TAMKANG JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 1999

ON THE ESSENTIAL SPECTRA OF GENERAL DIFFERENTIAL OPERATORS

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Abstract. In this paper, it is shown in the cases of one and two singular end-points and when all solutions of the equation M[u]-λuw=0, and its adjoint $M^+[v] - \lambda wv = 0$ are in $L^2_w(a,b)$ (the limit circle case) with $f \in L^2_w(a,b)$ for $M[u] - \lambda wu = wf$ that all well-posed extensions of the minimal operator $T_0(M)$ generated by a general ordinary quasi-differential expression M of nth order with complex coefficients have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum. This implies that all the regularly slovable operators have all the standard essential spectra to be empty. These results extend those of formally symmetric expression M studied in [1] and [12], and also extend those proved in [8] in the case of one singular end-point of the interval [a,b).

1. Introduction

The minimal operators $T_0(M)$ and $T_0(M^+)$ generated by a general ordinary quasi-differential expression M and its formal adjoint M^+ respectively, form an adjoint pair of closed, densely-defined operators in the underlying L^2_w-space, that is $T_0(M) \subset [T_0(M^+)]^*$. The operators which fulfill the role that the self-adjoint and maximal symmetric operators play in the case of a formally symmetric expression M are those which are regularly slovable with respect to $T_0(M)$ and $T_0(M^+)$. Such an operator S satisfies $T_0(M) \subset S \subset [T_0(M^+)]^*$ and for some $\lambda \in \mathbb{C}$, $(S - \lambda I)$ is a Fredholm operator of zero index, this means that S has the desirable Fredholm property that the equation $(S - \lambda I)u = f$ has a solution if and only if f is orthogonal to the solution space of $(S - \lambda I)v = 0$ and furthermore the solution spaces of $(S - \lambda I)u = 0$ and $(S^* - \overline{\lambda} I)v = 0$ have the same finite dimension. This notion was originally due to Visik in [15].

Akhiezer and Glazman [1] and Naimark [12] showed that the self-adjoint extensions of the minimal operator $T_0(M)$ generated by a formally symmetric differential expression M with maximal deficiency indices have reslovents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum. In [8] Ibrahim extend their results for a general ordinary quasi-differential expression M of nth order with complex coefficients in the case of one singular end-point.

Received May 28, 1998.
1991 *Mathematics Subject Classification.* Primary 3405, 34B25, 34C11, 34E10, 34E15, 3410.
Key words and phrases. Quasi-differential expressions, the joint field of regularity, minimal and maximal operators, regularly solvable operators, boundary conditions, essential spectra.

Our objective in this paper is to extend the results in [1], [8] and [12] for a general ordinary quasi-differential expression M in the cases of one and two singular end-points by considering that $f \in L^2_w(a,b)$ for $M[u] - \lambda wu = wf$ and when all solutions of the equations $M[u] - \lambda wu = 0$ and $M^+[v] - \overline{\lambda} wv = 0$ are in $L^2_w(a,b)$ for some (and hence all $\lambda \in \mathbb{C}$).

We deal throughout with a quasi-differential expression M of arbitrary order n defined by a general Shin-Zettl matrix, and the minimal operator $T_0(M)$ generated by $w^{-1}M[.]$ in $L^2_w(I)$, where w is a positive weight function on the underlying interval I. The end-points a and b of I may be regular or singular end-points.

2. Preliminaries

We begin with a brief survey of adjoint pairs of operators and their associated regularly solvable operators; a full treatment may be found in [2, Chapter III], [3] and [8].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$ respectively, and $N(T)$ will denote its null space. The *nullity* of T, written nul(T), is the dimenssion of $N(T)$ and the deficiency of T, def(T), is the co-dimension of $R(T)$ in H; if T is densely defined and $R(T)$ is closed, then def$(T) = nul(T^*)$. The Fredholm domain of T is (in the notation of [2]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values $\lambda \in \mathbb{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The *index* of $(T - \lambda I)$ is the number ind $(T - \lambda I) = nul(T - \lambda I) - \text{def}(T - \lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely-defined operators A, B in H are said to form an *adjoint pair* if $A \subset B^*$ and consequently $B \subset A^*$, equivalently, $(Ax, y) = (x, By)$, for all $x \in D(A)$ and $y \in D(B)$, where $(.,.)$ denotes the inner-product on H.

The *joint field of regularity* $\Pi(A, B)$ of A and B is the set of $\lambda \in \mathbb{C}$ which are such that $\lambda \in \Pi(A)$, the field of regularity of A, $\overline{\lambda} \in \Pi(B)$ and def$(A - \lambda I)$ and def$(B - \overline{\lambda} I)$ are finite. An adjoiont pair A, B is said to be compatible if $\Pi(A, B) \neq \phi$. Recall that $\lambda \in \Pi(A)$ if and only if there exists a positive constant $K(\lambda)$ such that ,

$$||(A - \lambda I)x|| \geq K(\lambda)||x|| \qquad \text{for all} \quad x \in D(A),$$

or equivalently, on using the Closed-Graph Theorem, $nul(A - \lambda I) = 0$ and $R(A - \lambda I)$ is closed.

A closed operator S in H is said to be *regularly slovable* with respect to the compatible adjoint pair A, B if $A \subset S \subset B^*$ and $\Pi(A, B) \cap \Delta_4(S) \neq \phi$, where

$$\Delta_4(S) = \{\lambda : \lambda \in \Delta_3(S), \ \text{ind}(A - \lambda I) = 0\}.$$

If $A \subset S \subset B^*$ and the resolvent set $\rho(S)$ (see [2]) of S is non-empty, S is said to be *well-posed* with respect to A and B. Note that if $A \subset S \subset B^*$ and $\lambda \in \rho(S)$, then $\lambda \in \Pi(A)$ and $\overline{\lambda} \in \rho(S^*) \subset \Pi(B)$ so that if def$(A - \lambda I)$ and def$(B - \overline{\lambda} I)$ are finite, then A and B are compatible; in this case S is regularly slovable with respect to A and B. The terminology *"regularly slovable"* comes from Visik s'paper [15], while the notion of *"well-posed"* was introduced by Zhikhar in his work on J-self-adjoint operators in [19]. The complement of $\rho(S)$ in $\mathbb{C}$ is called the spectrum of S and written $\sigma(S)$. The point spectrum $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$ and residual spectrum $\sigma_r(S)$ are the following subsets of $\sigma(S)$:

(i) $\lambda \in \sigma_p(S)$ if and only if $R(S-\lambda I) = \overline{R(S-\lambda I)} \subset H$,
(ii) $\lambda \in \sigma_c(S)$ if and only if $R(S-\lambda I) \subset \overline{R(S-\lambda I)} = H$,
(iii) $\lambda \in \sigma_r(S)$ if and only if $R(S-\lambda I) \subset \overline{R(S-\lambda I)} \subset H$,

For a closed operator S we have

$$\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S).$$

An important subset of the spectrum of a closed densely-defined T in H is the so-called *essential spectrum*. The various essential spectra of T are defined as in [2, Chapter IX] to be the sets,

$$\sigma_{ek}(T) = \mathbb{C} \backslash \Delta_k(T), \quad k = 1,2,3,4,5; \tag{2.1}$$

$\Delta_3(T)$ and $\Delta_4(T)$ have been defined earlier.

The sets $\sigma_{ek}(T)$ are closed and $\sigma_{ek}(T) \subset \sigma_{ej}(T)$ if $k < j$. The inclusion being strict in general. We refer the reader to [2, Chapter IX] for further information about the sets $\sigma_{ek}(T)$.

