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ON DEGREE OF APPROXIMATION OF FUNCTIONS BELONGING TO THE WEIGHTED $(L^p, \xi(t))$ CLASS BY $(C,1)(E,1)$ MEANS

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Abstract. In this paper, a theorem on the degree of approximation of the function belonging to the weighted class $W(L^P, \xi(t))$ by (C, 1) (E,1) means is established.

1. Definitions and Notations

Let $f(t)$ be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of $f(t)$ is given by

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt). \tag{1.1}$$

We define $\| \ \|_p$ by

$$\|f\|_p = \Big(\int_0^{2\pi} |f(x)|^p dx\Big)^{\frac{1}{p}}, \qquad p \geq 1 \tag{1.2}$$

and let the degree of approximation $E_n(f)$ be given by

$$E_n(f) = \text{Min}\|f - T_n\|_p, \tag{1.3}$$

where $T_n(x)$ is a trigonometric polynomial of degree n. A function $f \in \text{Lip}\ \alpha$ if

$$f(x+t) - f(x) = 0(|t|^\alpha) \quad \text{for } o < \alpha \leq 1 \tag{1.4}$$

$f(x) \in \text{Lip}\ (\alpha, p)$, for $a \leq x \leq b$, if

$$\Big(\int_a^b |f(x+t) - f(x)|^p dx\Big)^{\frac{1}{p}} \leq A|t|^\alpha, \qquad o < \alpha \leq 1,\ p \geq 1 \tag{1.5}$$

(definition 5.38 of McFadden (1942)). Given a positive increasing function $\xi(t)$ and an interger $p \geq 1$, $f(x) \in \text{Lip}\ (\xi(t), p)$ if

$$\Big(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\Big)^{\frac{1}{p}} = 0(\xi(t)) \tag{1.6}$$

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and that $f(x) \in W(L^p, \xi(t))$ if

$$\Big(\int_0^{2\pi} |f(x+t) - f(x)|^p \sin^{\beta p} x dx\Big)^{\frac{1}{p}} = 0(\xi(t)), \quad (\beta \geq 0). \tag{1.7}$$

In case $\beta = 0$, we find that $W(L^p, \xi(t))$ coincides with the class Lip $(\xi(t), p)$. If

$$E_n^1 = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} s_k \longrightarrow s, \quad \text{as} \quad n \to \infty \tag{1.8}$$

then an infinite serise $\sum_{n=0}^{\infty} a_n$ with the parital sums s_n is said to be summable $(E, 1)$ to the definite number s. The $(C, 1)$ transform of the $(E, 1)$ transform E_n^1 defines the $(C, 1)(E, 1)$ transform of the partial sum s_n of the serise $\sum_{n=0}^{\infty} a_n$. Thus if

$$(CE)_n^1 = \frac{1}{n} \sum_{k=1}^{n} E_k^1 \longrightarrow s, \quad \text{as } n \longrightarrow \infty \tag{1.10}$$

where E_n^1 denotes the $(E, 1)$ transform of s_n, then the serise $\sum_{n=0}^{\infty} a_n$ is said to be summable by $(C, 1)(E, 1)$ means or simply summable $(C, 1)(E, 1)$ to s.

We shall use following notations:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

2. The degree of approximation of functions belonging to Lip α, Lip (α, p) and weighted class $W(L^p, \xi(t))$ by Nörlund means has been discussed by a number of researchers like Qureshi (1981, 1982), Khan (1974) and Neha (1990). But till now no work seems to have been done to obtain the degree of approximation of the functions by product summabilities means of the form $(C, 1)(E, 1)$. In an attempt to make an advance study in this direction, one theorem on degree of approximation of functions of $W(L^p, \xi(t))$ class have been obtained in the following form:

Theorem. *If a function f, 2π-periodic, belonging to the weighted $W(L^p, \xi(t))$ class, then its degree of approximation is given by*

$$||f - (CE)_n^1|| = 0\Big(\xi(\frac{1}{n}) n^{\beta + \frac{1}{p}}\Big),$$

provided $\xi(t)$ satisfies the following conditions:

$$\Big\{\int_0^{\frac{1}{n}} \Big(\frac{t\phi(t)}{\xi(t)}\Big)^p \sin^{\beta p} t dt\Big\}^{\frac{1}{p}} = 0\Big(\frac{1}{n}\Big) \tag{2.1}$$

$$\Big\{\int_{\frac{1}{n}}^{\pi} \Big(\frac{t^{-\delta}\phi(t)}{\xi(t)}\Big)^p dt\Big\}^{\frac{1}{p}} = 0\Big(n^{\delta}\Big) \tag{2.2}$$

where δ is an arbitrary number such that $q(1-\delta)$ - $1 > 0$, conditions (2.1) and (2.2) holds uniformly in x and $(CE)_n^1$ are $(C,1)(E,1)$ means of the series (1.1).

