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FORCED OSCILLATIONS OF NONLINEAR HYPERBOLIC EQUATIONS WITH FUNCTIONAL ARGUMENTS

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Abstract. In this paper, sufficient conditions for the forced oscillations of hyperbolic equations with functional arguments of the form

$$\frac{\partial^2}{\partial t^2}u(x,t)=a(t)\Delta u(x,t)+\sum_{i=1}^{m}a_i(t)\Delta u(x,\rho_i(t))-\sum_{j=1}^{k}q_j(x,t)f_j(u(x,\sigma_j(t)))+f(x,t),$$
$$(x,t)\in\Omega\times[0,\infty),$$

are obtained, where Ω is a bounded domain in R^n with piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in Euclidean n-space R^n.

1. Introduction

Partial differential equations with functional arguments have been studied extensively for the past few years. However, only a few papers [1-6] have been published on the oscillation theory of hyperbolic equations with functional arguments. In this paper, we studey the forced oscillations of hyperbolic equations with functional arguments of the form

$$\frac{\partial^2}{\partial t^2}u(x,t)=a(t)\Delta u(x,t)+\sum_{i=1}^{m}a_i(t)\Delta u(x,\rho_i(t))-\sum_{j=1}^{k}q_j(x,t)f_j(u(x,\sigma_j(t)))+f(x,t),\quad (1)$$

$(x,t)\in\Omega\times[0,\infty)\equiv G$, where Ω is a bounded domain in R^n with piecewise smooth boundary $\partial\Omega$ and Δ is the Laplacian in Euclidean n-space R^n.

Suppose that the following conditions (C) hold:

(C_1) $a, a_i\in C([0,\infty);[0,\infty))$, $i=1,2,\dots,m$;

(C_2) $\rho_i,\sigma_j\in C([0,\infty);R)$, $\lim_{t\to\infty}\rho_i(t)=\lim_{t\to\infty}\sigma_j(t)=\infty$, $i=1,2,\dots,m$; $j=1,2,\dots,k$;

(C_3) $q_j\in C(\overline{\Omega}\times[0,\infty);[0,\infty))$ and $q_j(t)=\min_{x\in\overline{\Omega}}q_j(x,t)$, $j=1,2,\dots,k$, $f\in C(\overline{\Omega}\times[0,\infty);R)$;

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(C_4) $f_j \in C(R,R)$, f_j are positive and convex in $(0,\infty)$ and $f_j(-u) = -f_j(u)$ for $u \geq 0$, $j = 1,2,\ldots,k$.

Our aim is to establish sufficient conditions under which every (classical) solution $u(x,t)$ of (1) satisfying a certain boundary condition is oscillatory on $\Omega \times [0,\infty)$ in the sense that $u(x,t)$ has a zero on $\Omega \times [0,\infty)$ for every $t > 0$. We consider two kinds of boundary conditions

$$\frac{\partial}{\partial N}u(x,t) + \mu u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \tag{2}$$

where N is the unit exterior normal vector to $\partial\Omega$ and $\mu(x,t)$ is a nonnegative continuous function on $\partial\Omega \times [0,\infty)$, and

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty). \tag{3}$$

In the following section 2 and section 3 the sufficient conditions are obtained for the oscillation of solutions of the problem (1), (2) and the problem (1), (3) in domain G. Note that the conditions for the oscillations for $f_j(u) = u$, $j = 1,2,\ldots,k$, have been obtained in the work of [5].

2. Oscillation of Problem (1), (2)

Theorem 2.1. *Suppose that* (C) *hold and that*

(C_5) *$f_j(u)$ are increasing in $(0,\infty)$, $j = 1,2,\ldots,k$;*

(C_6) *there exists a nonnegative oscillatory function $\eta \in C^2(R;[0,\infty))$ such that $\eta''(t) = \int_\Omega f(x,t)dx$ and $\lim_{t\to\infty}\eta(t) = 0$.*

Then every solution $u(x,t)$ of the problem (1), (2) *is oscillatory in G if the differential ineuqality*

$$y''(t) + \sum_{j=1}^{k} q_j(t) f_j(y(\sigma_j(t))) \leq 0, \quad t \geq 0, \tag{4}$$

has no eventually positive solutions.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1), (2) which has no zero in $\Omega \times [t_0,\infty)$ for some $t_0 \geq 0$. Without loss generality, we may assume that $u(x,t) > 0$ in $\Omega \times [t_0,\infty)$. From ($C_2$) there exists a $t_1 \geq t_0$ such that $u(x,\rho_i(t)) > 0$, $i = 1,2,\ldots,m$ and $u(x,\sigma_j(t)) > 0$, $j = 1,2,\ldots,k$, in $\Omega \times [t_1,\infty)$.

