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A GENERAL METHOD FOR CONSTRUCTING REGULAR SUMMATION MATRIX

YIN DONGSHENG

Abstract. This note gives a general method for constructing regular summation matrix, specially, from which one can obtain the results of [2] naturally.

Denote $N = \{0, 1, 2, \dots\}$.

$A = (a_{nk})$, $n, k \in N$, is a lower infinite triangular matrix, i.e., $a_{nk} = 0$ if $k > n$. For sequence $\{S_n\}$, define $\{S_n^A\} : S_n^A := \sum_{k=0}^{n} a_{nk} S_k$, $n \in N$, we will say that $\{S_n\}$ is summable with sum s by the method defined by A, if $\lim_{n\to\infty} S_n^A = s$. If the summation method defined by A is regular, viz., $S_n \to s$ implies $S_n^A \to s$, then, for simplicity, we call A regular summation matrix.

Theorem. *If three functions $F(n,i), Q(n,i), \lambda(k)$ satisfy*

(1) $F(n, n+1) = 0$;

(2) $Q(n,i) \neq 0$ *and* $Q(n,i) \to \infty$ $(n \to \infty)$;

(3) $F(n,i)/Q(n,i)$ *and* $D(n,k) := Q(n,k+1)\lambda(k) - F(n,k+1)\lambda(k+1)$ are bounded functions of n;

(4) $\lambda(0) = 1$, define

$$\prod_{i=1}^{0} F(n,i) := \prod_{i=1}^{0} Q(n,i) := 1, \textit{ and denote}$$

$$C_{nk} := \Big[\prod_{i=1}^{k} \frac{F(n,i)}{Q(n,i)}\Big]\Big[\lambda(k) - \frac{F(n,k+1)}{Q(n,k+1)}\lambda(k+1)\Big],$$

where, n, i, k all $\in N$, then, (C_{nk}) is a regular summation matrix.

Proof.

$$\begin{aligned}\sum_{k=0}^{n} C_{nk} &= \sum_{k=0}^{n} \Big[\lambda(k) \prod_{i=1}^{k} \frac{F(n,i)}{Q(n,i)} - \lambda(k+1) \prod_{i=1}^{k+1} \frac{F(n,i)}{Q(n,i)}\Big] \\ &= \lambda(0) \prod_{i=1}^{0} \frac{F(n,i)}{Q(n,i)} - \lambda(n+1) \prod_{i=1}^{n+1} \frac{F(n,i)}{Q(n,i)}\end{aligned}$$

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$$= 1, (\text{according to (1), (4) and the definition of } \prod_{1}^{0}).$$

$$\begin{aligned}\lim_{n\to\infty} C_{nk} &= \lim_{n\to\infty} \Big[\prod_{i=1}^{k} \frac{F(n,i)}{Q(n,i)}\Big] \frac{Q(n,k+1)\lambda(k) - F(n,k+1)\lambda(k+1)}{Q(n,k+1)} \\ &= \lim_{n\to\infty} \frac{1}{Q(n,k+1)} \Big[D(n,k) \prod_{i=1}^{k} \frac{F(n,i)}{Q(n,i)}\Big] \\ &= 0, \ (\text{according to (2) and (3)})\end{aligned}$$

Namely, (C_{nk}) satisfies the conditions of the well-known Toeplitz theorem[1], So, (C_{nk}) is a regular summation matrix. If one want to construct a concrete regular summation matrix, he(she) can first choose two functions $F(n,i)$ and Q(n,i) which satisfy the conditions of above theorem, then determines $\lambda(k)$ of above theorem by using the second part of (3).

Example 1. Take $F(n,i) = r - rq^{n-i+1}$, $Q(n,i) = \mu(n,i) - q^{n+c}$, where, r, q, c, are real numbers, and $r \neq 0$, $q > 1$, $\mu(n,i)$ is a bounded function of n (for all $i \in N$) with $\mu(n,i) \neq q^{n+c}$. Obviously, $F(n,i)$, $Q(n,i)$ meet the requirements of above theorem.

$$\begin{aligned}D(n,k) &= Q(n,k+1)\lambda(k) - F(n,k+1)\lambda(k+1) \\ &= \lambda(k)[\mu(n,k+1) - q^{n+c}] - \lambda(k+1)(r - rq^{n-k}) \\ &= [\lambda(k)\mu(n,k+1) - r\lambda(k+1)] + q^n[r\lambda(k+1)q^{-k} - \lambda(k)q^c]\end{aligned}$$

is a bounded function of $n \Rightarrow r\lambda(k+1)q^{-k} - \lambda(k)q^c = 0 \Rightarrow \lambda(k+1) = \frac{1}{r}\lambda(k)q^{k+c}$, $\lambda(0) = 1 \Rightarrow \lambda(k) = \frac{1}{r^k}q^{\binom{k}{2}+kc}$ by induction for k.

$\lambda(k)$ is determined uniquely by $F(n,i)$ and $Q(n,i)$ here.

Above three functions $F(n,i), Q(n,i)$ and $\lambda(k)$ give a regular summation matix (C_{nk}). When $r = 1, c = 0$ and $\mu(n,i) = \mu_i(\mu_i$ is real number independent of n such that $\mu_{k+1} < q^k)$, this example becomes theorem 2 of [2].

Example 2. Take $F(n,i) = r(n-i+1)$, $Q(n,i) = n + \mu(n,i)$, where, $r \neq 0$, $\mu(n,i)$ is a bounded function of n with $n + \mu(n,i) \neq 0$. Similarly to example 1, we can obtain uniquely $\lambda(k) = \frac{1}{r^k}$.

When $r = 1$ and $\mu(n,i) = \lambda_i$ (λ_i is real number independent of n such that $\lambda_{k+1} > -k$), this example becomes theorem 1 of [2].

Remark. Interested reader can consider the relations between $(F(n,i), Q(n,i))$ and $\lambda(k)$ as well as matrix (C_{nk}) and triple $(F(n,i), Q(n,i), \lambda(k))$ systematically.

References

[1] R. V. Gramkrelidze (ed.), Analysis 1. Berlin Heidelberg: Springer-Verlag, 1989. 10-12.
[2] W. C. Chu and L. C. Hsu, *A note on a general class of arithmetic means*, Tamkang Journal of Mathematics **26** (1995), 155-157.

Institute of Mathematics, Dalian University of Technology, 116024, Liaoning Province, China.