We now turn to the quasi-differential expressions defined in terms of a Shin-Zettl matrix A on an open interval I, where I denotes an open interval with left end-point a and right end-point b, $(-\infty \le a < b \le \infty)$. The set $Z_n(I)$ of Shin-Zettl matrices on I consists of nxn-matrices $A = \{a_{rs}\}$ whose entries are complex-valued functions on I which satisfy the following conditions:

$$\left.\begin{array}{ll} a_{rs} \in L^1_{\text{loc}}(I) & (1 \le r,s \le n, n \ge 2) \\ a_{r,r+1} \ne 0 \quad \text{a.e. on I} & (1 \le r \le n-1) \\ a_{rs} = 0 \quad \text{a.e.on I} & (2 \le r+1 < s \le n) \end{array}\right\} \tag{2.2}$$

For $A \in Z_n(I)$ the *quasi-derivatives* associated with A are defined by:

$$\left.\begin{array}{l} y^{[0]} := y \\ y^{[r]} := a^{-1}_{r,r+1}\{(y^{[r-1]})' - \sum\limits_{s=1}^{r} a_{rs} y^{[s-1]}\}, \quad (1 \le r \le n-1) \\ y^{[n]} := (y^{[n-1]})' - \sum\limits_{s=1}^{n} a_{ns} y^{[s-1]} \end{array}\right\} \tag{2.3}$$

where the prime $'$ denotes *differentiation.*

The quasi-differential expression M associated with A is given by,

$$M[y] := i^n y^{[n]}, \tag{2.4}$$

This being defined on the set

$$V(M) := \{y : y^{[r-1]} \in AC_{\text{loc}}(I), r = 1, 2, \ldots, n\}, \tag{2.5}$$

where $AC_{\text{loc}}(I)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I.

The *formal adjoint* M^+ of M is defined by the matrix $A^+ \in Z_n(I)$ given by:

$$A^+ = -J_{nxn}^{-1} A^* J_{nxn'} \tag{2.6}$$

where A^* is the conjugate transpose of A and J_{nxn} is the non-singular nxn-matrix,

$$J_{nxn} = \Big((-1)^r \delta_{r,n+1-s}\Big), \quad (1 \le r, s \le n), \tag{2.7}$$

δ being the Kronecker delta. If $A^+ = \{a_{rs}^+\}$, then it follows that,

$$a_{rs}^+ = (-1)^{r+s+1} \overline{a}_{n-s+1,n-r+1}. \tag{2.8}$$

The *quasi-derivatives* associated with A^+ are therefore:

$$\left.\begin{array}{l} y_+^{[0]} := y, \\ y_+^{[r]} := ((\overline{a}_{n-r,n-r-1})^{-1}\{(y_+^{[r+1]})' - \sum\limits_{s=1}^{r}(-1)^{r+s+1}\overline{a}_{n-s+1,n-r+1} y_+^{[s+1]}\} \\ \quad (1 \le r \le n-1), \\ y_+^{[n]} := (y_+^{[n-1]})' - \sum\limits_{s=1}^{n}(-1)^{r+s+1}\overline{a}_{n-s+1,1} y_+^{[s-1]}, \\ \quad \ldots \end{array}\right\} \tag{2.9}$$

and

$$M^+[y] := i^n y_+^{[n]} \quad \text{for all} \quad y \in V(M^+); \tag{2.10}$$

$$V(M^+) := \{y : y_+^{[r-1]} \in AC_{\text{loc}}(I), \quad r = 1, 2, \ldots, n\}. \tag{2.11}$$

Note that, $(A^+)^+ = A$ and so $(M^+)^+ = M$. We refer to [2], [3], [5], [6] and [8] for a full account of the above and subsequent results on quasi-differential expressions. For $u \in V(M)$, $v \in V(M^+)$ and $\alpha, \beta \in I$, we have Green's formula,

$$\int_\alpha^\beta \{\overline{v} M[u] - u\overline{M^+[v]}\}dx = [u, v](\beta) - [u, v](\alpha), \tag{2.12}$$

where,

$$\begin{aligned} [u, v](x) &= i^n \Big(\sum_{r=0}^{n-1} (-1)^{n+r+1} u^{[r]}(x) \overline{v}_+^{[n-r-1]}(x)\Big) \\ &= (-i)^n \Big(u, \ldots, u^{[n-1]}\Big) J_{nxn} \begin{bmatrix} \overline{v}_+ \\ \vdots \\ \overline{v}_+^{[n-1]} \end{bmatrix} (x); \end{aligned} \tag{2.13}$$

see [8], [9] and [17, Corollary 1].

Let the interval I have end-point a, b ($-\infty \leq a < b \leq \infty$), and let w be a function which satisfies:

$$w > 0 \quad \text{a.e. on } I, \quad w \in L^1_{\text{loc}}(I). \tag{2.14}$$

The equation,

$$M[y] - \lambda w y = 0 \quad (\lambda \in \mathbb{C}) \tag{2.15}$$

on I is said to be *regular* at the left end-point a, if a is finite and for all $X \in (a, b)$,

$$a \in \mathbb{R}, w, a_{rs} \in L^1(a, X), (r = 1, 2, \dots, n). \tag{2.16}$$

Otherwise (2.15) is said to be *singular* at a. Similarly we define the terms regular and singular at b. If (2.15) is regular on (a,b), then we have

$$a, b \in \mathbb{R} \quad w, a_{rs} \in L^1(a, b), \quad (r, s = 1, 2, \dots, n). \tag{2.17}$$

Note that, in view of (2.8), an end-point of I is regular for (2.15), if and only if it is regular for the equation,

$$M^+[y] - \overline{\lambda} w y = 0 \quad (\lambda \in \mathbb{C}) \tag{2.18}$$

Let $H = L^2_w(a, b)$, denote the usual-weighted L^2 -space with inner-product,

$$(f, g) = \int_I f(x)\overline{g(x)} w(x) dx, \tag{2.19}$$

and norm $||f|| := (f, f)^{1/2}$; this is a Hilbert space on identifying functions which differ only on null sets. Set,

$$\left.\begin{array}{l} D(M) := \{u : u \in V(M), u \text{ and } w^{-1}M[u] \in L^2_w(a, b)\}, \\ D(M^+) := \{v : v \in V(M^+), v \text{ and } w^{-1}M^+[v] \in L^2_w(a, b)\}. \end{array}\right\} \tag{2.20}$$

Note that, at a regular end-point a, say, $u^{[r-1]}(a)(v_+^{[r-1]}(a))$ is defined for all $u \in V(M)(v \in V(M^+)), r = 1, 2, \dots, n$. The *manifolds* $D(M)$ and $D(M^+)$ of $L^2_w(a, b)$ are the domains of the so-called *maximal operators* $T(M)$ and $T(M^+)$ respectively, defined by:

$$T(M)u := w^{-1}M[u](u \in D(M)) \quad \text{and} \quad T(M^+)v = w^{-1}M^+[v](v \in V(M^+).$$