3. Estimate

We shall require the following estimate :

$$1-\cos^n\left(\frac{t}{2}\right)\cos\left(\frac{nt}{2}\right)=0(n^2t^2) \quad \text{for} \quad 0\le t\le \frac{1}{n} \tag{3.1}$$

For $0\le t\le \frac{1}{n}$, we have

$$\begin{aligned}1-\cos^n\left(\frac{t}{2}\right)\cos\left(\frac{nt}{2}\right)&=1-\left\{1-\left(\frac{t^2}{8}-\cdots\right)\right\}^n\left(1-\frac{n^2t^2}{8}+\cdots\right)\\&=1-\left\{1-\frac{nt^2}{8}\right\}\left(1-\frac{n^2t^2}{8}\right)\\&=1-\left\{1-\frac{n^2t^2}{8}-\frac{nt^2}{8}+\frac{n^3t^4}{64}\right\}\\&=\frac{n^2t^2}{8}\left(1+\frac{1}{n}\right)\\&=0(n^2t^2)\end{aligned}$$

4. Proof of Theorem

Following Titchmarsh $(1939, p.403)$ and using Riemann-Lebesgue theorem, the partial sum s_n of the series (1.1) at $t=x$ may be written as

$$s_n-f(x)=\frac{1}{\pi}\int_0^\pi\frac{\phi(t)}{t}\sin nt\,dt+O(1)$$

The $(E,1)$ transform E_n^1 of s_n is given by

$$\begin{aligned}E_n^1-f(x)&=\frac{2^{-n}}{\pi}\int_0^\pi\frac{\phi(t)}{t}\left\{\sum_{k=0}^n\binom{n}{k}\sin(kt)\right\}dt\\&=\frac{2^{-n}}{\pi}\int_0^\pi\frac{\phi(t)}{t}I_p\left\{\sum_{k=0}^n\binom{n}{k}e^{ikt}\right\}dt\\&=\frac{1}{\pi}\int_0^\pi\frac{\phi(t)}{t}\cos^n\left(\frac{t}{2}\right)\sin\left[\frac{nt}{2}\right]dt\end{aligned}$$

Denoting the $(C,1)$ transform of E_n^1 is the $(C,1)(E,1)$ transform of s_n by $(CE)_n^1$, we have by reqularity of $(C,1)$ transform,

$$(CE)_n^1-f(x)=\frac{1}{n\pi}\int_0^\pi\frac{\phi(t)}{t}\left[\sum_{k=1}^n\cos^k\left(\frac{t}{2}\right)\sin\left(\frac{kt}{2}\right)\right]dt$$

$$= \frac{1}{n\pi}\int_0^\pi \frac{\phi(t)}{t} I_p\Big[\sum_{k=1}^{n} \cos^k\left(\frac{t}{2}\right) e^{\frac{ikt}{2}}\Big] dt$$

$$= \frac{1}{n\pi}\int_0^\pi \frac{\phi(t)}{t} I_p\left[\frac{e^{it/2}\cos\left(\frac{t}{2}\right)\left\{1-\left(e^{it/2}\cos\left(\frac{t}{2}\right)\right)^n\right\}}{\tan\left(\frac{t}{2}\right)}\right] dt$$

$$= O[I_1 + I_2], \qquad \text{say}$$

Now,

$$I_1 \le \frac{1}{n}\int_0^{1/n} \frac{\phi(t)}{t^2}\Big(1-\cos^n\left(\frac{t}{2}\right)\cos\left(\frac{nt}{2}\right)\Big) dt$$

Applying Hölder's inequality and the fact that $\phi(t) \in W(L^p, \xi(t))$, we have

$$I_1 \le \frac{1}{n}\Big[\int_0^{1/n}\Big(\frac{t\phi(t)}{\xi(t)}\sin^\beta t\Big)^p dt\Big]^{1/p}\left[\int_0^{1/n}\left\{\frac{\xi(t)}{t^3}\frac{\left(1-\cos^n\left(\frac{t}{2}\right)\cos\left(\frac{nt}{2}\right)\right)}{\sin^\beta t}\right\}^q dt\right]^{1/q}$$

$$= O\Big(\frac{1}{n^2}\Big) O\Big[\int_0^{1/n}\Big(\frac{\xi(t)n^2t^2}{t^3t^\beta}\Big]^q dt\Big]^{1/q} \quad \text{by (2.1) \& (3)}$$