Integrating (1) with respect to x over the domain Ω, we have

$$\begin{aligned}\frac{d^2}{dt^2}\Big(\int_\Omega u(x,t)dx\Big) &= a(t)\int_\Omega \Delta u(x,t)dx + \sum_{i=1}^{m} a_i(t)\int_\Omega \Delta u(x,\rho_i(t))dx \\ &\quad - \sum_{j=1}^{k}\int_\Omega q_j(x,t) f_j(u(x,\sigma_j(t)))dx + \int_\Omega f(x,t)dx. \end{aligned}\tag{5}$$

Green's formula yields

$$\int_{\Omega} \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} dS = -\int_{\partial\Omega} \mu u dS \le 0, \quad t \ge t_i; \tag{6}$$

$$\int_{\Omega} \Delta u(x,\rho_i(t))dx = \int_{\partial\Omega} \frac{\partial}{\partial N} u(x,\rho_i(t))dS = -\int_{\partial\Omega} \mu(x,\rho_i(t))u(x,\rho_i(t))dS \le 0, \ t \ge t_1,$$
$$i = 1,2,\ldots,m. \tag{7}$$

From conditions (C_3), (C_4) and Jensen's inequality, it follows that

$$\int_{\Omega} q_j(x,t) f_j(u(x,\sigma_j(t)))dx \ge q_j(t) \int_{\Omega} f_j(u(x,\sigma_j(t)))dx$$
$$\ge q_j(t) \int_{\Omega} dx \cdot f_j(\int_{\Omega} u(x,\sigma_j(t))dx(\int_{\Omega} dx)^{-1}), \qquad t \ge t_1, \ j = 1,2,\ldots,k. \tag{8}$$

Combining (6), (7) and (8), we obtain

$$\frac{d^2}{dt^2}(\int_{\Omega} u(x,t)dx) \le -\sum_{j=1}^{k} q_j(t) \int_{\Omega} dx f_j(\int_{\Omega} u(x,\sigma_j(t)dx(\int_{\Omega} dx)^{-1}) + \int_{\Omega} f(x,t)dx, \quad t \ge t_i. \tag{9}$$

Set

$$y(t) = \frac{1}{|\Omega|}(\int_{\Omega} u(x,t)dx - \eta(t)), \quad t \ge t_1, \tag{10}$$

where $|\Omega| = \int_{\Omega} dx$. Then from (9) we obtain that

$$y''(t) \le -\sum_{j=1}^{k} q_j(t) f_j(\int_{\Omega} u(x,\sigma_j(t))dx \frac{1}{|\Omega|}), \quad t \ge t_1. \tag{11}$$

By (10) and (C_5), we have

$$f_j(\int_{\Omega} u(x,\sigma_j(t))dx \cdot \frac{1}{|\Omega|}) = f_j\Big(y(\sigma_j(t)) + \frac{1}{|\Omega|}\eta(\sigma_j(t))\Big) \ge f_j(y(\sigma_j(t)),$$
$$t \ge t_1, \quad j = 1,2,\ldots,k, \tag{12}$$

Consequently, we get

$$y''(t) + \sum_{j=1}^{k} q_j(t) f_j(y(\sigma_j(t))) \le 0, \quad t \ge t_1,$$

which contradicts assumption that (4) has no eventually positive solution.

In case $u(x,t) < 0$, then the function $-u(x,t)$ is a positive solution of the problem

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(x,t) = a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,\rho_i(t)) - \sum_{j=1}^{k} q(x,t) f_j(u(x,\sigma_j(t))) - f(x,t) \\ \qquad\qquad (x,t) \in \Omega \times [0,\infty) \equiv G, \\ \frac{\partial u}{\partial N} + \mu u = 0, \text{ on } \partial\Omega \times [0,\infty). \end{cases}$$

Now set $y(t) = \frac{1}{|\Omega|}(\int_\Omega -u(x,t)dx - \eta(t))$, $t \geq t_1$, and use an argument similar to the one used earlier to arrive at a contradiction. This completes the proof.