For the regular problem the *minimal operators* $T_0(M)$ and $T_0(M^+)$ are the restrictions of $w^{-1}M[.]$ and $w^{-1}M^+[.]$ to the subspaces,

$$\left.\begin{array}{l} D_0(M) := \{u : u \in D(M), u^{[r-1]}(a) = u^{[r-1]}(b) = 0, r = 1, 2, \dots, n\} \\ D_0(M^+) := \{v : v \in D(M^+), v_+^{[r-1]}(a) = v_+^{[r-1]}(b) = 0, r = 1, 2, \dots, n\} \\ \dots \end{array}\right\} \tag{2.21}$$

respectively. The subspaces $D_0(M)$ and $D_0(M^+)$ are dense in $L^2_w(a,b)$ and $T_0(M)$, $T_0(M^+)$ are closed operators (see [17, Section 3]). In the singular problem we first introduce operators $T'_0(M)$, $T'_0(M^+)$, where $T'_0(M)$ is the restriction of $w^{-1}M[.]$ to

$$D'_0(M) := \{u : u \in D(M), supp(u) \subset (a,b)\}, \tag{2.22}$$

and with $T'_0(M^+)$ defined similarly. These operators are densely-defined and closable in $L^2_w(a,b)$ and we define the minimal operators $T_0(M)$, $T_0(M^+)$ to be their respective closures (cf [8] and [17, Section 5]). We denote the domains of $T_0(M)$ and $T_0(M^+)$ by $D_0(M)$ and $D_0(M^+)$ respectively. It can be shown that, if (2.15) is regular at a,

$$\left.\begin{array}{l} u \in D_0(M) \longrightarrow u^{[r-1]}(a) = 0 (r = 1, 2, \ldots, n), \\ v \in D_0(M^+) \longrightarrow v_+^{[r-1]}(a) = 0 (r = 1, 2, \ldots, n). \end{array}\right\} \tag{2.23}$$

Moreover, in both the regular and singular problems we have,

$$T_0^*(M) = T(M^+), T^*(M) = T_0(M^+), \tag{2.24}$$

see [17, Section 5] in the case when $M^+ = M$, and compare with the treatment in [2, Section III. 10. 3] in the general case. Note that $T_0(M)$ is a closed densely-defined operator in H.

In the case of two singular end-points, the problem on (a,b) is effectively reduced to the problems with one singular end-point on the intervals (a,c] and [c,b), where $c \in (a,b)$. We denote by $T(M;a)$, $T(M;b)$ the maximal operators with domains $D(M;a)$ and $D(M;b)$, and denote by $T_0(M;a)$ and $T_0(M;b)$ the closures of the operators $T'_0(M;a)$ and $T'_0(M;b)$ defined in (2.22), on the intervals (a,c] and [c,b) respectively (see [5], [11], [12] and [16]).

Let $\widetilde{T}'_0(M)$ be the orthogonal sum,

$$\widetilde{T}'_0(M) = T'_0(M;a) \oplus T'_0(M;b) \text{ in } L^2_w(a,b) = L^2_w(a,c) \oplus L^2_w(c,b),$$

$\widetilde{T}'_0(M)$ is densely-defined and closable in $L^2_w(a,b)$ and its closure is given by

$$\widetilde{T}'_0(M) = T_0(M;a) \oplus T_0(M;b).$$

Also,

$$\text{nul}[\widetilde{T}_0(M) - \lambda I] = \text{nul}[T_0(M;a) - \lambda I] + \text{nul}[T_0(M;b) - \lambda I],$$

$$\text{def}[\widetilde{T}_0(M) - \lambda I] = \text{def}[T_0(M;a) - \lambda I] + \text{def}[T_0(M;b) - \lambda I],$$

and $R[\widetilde{T}_0(M) - \lambda I]$ is closed if, and only if, $R[T_0(M;a) - \lambda I]$ and $R[T_0(M;b) - \lambda I]$ are both closed. These results imply in particular that,

$$\Pi[\widetilde{T}_0(M)] = \Pi[T_0(M;a)] \cap \Pi[T_0(M;b)],$$

Remark 2.1. If S^a is a regularly solvable extension of $T_0(M;a)$ and S^b is a regularly solvable extension of $T_0(M;b)$, then $S = S^a \oplus S^b$ is regularly solvable extension of $\widetilde{T}_0(M)$. We refer to [2, §III.10.4] for more details.

Next, we state the following results; the proof is similar to that in [2, Section III.10.4], [9] and [12].

Theorem 2.2. *$\widetilde{T}_0(M) \subset T_0(M), T(M) \subset T(M;a) \oplus T(M;b)$ and*

$$\dim\{D[T_0(M)]/D[\widetilde{T}_0(M)]\} = n.$$

If $\lambda \in \Pi[\widetilde{T}_0(M)] \cap \Delta_3[T_0(M) - \lambda I]$, then

$$\operatorname{ind}[T_0(M) - \lambda I] = n - \operatorname{def}[T_0(M;a) - \lambda I] - \operatorname{def}[T_0(M;b) - \lambda I],$$

and in particular, if $\lambda \in \Pi[T_0(M)]$,

$$\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M;a) - \lambda I] + \operatorname{def}[T_0(M;b) - \lambda I] - n. \tag{2.25}$$

Remark 2.3. It can be shown that,

$$\left.\begin{array}{l} D[\widetilde{T}_0(M)] = \{u : u \in D[T_0(M)] \quad \text{and } u^{[r-1]}(c) = 0, r = 1, 2, \ldots, n\} \\ D[\widetilde{T}_0(M^+)] = \{v : v \in D[T_0(^+M)] \quad \text{and } v_+^{[r-1]}(c) = 0, r = 1, 2, \ldots, n\} \\ \cdots \end{array}\right\}; \tag{2.26}$$

see [2, Section III.10.4].

3. Some Technicel Lemmas

Let $\phi_k(t, \lambda)$ for $k = 1, \ldots, n$ be the solutions of the homogeneous equation (2.15) satisfying

$$\phi_j^{[k-1]}(t_0, \lambda) = \delta_{jk}, j, k = 1, \ldots, n \text{ for fixed } t_0, a < t_0 < b.$$

Then $\phi_j^{[k-1]}(t_0, \lambda)$ is continuous in (t, λ) for $a < t < b$, $|\lambda| < \infty$, and for fixed t it is entire in λ. Let $\phi_k^+(t, \lambda)$ for $k = 1, \ldots, n$ be the solutions of the homogeneous equation (2.18) satistying

$$(\phi_k^+)^{[r]}(t_0, \lambda) = (-1)^{k+r}\delta_{k,n-r}, \quad \text{for fixed} \quad t_0 \in [a, b),$$
$$k = 1, \ldots, n; r = 1, \ldots, n-1.$$

Suppose $a < c < b$. According to Gilbert [7, Section 3] and Zettl in [17, Theorem 3], a solution of $M[u] - \lambda wu = wf$, $f \in L^1_w(a, b)$ satisfying $\phi^{[r]}(c) = 0, r = 0, \ldots, n-1$ is given by

$$\phi(t) = (1/(i^n)) \sum_{j,k=1}^{n} \xi^{jk} \phi_j(t, \lambda) \int_c^t \overline{\phi_k^+(s, \lambda)} f(s) w(s) ds, \tag{3.1}$$

where $\phi_k^+(t, \lambda)$ stands for the complex conjugate of $\phi_k(t, \lambda)$, and for each j, k, ξ^{jk} is a constant which is independent of t, λ (but does depend in general on t_0).

The variation of parameters formula for a general ordinary quasi-differential equation is given by the following lemma:

Lemma 3.1. *For f locally integrable, the sloution $\phi(t,\lambda)$ of the quasi-differential equation $M[y]-\lambda wy = wf$ satisfying:*

$$\phi^{[r]}(t_0,\lambda) = \alpha_{r+1}(\lambda) \quad \text{for all} \quad r = 0,1,\ldots,n-1, t_0 \in [a,b)$$

is given by,

$$\phi(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j(t,\lambda_0) + ((\lambda-\lambda_0)/(i^n)).$$
$$.(\sum_{j,k=1}^{n} \xi^{jk}\phi_j(t,\lambda_0)\int_a^t \overline{\phi_k^+(s,\lambda_0)}f(s)w(s)ds), \tag{3.2}$$

for some constants $\alpha_1(\lambda),\ldots,\alpha_n(\lambda) \in \mathbb{C}$.

