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ON THE ABSOLUTE SUMMABILITY FACTORS OF TYPE (A, B)

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Abstract. In this paper we establish a relation between the $\varphi - |\bar{N}, p_n; \delta|_k$ and $\psi - |\bar{N}, q_n; \delta|_k$ summability methods, which generalizes a result of Mishra [2].

1. Introduction

Let (φ_n) be a sequence of positive real numbers and let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n). Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \ i \geq 1). \tag{1}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{2}$$

defines the sequence (t_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n).

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty \tag{3}$$

and it is said to be summable $\varphi - |\bar{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [3])

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} |t_n - t_{n-1}|^k < \infty. \tag{4}$$

If we take $\delta = 0$ and $\varphi_n = \frac{P_n}{p_n}$, then $\varphi - |\bar{N}, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n|_k$ summability.

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If $\sum a_n\lambda_n$ is summable by a method B whenever $\sum a_n$ is summable by a method A, then we say that the factor λ_n is of type (A,B) and write

$$\lambda_n \in (A,B). \tag{5}$$

2. The Following Theorem Is Known

Theorem A.([2]) *Let the sequences (p_n) and (q_n) be such that $p_n > 0$, $q_n > 0$, $P_n \to \infty$, $Q_n \to \infty$ and*

$$P_n/p_n = O(Q_n/q_n). \tag{6}$$

Then in order that

$$\lambda_n \in (|\bar{N}, q_n|_k, |\bar{N}, p_n, |_k), \quad k \geq 1 \tag{7}$$

it is sufficient that

$$\lambda_n = O(q_n P_n/p_n Q_n) \tag{8}$$

and

$$P_n Q_{n-1}\Delta\lambda_n + (q_n P_n - p_n Q_n)\lambda_n = O(q_n P_n), \tag{9}$$

where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

3. The Object of This Paper is to Generalize Above Theorem in the Following Form

Theorem. *Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let (φ_n) and (ψ_n) be sequences of positive numbers such that*

$$\varphi_n = O(\psi_n). \tag{10}$$

Let the sequences (p_n) and (q_n) be such that $p_n > 0$, $q_n > 0$, $P_n \to \infty$, $Q_n \to \infty$ and (6) is satisfied. If

$$\sum_{n=v+1}^{\infty} \frac{\varphi_n^{\delta k+k-1} p_n^k}{P_n^k P_{n-1}} = O\{\frac{\varphi_v^{\delta k+k-1} p_v^{k-1}}{P_v^k}\}, \tag{11}$$

then in order that

$$\lambda_n \in (\psi - |\bar{N}, q_n; \delta|_k, \varphi - |\bar{N}, p_n; \delta|_k), \tag{12}$$

it is sufficient that the conditions (8) and (9) hold.

It may be remarked that, in this thorem, if we take $\delta = 0$, $\varphi_n = \frac{P_n}{p_n}$ for $\varphi - |\bar{N}, p_n; \delta|_k$ and $\delta = 0$, $\psi_n = \frac{Q_n}{q_n}$ for $\psi - |\bar{N}, q_n; \delta|_k$, then we get Theorem A. In this case condition (11) reduces to

$$\sum_{n=v+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O(\frac{1}{P_v})$$

which always holds.

4. Proof of the Theorem

Let

$$T_n = \frac{1}{Q_n}\sum_{v=0}^{n} q_v s_v = \frac{1}{Q_n}\sum_{v=0}^{n}(Q_n - Q_{v-1})a_v. \tag{13}$$

Write $T_n - T_{n-1} = b_n$ (write $T_{-1} = 0$) so that $T_n = b_0 + b_1 + b_2 + \cdots + b_n$. Thus we suppose that in seies form the $(\bar{N}, q_n)$ transform of $\sum a_v$ is $\sum b_n$. In a similar way suppose that in series form the $(\bar{N}, p_n)$ transform of $\sum a_v\lambda_v$ is $\sum c_n$. Now we have for $n \geq 1$

$$b_n = \frac{q_n}{Q_nQ_{n-1}}\sum_{v=1}^{n} Q_{v-1}a_v \tag{14}$$

which gives

$$a_n = (\frac{Q_n}{q_n})b_n - (\frac{Q_{n-2}}{q_{n-1}})b_{n-1}. \tag{15}$$

Replacing a_v by $a_v\lambda_v$ and interchanging p, P with q, Q we have for $n \geq 1$

$$c_n = \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\lambda_v. \tag{16}$$

Substituting (15) in (16), we get

$$\begin{aligned} c_n &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n} P_{v-1}\lambda_v((Q_v/q_v)b_v - (Q_{v-2}/q_{v-1})b_{v-1}) \\ &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} b_v/q_v(P_{v-1}Q_v\lambda_v - P_vQ_{v-1}\lambda_{v+1}) + (p_nQ_n/q_nP_n)\lambda_nb_n \\ &= \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} b_v/q_v(P_vQ_{v-1}\Delta\lambda_v + (q_vP_v - p_vQ_v)\lambda_v) + (p_nQ_n/q_nP_n)\lambda_nb_n \quad (17) \end{aligned}$$

Now using (8) and (9) in (17) we have

$$\begin{aligned} c_n &= O(\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} b_vP_v) + O(b_n) \\ &= O(C_{n,1}) + O(C_{n,2}). \end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned} \{\sum_{v=1}^{n-1}|b_v|P_v\}^k &\leq \sum_{v=1}^{n-1}\{|b_v|^kP_v^k/p_v^{k-1}\}\{\sum_{v=1}^{n-1}p_v\}^{k-1} \\ &\leq P_{n-1}^{k-1}\sum_{v=1}^{n-1}|b_v|^kP_v^k/p_v^{k-1} \end{aligned}$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|C_{n,1}|^k &= \sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}|b_v|P_v\right|^k\\ &\le \sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}\frac{p_n^k}{P_n^kP_{n-1}}\sum_{v=1}^{n-1}|b_v|^kP_v^k/p_v^{k-1}\\ &= \sum_{v=1}^{\infty}|b_v|^kP_v^k/p_v^{k-1}\sum_{n=v+1}^{\infty}\frac{\varphi_n^{\delta k+k-1}p_n^k}{P_n^kP_{n-1}}\\ &= O\{\sum_{v=1}^{\infty}\varphi_v^{\delta k+k-1}|b_v|^k\}\\ &= O\{\sum_{v=1}^{\infty}\psi_v^{\delta k+k-1}|b_v|^k\} \quad \text{(by (10) and by (11))}\\ &< \infty.\end{aligned}$$

Now, in view of (4) we have

$$\begin{aligned}\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|C_{n,2}|^k &= O\{\sum_{n=1}^{\infty}\varphi_n^{\delta k+k-1}|b_n|^k\}\\ &= O\{\sum_{n=1}^{\infty}\psi_n^{\delta k+k-1}|b_n|^k\} \quad \text{(by (10))}\\ &< \infty,\end{aligned}$$

by the assumption that $\sum a_n$ is summable $\psi - |\bar{N}, q_n; \delta|_k$. This completes the proof of Theorem.

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