Lemma 2.1.[5] *Suppose that $y \in C^2([t_0,\infty);R)$ and that*

$$y(t) > 0, y'(t) > 0 \text{ and } y''(t) \leq 0, \quad t \geq t_0 > 0. \tag{13}$$

Then for any $\lambda \in (0,1)$ there exists a number $t_1 \geq t_0$, such that

$$y(t) \geq \lambda t y'(t) \text{ for } t \geq t_1. \tag{14}$$

Theorem 2.2. *Let the conditions* (C) *hold and that*
(C_7) *there exists a positive constant M such that $\frac{f_j(u)}{u} \geq M$ for $u > 0$;*
(C_8) *there exists an oscillatory function $\eta \in C^2(R;R)$ such that*

$$\eta''(t) = \int_\Omega f(x,t)dx \text{ and } \lim_{t\to\infty} \eta(t) = 0.$$

Then every solution $u(x,t)$ of the problem (1), (2) is oscillatory in G if the differential inequality

$$y''(t) + \lambda M \sum_{j=1}^{k} q_j(t) y(\sigma_j(t)) \leq 0, \quad t \geq 0, \tag{15}$$

has no eventually positive solutions for some $\lambda \in (0,1)$.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1), (2) which has no zero in $\Omega \times [t_0,\infty)$ for some $t_0 \geq 0$. Without loss generality, we may assume that $u(x,t) > 0$ in $\Omega \times [t_0,\infty)$. Form ($C_2$) there exists a $t_1 \geq t_0$ such that $u(x,\rho_i(t)) > 0$, $i = 1,2,\ldots,m$, and $u(x,\sigma_j(t)) > 0$, $j = 1,2,\ldots,k$, in $\Omega \times [t_1,\infty)$.

Integrating (1) with respect to x over the domain Ω, we have

$$\begin{aligned}\frac{d^2}{dt^2}(\int_\Omega u(x,t)dx) &= a(t)\int_\Omega \Delta u(x,t)dx + \sum_{i=1}^{m} a_i(t)\int_\Omega \Delta u(x,\rho_i(t))dx \\ &\quad - \sum_{j=1}^{k}\int_\Omega q_j(x,t) f_j(u(x,\sigma_j(t)))dx + \int_\Omega f(x,t)dx. \end{aligned}\tag{16}$$

Green's formula yields

$$\int_\Omega \Delta u(x,t)dx = \int_{\partial\Omega} \frac{\partial u}{\partial N} dS = -\int_{\partial\Omega} \mu u dS \leq 0, \quad t \geq t_1; \tag{17}$$

$$\begin{aligned}\int_\Omega \Delta u(x,\rho_i(t))dx = \int_{\partial\Omega} \frac{\partial}{\partial N} u(x,\rho_i(t))dS = -\int_{\partial\Omega} \mu(x,\rho_i(t))u(x,\rho_i(t))dS \leq 0, \\ t \geq t_1, \quad i = 1,2,\ldots,m. \end{aligned}\tag{18}$$

From conditions (C_3), (C_4), (C_7) and Jensen's inequality, it follows that

$$\int_\Omega q_j(x,t) f_j(u(x,\sigma_j(t)))dx \geq q_j(t)\int_\Omega f_j(u(x,\sigma_j(t)))dx$$
$$\geq Mq_j(t)\int_\Omega u(x,\sigma_j(t)dx, \quad t \geq t_i, \quad j = 1,2,\dots,k. \tag{19}$$

Thus we combine (17), (18) and (19) and get

$$\frac{d^2}{dt^2}\left(\int_\Omega u(x,t)dx\right) \leq -M\sum_{j=1}^{k} q_j(t)\int_\Omega u(x,\sigma_j(t))dx + \int_\Omega f(x,t)dx, \quad t \geq t_1. \tag{20}$$

Set

$$y(t) = \int_\Omega u(x,t)dx - \eta(t), \quad t \geq t_1, \tag{21}$$

from (20) we obtain

$$y''(t) \leq -M\sum_{j=1}^{k} q_j(t)\int_\Omega u(x,\sigma_j(t))dx \leq 0, \quad t \geq t_1. \tag{22}$$