Proof. See [4], [8], [12] and [17].

Lemma 3.1. contain the following lemma as a special case.

Lemma 3.2. *Suppose f is locally $L_w^1(a,b)$ funciton and $\phi(t,\lambda)$ is the solution of $M[y]-\lambda wy = wf$ satisfying:*

$$\phi^{[r]}(t_0,\lambda) = \alpha_{r+1}(\lambda) \quad \text{for all} \quad r = 0,1,\ldots,n-1, t_0 \in [a,b).$$

Then

$$\phi^{[r]}(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j^{[r]}(t,\lambda_0) + ((\lambda-\lambda_0)/(i^n)).$$
$$.(\sum_{j,k=1}^{n} \xi^{jk}\phi_j^{[r]}(t,\lambda)\int_a^t \overline{\phi_k^+(s,\lambda_0)}f(s)w(s)ds), \tag{3.3}$$

for $r = 0,1,\ldots,n-1$; see [18].

Lemma 3.3. [8,Proposition 3.24]. *Suppose that for some $\lambda_0 \in \mathbb{C}$ all solutions of*

$$M[\phi] - \lambda_0 w\phi = 0 \quad \text{and} \quad M^+[\phi^+] - \overline{\lambda}_0 w\phi^+ = 0$$

are in $L_w^2(a,b)$. Then, all solutions of

$$M[\phi] - \lambda w\phi = 0 \quad \text{and}\ M^+[\phi^+] - \overline{\lambda} w\phi^+ = 0$$

are in $L_w^2(a,b)$ for every complex number $\lambda \in \mathbb{C}$.

Lemma 3.4. *Suppose that for some complex number $\lambda_0 \in \mathbb{C}$ all solutions of the equations*

$$M[\phi] - \lambda_0 w\phi = 0 \quad and \quad M^+[\phi^+] - \overline{\lambda}_0 w\phi^+ = 0, \tag{3.4}$$

are in $L^2_w(a,b)$. Suppose $f \in L^2_w(a,b)$. Then all solutions of the equation $M[\phi] - \lambda w\phi = wf$ are in $L^2_w(a,b)$ for all $\lambda \in \mathbb{C}$.

Proof. Let $\{\phi_1(.,\lambda_0),\dots,\phi_n(.,\lambda_0)\}$ and $\{\phi_1^+(.,\lambda_0),\dots,\phi_n^+(.,\lambda_0)\}$ be two sets of linearly independent solutions of the equations in (3.4). Then for any sloution $\phi(t,\lambda)$ of $M[\phi] - \lambda w\phi = wf$ which may be written as follows $M[\phi] - \lambda_0 w\phi = (\lambda - \lambda_0)w\phi + wf$, it follows from (3.2) that,

$$\begin{aligned}\phi(t,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j(t,\lambda_0) + \frac{1}{i^n}\Big(\sum_{j,k=1}^{n} \xi^{jk}\phi_j(t,\lambda_0) \int_a^t \overline{\phi_k^+(s,\lambda_0)}.\\ .[(\lambda-\lambda_0)\phi(s,\lambda) + f(s)]w(s)ds\Big).\end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned}|\phi(t,\lambda)| \le \sum_{j=1}^{n} |\alpha_j(\lambda)||\phi_j(t,\lambda_0)| + \sum_{j,k=1}^{n} |\xi^{jk}||\phi_j(t,\lambda_0)| \int_a^t |\overline{\phi_k^+(s,\lambda_0)}|.\\ .(|\lambda-\lambda_0)||\phi(s,\lambda)| + |f(s)|)w(s)ds).\end{aligned} \tag{3.6}$$

Since $f \in L^2_w(a,b)$ and $\phi_k^+(t,\lambda_0) \in L^2_w(a,b)$ for some $\lambda_0 \in \mathbb{C}$, $k = 1,\dots,n$, then $\phi_k^+(t,\lambda_0)f \in L^1_w(a,b)$ for some $\lambda_0 \in \mathbb{C}$ and $k == 1,\dots,n$. Setting

$$c_j(\lambda) = \sum_{j,k=1}^{n} |\xi^{jk}| \int_a^b \overline{\phi_k^+(s,\lambda_0)}f(s)w(s)ds, \quad k = 1,\dots,n, \tag{3.7}$$

then,

$$\begin{aligned}|\phi(t,\lambda)| \le \sum_{j=1}^{n}\{|\alpha_j(\lambda)| + c_j(\lambda)\}|\phi_j(t,\lambda_0)| + |\lambda-\lambda_0| \sum_{j,k=1}^{n} |\xi^{jk}|.\\ \cdot(|\phi_j(t,\lambda_0)| \int_a^t |\overline{\phi_k^+(s,\lambda_0)}||\phi(s,\lambda)|)w(s)ds.\end{aligned} \tag{3.8}$$

On application of the Cauchy-Schwartz inequality to the integral in (3.8), we get

$$\begin{aligned}|\phi(t,\lambda) \le \sum_{j=1}^{n}(|\alpha_j(\lambda)| + c_j(\lambda))|\phi_j(t,\lambda_0)| + |\lambda-\lambda_0| \sum_{j,k=1}^{n} |\xi^{jk}|.\\ .|\phi_j(t,\lambda_0)|\Big(\int_a^t |\overline{\phi_k^+(s,\lambda_0)}|^2 w(s)ds\Big)^{\frac{1}{2}}\Big(\int_a^t |\phi(s,\lambda)|^2 w(s)ds\Big)^{\frac{1}{2}}.\end{aligned}$$

From the inequality,

$$(u+v)^2 \le 2(u^2+v^2), \tag{3.9}$$

it follows that,

$$|\phi(t,\lambda)|^2 \leq 4\sum_{j=1}^{n}(|\alpha_j(\lambda)|^2 + c_j^2(\lambda))|\phi_j(t,\lambda_0)|^2 + 4|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n} |\xi^{jk}|^2.$$
$$\cdot|\phi_j(t,\lambda_0)|^2 (\int_a^t \overline{|\phi_l^+(s,\lambda_0)|^2 w(s)ds})(\int_a^t |\phi(s,\lambda)|^2 w(s)ds).$$

By hypothesis there exist positive constants K_0 and K_1 such that,

$$||\phi_j(.,\lambda_0)||L_w^2(a,b) \leq K_0 \quad \text{and} \quad \phi_k^+(.,\lambda_0)||L_w^2(a,b) \leq K_1 \quad j,k=1,\ldots,n. \tag{3.10}$$

Hence,

$$|\phi(t,\lambda)|^2 \leq 4\sum_{j=1}^{n}(|\alpha_j(\lambda)|^2 + c_j^2(\lambda))|\phi_j(t,\lambda_0)|^2 +$$
$$+(4K_1^2|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n} |\xi^{jk}|^2 \Big[|\phi_j(t,\lambda_0)|^2 \int_a^t |\phi(s,\lambda)|^2 w(s)ds)\Big] \tag{3.11}$$

Integrating the inequality in (3.11) between a and t, we obtain

$$\int_a^t |\phi(s,\lambda)|^2 w(s)ds \leq K_2 +$$
$$+\Big(4K_1^2|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n} |\xi^{jk}|^2\Big) \int_a^t |\phi_j(s,\lambda_0)|^2 \Big[\int_a^s |\phi(\tau,\lambda)|^2 w(\tau)d\tau\Big] w(s)ds,$$

where

$$K_2 = 4K_0^2 \sum_{j=1}^{n} \Big(|\alpha_j(\lambda)|^2 + c_j^2(\lambda)\Big).$$

Now, on using Growall's inequality, it follows that,

$$\int_a^t |\phi(s,\lambda)|^2 w(s)ds \leq K_2 exp(4K_1^2|\lambda-\lambda_0|^2 \sum_{j,k=1}^{n} |\xi^{jk}|^2 \int_a^t |\phi_j(s,\lambda_0)|^2 w(s)ds).$$

Since $\phi_j(.,\lambda_0) \in L_w^2(a,b)$ for some $\lambda_0 \in \mathbb{C}$ and for $j=1,\ldots,n$, then, $\phi(t,\lambda) \in L_w^2(a,b)$.

