&\quad -(p_n - q_{n-\tau+\sigma}) \sum_{i=n-\tau+\sigma}^{n-1} \frac{q_{i-\tau}}{p_{i-\sigma} - q_{i-\tau}}(-\Delta z_{i-\sigma}) \\
&\geq (p_n - q_{n-\tau+\sigma})z_{n-\tau} + h_1\Delta z_{n-\xi} + h_2 \sum_{i=n-\tau+\sigma}^{n-1} \Delta z_{i-\sigma} \\
&= -(p_n - q_{n-\tau+\sigma} + h_2)z_{n-\tau} + h_2 z_{n-\sigma} + h_1\Delta z_{n-\xi},
\end{aligned}$$

so that

$$\Delta(z_n - h_1 z_{n-\xi}) + (p_n - q_{n-\tau+\sigma} + h_2)z_{n-\tau} - h_2 z_{n-\sigma} \geq 0$$

for all large n. This shows that $\{-z_n\}$ is an eventually positive solution of the inequality

$$\Delta(z_n - h_1 z_{n-\xi}) + (p_n - q_{n-\tau+\sigma} + h_2)z_{n-\tau} - h_2 z_{n-\sigma} \leq 0.$$

By Corollary 1 in [1] mentioned above, we see that the equation

$$\Delta(z_n - h_1 z_{n-\xi}) + (p_n - q_{n-\tau+\sigma} + h_2)z_{n-\tau} - h_2 z_{n-\sigma} = 0$$

has an eventually positive solution. This is contrary to the conclusion of Lemma 1.

As our final example, consider the equation

$$\Delta\Big(x_n - \frac{n+2}{2(n+1)}x_{n-1}\Big) + \Big(\frac{1}{2} + \frac{1}{n^\beta}\Big)x_{n-2} - \frac{1}{2}x_{n-1} = 0.$$

Since $r_n = (n+2)/(2n+2)$, $p_n = 1/2 + 1/n^\beta$, $q_n = 1/2$, $\xi = \sigma = 1$, $\tau = 2$, we see that

$$r_n + \sum_{i=n-\tau+\sigma}^{n-1} q_i = \frac{n+2}{2(n+1)} + \frac{1}{2} \geq 1.$$

If we take $h_1 = 1/2$ and $h_2 = 1/2$, then

$$h_1 + h_2(\tau - \sigma) = 1,$$
$$h_1(p_{n-\xi} - q_{n-\tau-\xi-\sigma}) = \frac{1}{2(n-1)^\beta},$$
$$r_{n-\tau}(p_n - q_{n-\tau-\sigma}) = \frac{1}{2(n-1)n^{\beta-1}},$$
$$q_{n-\tau}(p_n - q_{n-\tau+\sigma}) = \frac{1}{2n^\beta},$$

and

$$h_2(p_{n-\sigma} - q_{n-\tau}) = \frac{1}{2(n-1)^\beta}.$$

Thus the assumptions (4), (8) and (9) in Corollary 2 are satisfied for all large n when $1 < \beta < 2$. Furthermore, as already seen in the previous example, condition (5) is satisfied. Hence all its solutions oscillate. The same conclusion cannot be drawn form those in [1,4] when $3/2 < \beta < 2$.

References

[1] W. T. Li and S. S. Cheng, *On a neutral difference equation with positive and negative coefficients*, Southeast Asia Bull. Math., to appear.

[2] J. S. Yu and Z. C. Wang, *Asymptotic behavior and oscillation in neutral dealy difference equations*, Funcialaj Ekvacioj **37** (1994), 241-248.

[3] G. Zhang and S. S. Cheng, *Oscillation criteria for a neutral difference equation with delay*, Appl. Math. Lett. **8** (1995), 13-17.

[4] M. P. Chen and B. G. Zhang, *Oscillation and comparison theorems of difference equations with positive and negative coefficients*, Bull. Institute of Math. Acad. Sinica **22** (1994), 295-306.

[5] G. Zhang and S. S. Cheng, *Elementary non-existence criteria for a recurrence relation*, Chinese J. Math. **24** (1996), 229-235.

Institute of Applied Mathematices, Gansu University of Technology, Lanzhou, Gansu, 730050, P. R. China.

Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R.O.C.