We claim that there is a number $t_2 \geq t_1$ such that $y(t) > 0$, $t \geq t_2$. In fact, if $y(t) \leq 0$ then $\int_\Omega u(x,t)dx \leq \eta(t)$, which is impossible in view the fact that $u(x,t) > 0$ and the function η is oscillatory. From (22) we have $y''(t) \leq 0$, $t \geq t_2$. Using the fact that $y(t) > 0$ and $y''(t) \leq 0$ we have $y'(t) > 0$, $t \geq t_2$. Now, since y is an increasing function and $\lim_{t\to\infty}\eta(t) = 0$, it follows from (21) that there is a number $t_3 \geq t_2$, by Lemma 2.1, such that

$$\int_\Omega u(x,\sigma_j(t))dx \geq \lambda y(\sigma_j(t)), \quad t \geq t_3, \quad j = 1,2,\dots,k.$$

Consequently, we get

$$y''(t) + \lambda M\sum_{j=1}^{k} q_j(t) y(\sigma_j(t)) \leq 0, \quad t \geq t_3, \tag{23}$$

which contrdaicts the assumption that (15) has no eventuully positive solution.

In case $u(x,t) < 0$, then the function $-u(x,t)$ is a positive solution of the problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}u(x,t) = a(t)\Delta u(x,t) + \sum_{i=1}^{m} a_i(t)\Delta u(x,\rho_i(t)) - \sum\limits_{j=1}^{k} q(x,t) f_j(u(x,\sigma_j(t)) - f(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \Omega \times [0,\infty) \equiv G, \\ \frac{\partial u}{\partial N} + \mu u = 0, \text{ on } \partial\Omega \times [0,\infty). \end{cases}$$

Now set $y(t) = \int_\Omega(-u(x,t)dx - \eta(t)$, $t \geq t_0$, and use arguments similar to the one used earlier to arrive at a contradiction. This completes the proof.

Theorem 2.3. *Suppose that conditions* (C), (C_7) *and* (C_8) *hold and that*

(C_9) $\sigma(t) = \max_{1\le j\le k}\{\sigma_j(t)\} \le t$, $\sigma'(t) \ge 0$, $t \ge t_0$ for some $t_0 \ge 0$.
If there exista a $\lambda \in (0,1)$ *such that*

$$\limsup_{t\to\infty} M\int_{\sigma(t)}^{t}\sum_{j=1}^{k} q_j(s)\sigma_j(s)ds > \frac{1}{\lambda^2}, \tag{24}$$

then every solution $u(x,t)$ *of the problem* (1), (2) *is oscillatory in* G.

Proof. On the contrary let $u(x,t)$ be a nonoscillatory solution of (1), (2), which we assume to be positive on $\Omega \times (0,\infty)$. Similarly to the proof of Theorem 2.2, we can prove that the function y defined by (21) satisfies the inequalities (13) and (15) for above $\lambda \in (0,1)$. By Lemma 2.1 we can choose a number t_1 sufficiently large such that $y(t) \ge \lambda t y'(t)$ for $t \ge t_1$, and

$$y(\sigma_j(t)) \ge \lambda\sigma_j(t)y'(\sigma_j(t)) \text{ for } t \ge t_1, \quad j = 1,2,\ldots,k.$$

Now, by (23) we can to get

$$y''(t) + \lambda^2 M\sum_{j=1}^{k} q_j(t)\sigma_j(t)y'(\sigma_j(t)) \le 0, \quad t \ge t_1.$$

Integrating the above inequality from $\sigma(t)$ to t we have

$$y'(t) - y'(\sigma(t)) + \lambda^2 M\int_{\sigma(t)}^{t}\sum_{j=1}^{k} q_j(s)\sigma_j(s)y'(\sigma_j(s))ds \le 0, \quad t \ge t_1.$$

Therefore,

$$\lambda^2 M\int_{\sigma(t)}^{t}\sum_{j=1}^{k} q_j(s)\sigma_j(s)y'(\sigma_j(s))ds \le 1 - \frac{y'(t)}{y'(\sigma(t))} < 1, \quad t \ge t_1.$$

And hence

$$\limsup_{t\to\infty} M\int_{\sigma(t)}^{t}\sum_{j=1}^{k} q_j(s)\sigma_j(s)y'(\sigma_j(s))ds \le \frac{1}{\lambda^2},$$

which violates the condition (24).