Remark. Lemma 3.4. also holds if the function f is bounded on [a,b).

Lemma 3.5. *Let $f \in L_w^2(a,b)$. Suppose for some $\lambda_0 \in \mathbb{C}$ that :*

(i) *All solutions of $M^+[\phi] - \overline{\lambda}_0 w\phi = 0$ are in $L_w^2(a,b)$,*

(ii) *$\phi_j^{[r]}(t,\lambda_0)$, $j=1,\ldots,n$ are bounded on [a,b) for some $r=0,1\ldots,n-1$. Then $\phi^{[r]}(t,\lambda) \in L_w^2(a,b)$ for any solution $\phi(t,\lambda)$ of the equation $M[\phi] - \lambda w\phi = wf$ for all $\lambda \in \mathbb{C}$.*

Proof. On using Lemma 3.2, the proof is similar to that in lemma 3.4 and therefore omitted.

Lemma 3.6. *Suppose that for some complex number $\lambda_0 \in \mathbb{C}$ all solutions of $M^+[v] - \overline{\lambda}_0 wv = 0$ are in $L^2_w(a,c)$, where $a < b < b$. Suppose $f \in L^2_w(a,b)$, then,*

$$\int_a^t \overline{\phi_j^+(s,\lambda)} w(s) f(s) ds, \quad j = 1, \ldots, n,$$

is continuous in (t,λ) for $a < t < b$, for all λ.

Proof. The proof follows from [7,Lemma 3.2.] and Lemma 3.4.

Lemma 3.7. [10, Theorem 4.1.]. *The point spectra $\sigma_p[T_0(M)]$ and $\sigma_p[T_0(M^+)]$ of $T_0(M)$ and $T_0(M^+)$ are empty.*

Lemma 3.8. [2, Lemma IX.9.11] *If $I = [a,b]$, with $-\infty < a < b < \infty$, then for any $\lambda \in \mathbb{C}$, the operator $[T_0(M) - \lambda I]$ has closed range, zero nullity and deficiency n. Hence*

$$\sigma_{ek}[T_0(M)] = \begin{cases} \emptyset & (k = 1,2,3) \\ \mathbb{C} & (k = 4,5) \end{cases}$$

4. The Case of One Singular End-Point

We see from (2.24) that $T_0(M) \subset T(M) = [T_0(M^+)]^*$ and hence $T_0(M)$ and $T_0(M^+)$ form an adjoint pair of closed, densely-defined operators in $L^2_w(a,b)$. By [3, Corollary III.3.2], $\text{def}[T_0(M) - \lambda I] + \text{def}[T_0(M^+) - \overline{\lambda} I]$ is constant on the joint field of regularity $\Pi[T_0(M), T_0(M^+)]$ and we have shown in [4] that,

$$n \leq \text{def}[T_0(M), \lambda I] + \text{def}[T_0(M^+) - \overline{\lambda} I] \leq 2n \text{ for all } \lambda \in \Pi(T_0(M), T_0(M^+)].$$

For $\Pi(T_0(M), T_0(M^+)] \neq \emptyset$ the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ are characterized by the following theorem which is proved for the general case in [4] and [8]; see also [2] and [3, Theorem 10.5]. We shall use the notation

$$[u,v](b) = \lim_{x \to b^-} [u,v](x), \quad u \in D(M) \text{ and } v \in D(M^+),$$

if b is a singular end-point of I, and similarly for $[u,v](a)$ if a is singular. Note that it follows from (2.12) that these limits exists for $u \in D(M)$ and $v \in D(M^+)$ since then $\overline{v} M[u]$ and $u \overline{M^+[v]}$ are both integrable by the Cauchy-Schwartz inequality.

Theorem 4.1. *Let $T_0(M)$ and $T_0(M^+)$ be compatible and suppose that $\text{def}(T_0(M) - \lambda I) = \text{def}(T_0(M^+) - \overline{\lambda} I) = n$ for all $\lambda \in \Pi[T_o(M), T_0(M^+)]$. Then every closed operator S which is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ is the restriction of $T(M)$ to the set of function $u \in D(M)$ which satisfy linearly independent boundary conditions*

$$[u, \phi_j](b) - [u, \phi_j](a) = 0, \quad (j = 1, 2, \ldots, n). \tag{4.1}$$

The set $\{\phi_1, \phi_2, \ldots, \phi_n\}$ is a basis for $\{D(S^*)/D_0(M^+)\}$ where $\dim\{D(S^*)/D_0(M^+)\} = \operatorname{def}(T_0(M^+) - \overline{\lambda} I)$, and S^* is the restriction of $T(M^+)$ to the set of functions $v \in D(M^+)$ which satisfy linearly independent boundary condtions

$$[\psi_j, v](b) - [\phi_j, v](a) = 0, \quad (j = 1, 2, \ldots, n). \tag{4.2}$$

The set $\{\psi_1, \psi_2, \ldots, \psi_n\}$ is a basis for $\{D(S)/D_0(M)\}$ where $\dim\{D(S)/D_0(M)\} = \operatorname{def}(T_0(M) - \lambda I)$ and

$$[\psi_j, \phi_k](b) - [\psi_j, \phi_k](a) = 0, \quad (j, k = 1, 2, \ldots, n). \tag{4.3}$$

Conversely, for arbitrary functions $\{\phi_1, \phi_2, \ldots, \phi_n\}$ $(\{\psi_1, \psi_2, \ldots, \psi_n\})$ in $D^+(D)$ which are linearly independent modulo $D_0^+(D_0)$, if $D_1(D_2)$ is the set of functions in $D(D^+)$ which satisfy (4.1) (4.2) and (4.3) is satisfied, then $S = T|D_1$ is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$, and $S^* = T(M^+)|D_2$.

S is self-adjoint (J-self-adjoint) if, and only if, $M = M^+ (M^+ = \overline{M})$ and $\psi_j = \phi_j (\psi_j = \overline{\phi}_j)$ for $j = 1, 2, \ldots, n$.

We shall now investigate in the case of one singular end-point that the resolvents of all well-posed extensions of the minimal operator $T_0(M)$, and we show that in the maximal case, i.e., when $\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = n$ for all $\lambda \in \Pi[T_0(M), T_0(M^+)]$, these resolvents are integral operators, in fact they are Hilbert-Schmidt integral operators by considering that the function f be in $L^2_w(a,b)$, i.e., is quadratically integrable over the interval $[a, b)$.

The following theorem is an extension of that proved in Akhiezer and Glazman [1, Vol. II] and Naimark [12, Vol. II] namely the case of self-adjoint extensions of the minimal operator and the function f has compact support interior to $[a, b)$, and also extends of that proved in [8, Theorem 3.27] with compact support of the function f.