Applying mean-value theorem, we have

$$I_1 = O\Big(\xi\Big(\frac{1}{n}\Big)\Big(\int_\epsilon^{1/n}\Big(\frac{dt}{t^{(\beta+1)q}}\Big)^{1/q}\Big] \quad \text{where } 0<\epsilon<\frac{1}{n}$$

$$= \Big(\xi\Big(\frac{1}{n}\Big)\Big(\Big[\frac{t^{-(\beta+1)q+1}}{-(\beta+1)q+1}\Big]_\epsilon^{1/n}\Big)^{1/q}\Big)$$

$$= O\Big(\xi\Big(\frac{1}{n}\Big)\Big(n^{(\beta+1)q-1)}\Big)^{1/q}\Big)$$

$$= O\Big(\xi\Big(\frac{1}{n}\Big)n^{\beta+1-1/q}\Big)$$

$$= O\Big(\xi\Big(\frac{1}{n}\Big)n^{\beta+1/p}\Big), \quad \text{where } \tfrac{1}{p}+\tfrac{1}{q}=1 \text{ such that } 1\le p\le\infty.$$

Therefore

$$I_1 = O\Big(\xi\Big(\frac{1}{n}\Big)n^{\beta+1/p}\Big)$$

Similarly, as above, we have

$$I_2 \le \frac{1}{n}\Big[\int_{1/n}^{\pi}\Big(\frac{t^{-\delta}\phi(t)}{\xi(t)}\Big)\sin^\beta t\Big)^p dt\Big]^{1/p}\left[\int_{1/n}^{\pi}\left\{\frac{\xi(t)\left(1-\cos^n\left(\frac{t}{2}\right)\cos\left(\frac{nt}{2}\right)\right)}{t^2t^{-\delta}\sin^\beta t}\right\}^q dt\right]^{1/q}$$

$$= O(n^{\delta-1})O(n^\beta)O\Big[\int_1^n\Big(\frac{\xi(\frac{1}{y})}{y^{-2}y^\delta}\Big)^q\frac{dy}{y^2}\Big)^{1/q}\Big]$$

$$= O(n^{\delta+\beta-1})\xi\Big(\frac{1}{n}\Big)n^{2-\delta-\frac{1}{q}}$$

$$= O\left[\xi\left(\frac{1}{n}\right)n^{\beta+1-\frac{1}{q}}\right]$$
$$= O\left[\xi\left(\frac{1}{n}\right)n^{\beta+\frac{1}{p}}\right]$$

Now

$$|(CE)^1_n - f(x)| = O\left(\xi\left(\frac{1}{n}\right)n^{\beta+\frac{1}{p}}\right)$$
$$\|(CE)^1_n - f(x)\| = O\left[\left\{\int_0^{2\pi}\left\{\xi\left(\frac{1}{n}\right)n^{\beta+\frac{1}{p}}\right\}^p dx\right\}^{1/p}\right]$$
$$= O\xi\left(\frac{1}{n}\right)n^{\beta+\frac{1}{p}}\left(\int_0^{2\pi} dx\right)^{1/p}\Big]$$
$$= O\left(\xi\left(\frac{1}{n}\right)n^{\beta+\frac{1}{p}}\right).$$

This completes the proof of the Theorem.

5. Following corollaries can be derived from the theorem:

Corollary 1. *If $\beta = 0$ and $\xi(t) = t^\alpha$ then degree of approximation of a function f belonging to the class $Lip(\alpha, p)$, $0 < \alpha \le 1$, is given by*

$$|(CE)^1_n - f(x)| = O\left[\left(\frac{1}{n}\right)^{\alpha-\frac{1}{p}}\right]$$

Proof. Since

$$|(CE)^1_n - f(x)| = O\left[\xi\left(\frac{1}{n}\right)n^{\beta+\frac{1}{p}}\right]$$
$$= O\left[\left(\frac{1}{n}\right)^\alpha n^{\frac{1}{p}}\right]$$
$$= O\left[\left(\frac{1}{n}\right)^{\alpha-\frac{1}{p}}\right]$$

which completes the proof.

Corollary 2. *If $p \to \infty$ in Corollary 1, then we have, for $0 < \alpha < 1$,*

$$|(CE)^1_n - f(x)| = O\left[\frac{1}{n^\alpha}\right].$$

Remark. An independent proof of Corollary 1 can be developed along the same lines as the theorem.

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