The proof of the case $u(x,t) < 0$ is similar and is omitted.

Corollary 2.1. *In addition to conditions* (C), *let* (C_7), (C_9) *hold and suppose that* $f(x,t) \equiv 0$. *If*

$$\limsup_{t\to\infty} M\int_{\sigma(t)}^{t}\sum_{j=1}^{k} q_j(s)\sigma_j(s)ds > 1, \ \textit{where } \sigma(t) = \max_{1\le j\le k}\{\sigma_j(t)\},$$

then every solution $u(x,t)$ of the problem (1), (2) *is oscillatory in G.*

Remark 1.1. Theorem 2.2 and Theorem 2.3 improved the results of Theorem 2.1 and Theorem 2.2 from [5] and Corollary 2.1 extended the corollary 2.1 in [5].

3. Oscilation of Problem (1),(3)

In the domain Ω we consider the following Dirichlet problem

$$\begin{cases} \Delta u + \alpha u = 0 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases} \tag{25}$$

where α is a contant. It is well know [7,8] that the least eigenvalue α_0 of the problem (25) is positive and the corresponding eigenfuction $\varphi(x)$ is positive on Ω.

Theorem 3.1. *Let the conditions* (C) *and* (C_5) *hold and that*
(C_{10}) *There exists a nonnegative oscillatory function $\eta \in C^2(R;[0,\infty))$ such that*

$$\eta''(t) = \int_\Omega f(x,t)\varphi(x)dx \text{ and } \lim_{t\to\infty}\eta(t) = 0.$$

Then every solution $u(x,t)$ of the problem (1), (3) is oscillatory in G if the differential inequality

$$y''(t) + \sum_{j=1}^{k} q_j(t)f_j(y(\sigma_j(t))) \le 0, \quad t \ge 0, \tag{26}$$

has no eventually positive solutions.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1), (3), which has no zero in $\Omega \times [t_0,\infty)$ for some $t_0 \ge 0$. Without loss generality, we may assume that $u(x,t) > 0$ in $\Omega \times [t_0,\infty)$. From ($C_2$) there exists a $t_1 \ge t_0$ such that $u(x,\rho_i(t)) > 0$, $i = 1,2,\ldots,m$, and $u(x,\sigma_j(t)) > 0$, $j = 1,2,\ldots,k$, in $\Omega \times [t_1,\infty)$.

Multiplying (1) by $\varphi(x)$ and integrating over Ω, we obtain

$$\frac{d^2}{dt^2}\Big(\int_\Omega u(x,t)\varphi(x)dx\Big) = a(t)\int_\Omega \Delta u(x,t)\varphi(x)dx + \sum_{j=1}^{m} a_i(t)\int_\Omega \Delta u(x,\rho_i(t))\varphi(x)dx$$
$$-\sum_{j=1}^{k}\int_\Omega q_j(x,t)f_j(u(x,\sigma_j(t)))\varphi(x)dx + \int_\Omega f(x,t)\varphi(x)dx, \ t \ge t_1. \tag{27}$$

Using Green's formula, it follows that

$$\int_\Omega \Delta u\varphi(x)dx = \int_\Omega u(x,t)\Delta\varphi(x)dx = -a_0\int_\Omega u(x,t)\varphi(x)dx \le 0, t \ge t_1. \tag{28}$$

$$\int_\Omega \Delta u(x,\rho_i(t))\varphi(x)dx = \int_\Omega u(x,\rho_i(t))\Delta\varphi(x)dx = -a_0\int_\Omega u(x,\rho_i(t))\varphi(x)dx \le 0, \ t \ge t_1. \tag{29}$$

From conditions (C_3), (C_4) and Jensen's inequality, it follows that

$$\int_\Omega q_j(x,t)f_j(u,(x,\sigma_j(t)))\varphi(x)dx \geq q_j(t)\int_\Omega f_j(u(x,\sigma_j(t)))\varphi(x)dx$$
$$\geq q_j(t)\int_\Omega \varphi(x)dx f_j(\int_\Omega u(x,\sigma_j(t))\varphi(x)dx(\int \varphi(x)dx)^{-1}), \quad j=1,2,\ldots,k, \quad t\geq t_1 \tag{30}$$