Theorem 4.2. *Suppose for an operator $T_0(M)$ with one sigular end-point that $\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = n$ for all $\lambda \in \Pi[T_0(M), T_0(M^+)]$, and let S be an aribtrary closed operator which is a well-posed extension of the minimal operator $T_0(M)$ and $\lambda \in \rho(S)$, then the resolvents R_λ and R^*_λ of S and S^* respectively are Hilbert-Schmidt integral operators whose kernels are continuous functions on $[a, b) \times [a, b)$ and satisfy,*

$$K(x, t, \lambda) = \overline{K^+(t, x, \overline{\lambda})},$$

and

$$\int_a^b \int_a^b |K(x, t, \lambda)|^2 w(x) w(t)\, dx dt < \infty.$$

Remark. An example of a closed operator which is a well-posed with respect to a compatible adjoint pair is given by the Visik extension (see [2, Theorem III.3.3] and [15, Theorem 1]). Note that if S is a well-posed, then $T_0(M)$ and $T_0(M^+)$ are a compatible adjoint pair and S is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$.

Proof. Let

$$\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = n \text{ for all } \lambda \in \Pi[T_0(M), T_0(M^+)],$$

then we choose a fundamental system of solutions $\{\phi_1(t,\lambda), \ldots, \phi_n(t,\lambda)\}$, $\psi_1(t,\lambda), \ldots, \psi_n(t,\lambda)\}$ of the equations,

$$M[\phi_j] - \lambda\phi_j w = 0, M^+[\psi_j] - \overline{\lambda}\psi_j w = 0, \quad j = 1, 2, \ldots, n \text{ on } [a,b), \tag{4.4}$$

so that $\{\phi_1(t,\lambda), \ldots, \phi_n(t,\lambda)\}$ and $\{\psi_1(t,\lambda), \ldots, \psi_n(t,\lambda)\}$ belong to $L^2_w(a,b)$, i.e., they are quadratically integrable in the interval $[a,b)$.

Let $R_\lambda = (S - \lambda I)^{-1}$ be the resolvent of any well-posed extensions S of the minimal operator $T_0(M)$. For $f \in L^2_w(a,b)$, we put $\phi(t,\lambda) = R_\lambda f$, then

$$M[\phi] - \lambda w\phi = wf$$

and consequently has a solution $\phi(t,\lambda)$ in the form

$$\begin{aligned}\phi(t,\lambda) &= \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j(t_0,\lambda) \\ &\quad + ((\lambda - \lambda_0)/i^n)\Big[\sum_{j,k=1}^{n} \xi^{jk}\phi_j(t_0,\lambda)\int_a^b \overline{\phi_k^+(s,\lambda_0)} f(s)w(s)ds\Big],\end{aligned} \tag{4.5}$$

for some constants $\alpha_1(\lambda), \ldots, \alpha_n(\lambda) \in \mathbb{C}$ (see Lemma 3.1). Since, $f \in L^2_w(a,b)$ and $\phi_k^+(.,\lambda_0) \in L^2_w(a,b)$, $k = 1, \ldots, n$ for some $\lambda_0 \in \mathbb{C}$, then $\phi_k^+(.,\lambda_0)f \in L^1_w(a,b)$, $k = 1, \ldots, n$ for some $\lambda_0 \in \mathbb{C}$, and hence the integral in the right-hand side of (4.5) will be finite.

To determine the constants $c_j(\lambda)$, $j = 1, \ldots, n$, let $\psi_k^+(t,\lambda)$, $k = 1, \ldots, n$ be a basis for $\{D(S^*)/D_0(M^+)\}$, then because $\phi(t,\lambda) \in D(S) \subset \rho(S) \subset \Delta_4(S)$, we have from Theorem 4.1 that

$$[\phi, \psi_k^+](b) - [\phi, \psi_k^+](a) = 0, \quad (k = 1, \ldots, n) \quad \text{on } (a,b), \tag{4.6}$$

and hence from (4.5), (4.6) and using Lemma 3.2, we have

$$[\phi, \psi_k^+](b) = \sum_{j=1}^{n}\Big[\alpha_j(\lambda) + [(\lambda - \lambda_0)/i^n)]\sum_{j,k=1}^{n}\xi^{jk}\int_a^b \overline{\phi_k^+(s,\lambda_0)} f(s)w(s)ds\Big][\phi_j, \psi_k^+](b),$$

$$[\phi, \psi_k^+](a) = \sum_{j=1}^{n}\alpha_j(\lambda)[\phi_j, \psi_k^+](a), \quad k = 1, \ldots, n.$$

By substituting these expressions into the conditions (4.6), we get

$$\Big[\sum_{j=1}^{n}\alpha_j(\lambda) + [(\lambda - \lambda_0)/i^n]\sum_{j,k=1}^{n}\xi^{jk}\int_a^b \overline{\phi_k^+(s,\lambda)} f(s)w(s)ds\Big][\phi_j, \psi_k^+](b)$$

$$= \sum_{j=1}^{n} c_j(\lambda)[\phi_j, \psi_k^+](a). \quad \text{This implies the system}$$

$$\sum_{j=1}^{n} c_j(\lambda)\Big([\phi_j, \psi_k^+]\Big)_a^b$$
$$= -[(\lambda-\lambda_0)/i^n]\Big[\sum_{j,k=1}^{n} \xi^{jk}[\phi_j, \psi_k^+](b)\int_a^b \overline{\phi_k^+(s,\lambda)} f(s)w(s)ds\Big], \tag{4.7}$$

in the variables $\alpha_j(\lambda)$, $j = 1, 2, \ldots, n$. The determinent of this system does not vanish (see [8, Theorem 3.27] and [12]). If we solve the system (4.7), we obtain

$$\alpha_j(\lambda) = [(\lambda-\lambda_0)/i^n]\Big[\int_a^b h_j(s,\lambda)f(s)w(s)ds\Big], \quad j = 1, 2, \ldots, n,$$

where $h_j(t,\lambda)$ is a solution of the system

$$\sum_{j=1}^{n} h_j(s,\lambda)\Big[[\phi_j, \psi_k^+]\Big]_a^b = -\sum_{j,k=1}^{n} \xi^{jk}[\phi_j, \psi_k^+](b)\overline{\phi_k^+(s,\lambda_0)}. \tag{4.8}$$

Since, the determinent of the above system (4.8) does not vanish, and the functions $\phi_k^+(t,\lambda_0)$, $k = 1, \ldots, n$ are continuous in the interval $[a, b)$, then the functions $h_j(t,\lambda)$ are also continuous in this interval. By substituting in formula (4.5) for the expressions $\alpha_j(\lambda)$, $(j = 1, \ldots, n)$, we get

$$\begin{aligned} R_\lambda f &= \phi(t,\lambda) \\ &= [(\lambda-\lambda_0)/i^n]\Big[\sum_{j,k=1}^{n} \phi_j(t,\lambda_0)\int_a^t [\xi^{jk}\overline{\phi_k^+(s,\lambda_0)} + h_j(s,\lambda)]f(s)w(s)ds \\ &\quad + \sum_{j=1}^{n} \phi_j(t,\lambda_0)\int_t^b h_j(s,\lambda)f(s)w(s)ds\Big]. \end{aligned} \tag{4.9}$$

Now, we put

$$K(t,s,\lambda) = \begin{cases} [(\lambda-\lambda_0)/i^n]\Big[\sum_{j=1}^n \phi_j(t,\lambda_0)h_j(s,\lambda)\Big] & \text{for } t < s \\ [(\lambda-\lambda_0)/i^n]\Big[\sum_{j,k=1}^n \phi_j(t,\lambda_0)(\xi^{jk}\overline{\phi^+(s,\lambda_0)} + h_j(s,\lambda))\Big] & \text{for } t > s \end{cases} \tag{4.10}$$

Formula (4.9) then takes the form,

$$R_\lambda f(t) = \int_a^b K(t,s,\lambda)f(s)w(s)ds \text{ for all } t \in [a,b), \tag{4.11}$$

i.e., R_λ is an integral operator with the kernel $K(t,s,\lambda)$ operating on the functions $f \in L_w^2(a,b)$. Similarly, the solutions $\phi^+(t,\lambda)$ of the equation $M^+[\psi] - \overline{\lambda}w\psi = wg$ has the form,

$$\phi^+(s,\lambda) = \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j^+(s,\lambda_0)$$
$$+[(\overline{\lambda}-\overline{\lambda}_0)/i^n]\Big[\sum_{j,k=1}^{n} \zeta^{jk}\phi_j^+(s,\lambda_0)\int_a^s \overline{\phi_k(t,\lambda_0)}g(t)w(t)dt\Big], \tag{4.12}$$

where $\phi_k(t,\lambda_0)$ and $\phi_j^+(s,\lambda_0)$, $j,k=1,\ldots,n$ are solutions of the equations in (4.4). The argument as before leads to,

$$R_\lambda^* g = \int_a^b K^+(s,t,\overline{\lambda})g(t)w(t)dt \text{ for } g \in L_w^2(a,b), \tag{4.13}$$

i.e., is an integral operator with kernel $K^+(s,t,\overline{\lambda})$ operating on the functions $g \in L_w^2(a,b)$, where

$$K^+(s,t,\overline{\lambda}) = \begin{cases} [(\overline{\lambda}-\overline{\lambda}_0)/i^n]\Big[\sum_{j=1}^n \phi_j^+(s,\lambda_0)h_j^+(t,\lambda)\Big] & \text{for } s<t \\ [(\overline{\lambda}-\overline{\lambda}_0)/i^n]\Big[\sum_{j,k=1}^n \phi_j^+(s,\lambda_0)(\zeta^{jk}\overline{\phi^+(t,\lambda_0)}+h_j^+(s,\lambda))\Big] & \text{for } s>t \end{cases} \tag{4.14}$$

and $h_j^+(t,\lambda)$ is a solutions of the system,

$$\sum_{j=1}^{n} \overline{h_j^+(t,\lambda)}\Big[[\psi_k,\phi_j^+]\Big]_a^b = -\sum_{j,k=1}^{n} \zeta^{jk}[\psi_k,\phi_j^+](b)\phi_k(t,\lambda_0). \tag{4.15}$$

From definitions of R_λ and $^*_\lambda$, it follows that,

$$(R_\lambda f,g) = \int_a^b \Big\{\int_a^b K(t,s,\lambda)f(s)w(s)ds\Big\}\overline{g(t)}w(t)dt$$
$$= \int_a^b \Big\{\int_a^b K(t,s,\lambda)\overline{g(t)}w(t)\Big\}f(s)w(s)ds = (f,R_\lambda^* g) \tag{4.16}$$

for any continuous functions $f,g \in H$, and by construction (see (4.10) and (4.14)) $K(t,s,\lambda)$ and $K^+(s,t,\overline{\lambda})$ are continuous functions on $[a,b)\times[a,b)$, and (4.16) gives us

$$K(t,s,\lambda) = \overline{K^+(s,t,\overline{\lambda})} \quad \text{for all } t,s \in [a,b)\times[a,b). \tag{4.17}$$

Since $\phi_j(t,\lambda)$, $\phi_k^+(s,\lambda) \in L_w^2(a,b)$ for $(j,k=1,\ldots,n)$ and for fixed s, $K(t,s,\lambda)$ is a linear combination of $\phi_j(t,\lambda)$, while, for fixed t, $K^+(s,t,\overline{\lambda})$, is a linear combination of $\phi_k^+(s,\lambda)$. Then we have,

$$\int_a^b |K(t,s,\lambda)|^2 w(t)dt < \infty, \quad \int_a^b |K^+(s,t,\overline{\lambda})|^2 w(s)ds < \infty, \quad a<s,t<b$$

and (4.17) implies that,

$$\int_a^b |K(t,s,\lambda)|^2 w(s)ds = \int_a^b |K^+(s,t,\overline{\lambda})|^2 w(s)ds < \infty,$$
$$\int_a^b |K^+(s,t,\overline{\lambda})|^2 w(t)dt = \int_a^b |K(t,s,\lambda)|^2 w(t)dt < \infty.$$

Now, it is clear from (4.8) that the functions $h_j(t,\lambda)$, $(j=1,\dots,n)$ belong to $L^2_w(a,b)$, since $h_j(t,\lambda)$ is a linear combination of the functions $\phi_j^+(t,\lambda)$ which lie in $L^2_w(a,b)$ and hence $h_j(t,\lambda)$ belong to $L^2_w(a,b)$. Similarly $h_j^+(t,\lambda)$ belong to $L^2_w(a,b)$.

By the upper half of the formula (4.10) and (4.14), we have

$$\int_a^b w(t)dt \int_t^b |K(t,s,\lambda)|^2 w(s)ds < \infty,$$

for the *inner* integral exists and is a linear combination of products $\phi_j(t,\lambda)\phi_k^+(s,\lambda)$ $(j,k=1,\dots,n)$ and these products are integrable because each of the factors belongs to $L^2_w(a,b)$. Then by (4.17), and by the upper half of (4.14) ,

$$\int_a^b w(t)dt \int_a^t |K(t,s,\lambda)|^2 w(s)ds = \int_a^b w(t)dt \int_a^t |K^+(s,t,\overline{\lambda})|^2 w(s)ds < \infty.$$

Hence, we also have,

$$\int_a^b \int_a^b |K(t,s,\lambda)|^2 w(t)w(s)dtds < \infty.$$

and the theorem is completely proved for any well-posed extension.

Remark 4.3. It follows immediately from Theorem 4.2 that, if for an operator $T_0(M)$ with one singular end-point that, $\operatorname{def}[T_0(M)-\lambda I] = \operatorname{def}[T_0(M^+)-\overline{\lambda}I] = n$ for all $\lambda \in \Pi[T_0(M), T_0(M^+)]$, and S is well-posed with respect to $T_0(M)$ and $T_0(M^+)$ with $\lambda \in \rho(S)$, then $R_\lambda = (S-\lambda I)^{-1}$ is a Hilbert-Schmidt integral operator. Thus it is a completely continuous operator, and consequently its spectrum is discrete and consists of isolated eigenvalues having finite algebraic (so geometric) multiplicity with zero as the only possible point of accumulation. Hence, the spectra of all well-posed operators S are discrete, i.e.,

$$\sigma_{ek}(S) = \emptyset, \quad \text{for } k = 1,2,3,4,5. \tag{4.18}$$

We refer to [2, Theorem IX.3.1] for more details.

5. The Case of Two Singular End-Points

For the case of two singular end-points, we consider our interval to be $I=(a,b)$ and denote by $T_0(M)$ and $T(M)$ the minimal and maximal operators. We see from (2.24) that $T_0(M) \subset T(M) \subset [T_0(M^+)]^*$ and hence $T_0(M)$, $T_0(M^+)$ form an adjoint pair of closed densely-defined operators in $L^2_w(a,b)$.