Now conbining (28), (29) and (30), we obtain

$$\frac{d^2}{dt^2}(\int_\Omega u(x,t)\varphi(x)dx) \leq -\sum_{j=1}^k q_i(t)\int_\Omega \varphi(x)dx f_j(\int_\Omega u(x,\sigma_j(t))\varphi(x)dx(\int_\Omega \varphi(x)dx)^{-1})$$
$$+\int_\Omega f(x,t)dx, \quad t\geq t_1. \tag{31}$$

Set

$$y(t) = (\int_\Omega \varphi(x)dx)^{-1}(\int_\Omega u(x,t)\varphi(x)dx - \eta(t)), \quad ,t\geq t_1, \tag{32}$$

by (31), we obtain

$$y''(t) \leq -\sum_{j=1}^k q_j(t)f_j(\int_\Omega u(x,\sigma_j(t))\varphi(x)dx(\int_\Omega \varphi(x)dx)^{-1}), \quad t\geq t_1. \tag{33}$$

And from (32) and (C_5) we have

$$f_j(\int_\Omega u(x,\sigma_j(t))\varphi(x)dx(\int_\Omega \varphi(x)dx)^{-1}) = f_j(y(\sigma_j(t)) + (\int_\Omega \varphi(x)dx)^{-1}\cdot\eta(\sigma_j(t)))$$
$$\geq f_j(y(\sigma_j(t))), \; t\geq t_1, \; j=1,2,\ldots,k. \tag{34}$$

Consequently, we get

$$y''(t) + \sum_{j=1}^k q_j(t)f_j(y(\sigma_j(t))) \leq 0, \quad t\geq t_1,$$

Which contradicts the assumption that (26) has no eventually positive solution. A similar proof can be given for the case $u(x,t)<0$. This completes the proof.

Theorem 3.2. *Let the conditions* (C) *and* (C_7) *hold and that*
(C_{12}) *There exists an oscillatory function* $\eta\in C^2(R;R)$ *such that*

$$\eta''(t) = \int_\Omega f(x,t)\varphi(x)dx \text{ and } \lim_{t\to\infty}\eta(t)=0.$$

Then every solution $u(x,t)$ *of the problem* (1), (3) *is oscillatory in* G *if the differential inequality*

$$y''(t) + \lambda M\sum_{j=1}^k q_j(t)y(\sigma_j(t)) \leq 0, \quad t\geq 0, \tag{35}$$

has no eventually positive solutions for some $\lambda \in (0,1)$.

Proof. Suppose to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1), (3) which has no zero in $\Omega \times [t_0, \infty)$ for some $t_0 \geq 0$. Without loss generality, we may assume that $u(x,t) > 0$ in $\Omega \times [t_0, \infty)$. From ($C_2$) there exists a $t_1 \geq t_0$ such that $u(x, \rho_i(t)) > 0$, $i = 1, 2, \ldots, m$ and $u(x, \sigma_j(t)) > 0$, $j = 1, 2, \ldots, k$, in $\Omega \times [t_1, \infty)$.

Multiplying (1) by $\varphi(x)$ and integrating over Ω, we obtain

$$\frac{d^2}{dt^2}\Big(\int_\Omega u(x,t)\varphi(x)dx\Big) = a(t)\int_\Omega \Delta u(x,t)\varphi(x)dx + \sum_{i=1}^m a_i(t)\int_\Omega \Delta u(x,\rho_i(t))\varphi(x)dx$$
$$- \sum_{j=1}^k \int_\Omega q_j(x,t) f_j(u(x,\sigma_j(t)))\varphi(x)dx + \int_\Omega f(x,t)\varphi(x)dx. \quad (36)$$

From Green's formala it follows that

$$\int_\Omega \Delta u\varphi(x)dx = \int_\Omega u(x,t)\Delta\varphi(x)dx = -\alpha_0 \int_\Omega u(x,t)\varphi(x)dx \leq 0, \quad t \geq t_i; \quad (37)$$
$$\int_\Omega \Delta u(x,\rho_i(t))\varphi(x)dx = \int_\Omega u(x,\rho_i(t))\Delta\varphi(x)dx$$
$$= -\alpha_0 \int_\Omega u(x,\rho_i(t))\varphi(x)dx \leq 0, \quad t \geq t_1,\ i = 1, 2, \ldots, m. \quad (38)$$

Moreover, from conditions (C_3), (C_4) and (C_7) and Jensen's inequality, it follows that

$$\int_\Omega q_j(x,t) f_j(u(x,\sigma_j(t)))\varphi(x)dx \geq q_j(t)\int_\Omega f_j(u(x,\sigma_j(t)))\varphi(x)dx$$
$$\geq q_j(t)\int_\Omega \varphi(x)dx f_j\Big(\int_\Omega u(x,\sigma_j(t))\varphi(x)dx\Big(\int_\Omega \varphi(x)dx\Big)^{-1}\Big)$$
$$\geq q_j(t)\int_\Omega \varphi(x)dx M \int_\Omega u(x,\sigma_j(t))\varphi(x)dx\Big(\int_\Omega \varphi(x)dx\Big)^{-1}$$
$$= M q_j(t)\int_\Omega u(x,\sigma_j(t))\varphi(x)dx,\ j = 1, 2, \ldots, k,\ t \geq t_1. \quad (39)$$

Then using (37), (38) and (39), we obtain

$$\frac{d^2}{dt^2}\Big(\int_\Omega u(x,t)\varphi(x)dx\Big) \leq -M\sum_{j=1}^k q_j(t)\int_\Omega u(x,\sigma_j(t))\varphi(x)dx + \int_\Omega f(x,t)\varphi(x)dx,$$
$$t \geq t_1. \quad (40)$$

Set

$$y(t) = \int_\Omega u(x,t)\varphi(x)dx - \eta(t), \quad t \geq t_1, \quad (41)$$

by (40) we obtain

$$y''(t) \le -M \sum_{j=1}^{k} q_j(t) \int_\Omega u(x, \sigma_j(t))\varphi(x)dx, \quad t \ge t_1. \tag{42}$$

We note that $\int_\Omega u(x,t)\varphi(x)dx > 0$, hence as in the proof of Theorem 2.2 we have

$$y(t) > 0, \quad t \ge t_2 \ge t_1, \tag{43}$$

and by (42) we have

$$y''(t) \le 0, \quad t \ge t_1 \tag{44}$$

Form (43) and (44) it follows that $y'(t) > 0$, $t \ge t_1$. Since y is an increasing function and $\lim_{t\to\infty} y(t) = 0$, we conclude from $\int_\Omega u(x,t)\varphi(x)dx = y(t) + \eta(t)$ that there exists a number $t_3 \ge t_2$ such that the following inequalities hold

$$\int_\Omega u(x,t)\varphi(x)dx \ge \lambda y(t), \quad t \ge t_3,$$
$$\int_\Omega u(x,\sigma_j(t))\varphi(x)dx \ge \lambda y(\sigma_j(t)), \quad t \ge t_3,\ j = 1, 2, \ldots, k.$$

Consequently, we get

$$y''(t) + \lambda M \sum_{j=1}^{k} q_j(t) y(\sigma_j(t)) \le 0, \quad t \ge t_3,$$

which contradicts the assumption that (35) has no eventually positive solution. A similar proof can be given for the case $u(x,t) < 0$. This completes the proof.

The proof of the following Theorem can be modelled on that of Theorem 3.2 and Theorem 2.3.

Theorem 3.3. *Suppose that the conditions* (C), (C_7), (C_9) *and* (C_{11}) *hold, and that there exists a* $\lambda \in (0,1)$ *such that*

$$\limsup_{t\to\infty} M \int_{\sigma(t)}^{t} \sum_{j=1}^{k} q_j(s)\sigma_j(s)ds > \frac{1}{\lambda^2},$$

then every solution of the problem (1), (3) *is oscillatory in* G.

Corollary 3.1. *Let conditions* (C), (C_7) *and* (C_9) *hold, and suppose that* $f(x,t) \equiv 0$. *If*

$$\limsup_{t\to\infty} M \int_{\sigma(t)}^{t} \sum_{j=1}^{k} q_j(s)\sigma_j(s)ds > 1,$$

then every solution of the problem (1), (3) *is oscillatory in* G.

4. Examples

Following are illustrative examples.

Example 1. Consider the hyperbolic equation

$$u_{tt}(x,t) = u_{xx}(x,t) + u_{xx}(x,t-3\pi) - e^{-t}u(x,t-3\pi) - u(x,t-4\pi)$$
$$-e^{-t}(1+\cos x)\cos t, (x,t) \in (0,\pi) \times [0,\infty), \tag{45}$$

with boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, \quad t \geq 0. \tag{46}$$

Here $a(t) = 1$, $a_1(t) = 1$, $\rho_1(t) = t - 3\pi$, $q_1(t) = e^{-t}$, $q_2(t) = 1$, $f_1(s) = s$, $f_2(s) = s$, $f(x,t) = -e^{-t}(1+\cos x)\cos t$, $\sigma_1(t) = t - 3\pi$, $\sigma_2(t) = t - 4\pi$, $\sigma(t) = t - 3\pi < t$. We note that

$$\int_\Omega f(x,t)dx = \int_0^\pi e^{-t}(1+\cos x)\cos t dx = -\pi e^{-t}\cos t \tag{47}$$

We can choose $\eta(t) = \frac{\pi}{2}e^{-t}\cos t$. It is easy to verify that all the hypothesises of Theorem 2.3 are satisfied and hence all the solutions of problem (45), (46) are oscillatory. One such solution is $u(x,t) = (1+\cos x)\cos t$.

Example 2. Consider the hyperbolic equation

$$u_{tt}(x,t) = u_{xx}(x,t) + 2e^{-\frac{\pi}{2}}u_{xx}(x,t-\frac{\pi}{2}) - 2e^{-\pi}u(x,t-\pi) - e^{-t}\cos t \sin x,$$
$$(x,t) \in (0,\pi) \times [0,\infty). \tag{48}$$

with boundary condition

$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0 \tag{49}$$

Here $\Omega = (0,\pi)$, $a(t) = 1$, $a_1(t) = 2e^{-\frac{\pi}{2}}$, $\rho_1(t) = t - \frac{\pi}{2}$, $q_1(t) = 2e^{-\pi}$, $f_1(s) = s$, $\sigma(t) = \sigma_1(t) = t - \pi < t$, $f(x,t) = -e^{-t}\cos t \sin x$. Moreover, the corresponding eigenvalue problem

$$\begin{cases} \Delta u + \alpha u = 0, & x \in (0,\pi) \\ u = 0, x = 0, \pi \end{cases} \tag{50}$$

has the eigenvalue $\alpha_0 = 1$ with the corresponding eigenfunction $\varphi(x) = \sin x > 0$ on $(0,\pi)$.

We note that

$$\int_\Omega f(x,t)\varphi(x)dx = -\int_0^x e^{-t}\cos t \sin^2 x dx = -\frac{\pi}{2}e^{-t}\cos t. \tag{51}$$

Choose the function $\eta(t) = \frac{\pi}{4}e^{-t}\sin t$. Now it is easily checked that the hypothesises of Theorem 3.3 are verified. Thus all the solutions of perоblem (48), (49) are oscillatory. One such solution is $u(x,t) = e^{-t}\cos t \sin x$.

References

[1] B. T. Cui, *Oscillation theorems of hyperbolic equations with deviating arguments*, Acta Sci. Math. (Szeged) **58**(1993), 159-168.
[2] B. S. Lalli, Y. H. Yu and B. T. Cui, *Oscillations of certain neutral differential equations with deviating arguments*, Bull. Austral. Math. Soc. **46**(1992), 373-380.
[3] B. S. Lalli, Y. H. Yu and B. T. Cui, *Oscillation of hyperbolic equations with functional arguments*, Appl. Math. Comput. **53**(1993), 97-110.
[4] D. D. Bainov and B. T. Cui, *Oscillation properties for damped hyperbolic equations with deviating arguments*, in Proceedings of The Third International Colloquium on Differential Equations, International Science Pubishers, 1993, 23-30(Netherlands).
[5] B. S. Lalli, Y. H. Yu and B. T. Cui, *Forced Oscillations of hyperbolic differential equations with deviating arguments*, Indian. J. Pure Appl. Math. **25**(1994), 387-397.
[6] B. T. Cui, *Oscillation properties of the solutions of hyperbolice equations with deviating arguments*, Demonstratio Mathematica **29**(1996), 61-68.
[7] N. Yoshida, *Forced oscillations of solutions of parabolic equations*, Bull. Austral. Math. Soc. **36**(1987), 287-294.
[8] B. T. Cui, *Oscillation theroems of nonlinear parabolic equations of neutral type*, Math. J. Toyama Univ. **14**(1991), 112-123.

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