Lemma 5.1. *For* $\lambda \in \Pi[T_0(M), T_0(M^+)]$, $\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I]$ *is constant and*

$$0 \leq \operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \leq 2n. \tag{5.1}$$

In the problem with one singular end-point,

$$n \leq \operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \leq 2n,$$

for all $\lambda \in \Pi[T_0(M), T_0(M^+)]$. *In the regular problem,*

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = 2n.$$

Proof. See [4] and [9, Lemma 3.1].
For $\lambda \in \pi[T_0(M), T_0(M^+)]$, we define r, s and m as follows:

$$\left.\begin{aligned} r = r(\lambda) :&= \operatorname{def}[T_0(M) - \lambda I] \\ &= \operatorname{def}[T_0(M;a) - \lambda I] + \operatorname{def}[T_0(M;b) - \lambda I] - n \\ &= r_1 + r_2 - n \\ s = s(\lambda) :&= \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \\ &= \operatorname{def}[T_0(M^+;a) - \overline{\lambda} I] + \operatorname{def}[T_0(M^+;b) - \overline{\lambda} I] - n \\ &= s_1 + s_2 - n, \\ \text{and} \qquad & \\ m :&= r + s. \end{aligned}\right\} \tag{5.2}$$

Since,

$$r = r_1 + r_2 - n, \quad s = s_1 + s_2 - n,$$

then,

$$\left.\begin{aligned} m = r + s &= (r_1 + r_2 - n) + (s_1 + s_2 - n) \\ &= (r_1 + s_1) + (r_2 + s_2) - 2n \\ &= m_1 + m_2 - 2n. \end{aligned}\right\} \tag{5.3}$$

Also, since, $n \leq m_i \leq 2n$ $(i = 1, 2,)$, then by Lemma 5.1 we have that,

$$0 \leq m \leq 2n. \tag{5.4}$$

For $\Pi[T_0(M), T_0(M^+)] \neq \emptyset$, the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ are characterized by the following theorem which was proved in [9, Theorem 3.2].

Theorem 5.2. *For* $\lambda \in \Pi[T_0(M), T_0(M^+)]$, *let* r *and* m *be defined by* (5.2) *and let* $\psi_j (j = 1, 2, \ldots, r)$, $\phi_k (k = r+1, \ldots, m)$ *be arbitrary functions satisfying:*

(i) $\{\psi_j : j = 1, \ldots, r\} \subset D(M)$ *is linearly independent modulo* $D_0(M)$ *and* $\{\phi_k : k = r+1, \ldots, m\} \subset D(M^+)$ *is linearly independent modulo* $D_0(M^+)$;

(ii) $[\psi_j, \psi_k](a) - [\psi_j, \phi_k](b) = 0$ $(j = 1, \ldots, r; k = r+1, \ldots, m)$.

Then the set,

$$\{u : u \in D(M), [u, \phi_k](a) - [u, \phi_k](b) = 0, k = r+1, \ldots, m\} \tag{5.5}$$

is the domain of an operator S which is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and

$$\{v : v \in D(M^+), [\psi_j, v](a) - [\psi_j, v](b) = 0, \quad j = 1, 2, \ldots, r\} \tag{5.6}$$

is the domain of S^*; moreover $\lambda \in \Delta_3(S)$.

Conversely, if S is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and $\lambda \in \Pi[T_0(M), T_0(M^+)] \cap \dot{\Delta}_4(S)$, then with r and m defined by (5.2) there exist funcitons $\psi_j (j = 1, \ldots, r)$, $\phi_k (k = r+1, \ldots, m)$ which satisfy (i) and (ii) and are such that (5.5) and (5.6) are the domains of S and S^* respectively.

S is self-adjoint if and only if $M = M^+$, $r = s$ and $\phi_k = \psi_{k-r} (k = r+1, \ldots, m)$; S is J-self-adjoint if $M = JM^+J$ (J is a complex conjugate), $r = s$ and $\phi_k = \overline{\psi}_{k-r} (k = r+1, \ldots, m)$.

For an operator $T_0(M)$ with two singular end-points, Theorem 4.2 remains true in its entirely, that is all well-posed extensions of the minimal operator $T_0(M)$ in the maximal case, i.e., when $r_1 = r_2 = n$ and $s_1 = s_2 = n$ in (5.2) have resolvents which are Hilbert-Schmidt integral operators and consequently have a wholly discrete spectrum, and hence Remark 4.3 also remain valid. This implies as in Corollary 5.3 below that all the regularly solvable operators have standard essential spectra to be empty. We refer to [1], [2], [12] and [16] for more details.

Now, we prove Theorem 4.2 in the case of two singular end-points.

Proof. Let,

$$\text{def}[T_0(M) - \lambda I] = \text{def}[T_0(M^+) - \overline{\lambda} I] = n \text{ for all } \lambda \in \Pi[T_0(M), T_0(M^+)],$$

then we choose a fundamental system of solutions

$$\phi_j(t,\lambda) = \begin{cases} \phi_j^a(t,\lambda) \text{ on } (a,c] \\ \phi_j^b(t;\lambda) \text{ on } [c,b) \end{cases} \quad \text{and} \quad \psi_j(t,\lambda) = \begin{cases} \psi_j^a(t,\lambda) \text{ on } (a,c] \\ \psi_j^b(t,\lambda) \text{ on } [c,b) \end{cases}$$

of the equations

$$M[\phi_j] - \lambda w \phi_j = 0, M^+[\psi_j] - \overline{\lambda} w \psi_j = 0, \quad j = 1, \ldots, n \text{ on } (a,b). \tag{5.7}$$

so that $\{\phi_1(t,\lambda), \ldots, \phi_n(t,\lambda)\}$ and $\{\psi_1(t,\lambda), \ldots, \psi_n(t,\lambda)\}$ belong to $L^2_w(a,b)$, i.e., they are quadratically integrable in the interval (a,b).

Let $R_\lambda = (S - \lambda I)^{-1}$ be the resolvent of any well-posed extension $S = S^a \oplus S^b$ of the minimal operator $T_0(M)$. For $f \in L^2_w(a,c) \oplus L^2_w(c,b)$, we put $\phi(t,\lambda) = R_\lambda f(t)$, then $M(\phi) - \lambda w\phi = wf$ and hence as in (4.5) we have,

$$\begin{aligned} R_\lambda f(t) &= \sum_{j=1}^{n} \alpha_j(\lambda)\phi_j(t_0,\lambda) \\ &+[(\lambda-\lambda_0)/i^n]\Big[\sum_{j,k=1}^{n} \xi^{jk}\phi_j(t_0,\lambda)\int_a^t \overline{\phi_k^+(s,\lambda_0)}f(s)w(s)ds\Big], \end{aligned} \tag{5.8}$$

for some constants $\alpha_1(\lambda), \ldots, \alpha_n(\lambda) \in \mathbb{C}$, where

$$\phi(t,\lambda) = \begin{cases} \phi^a(t,\lambda) \text{ on } (a,c] \\ \phi^b(t,\lambda) \text{ on } [c,b). \end{cases}$$

By proceeding as in Theorem 4.2, we find that

$$\alpha_j(\lambda) = [(\lambda-\lambda_0)/i^n]\Big[\int_a^b h_j(s,\lambda)f(s)w(s)ds\Big], \quad j = 1,\ldots,n,$$

where $h_j(t,\lambda)$ are continuous functions on the interval (a,b),

$$h_j(t,\lambda) = \begin{cases} h_j^a(t,\lambda) \text{ on } (a,c] \\ h_j^b(t,\lambda) \text{ on } [c,b), \quad j = 1,\ldots,n. \end{cases}$$

By substituting in (5.8) for the constants $\alpha_j(\lambda)$, $j = 1,\ldots,n$, we get,

$$R_\lambda f = \int_a^b K(t,s,\lambda)f(s)w(s)ds$$

where,

$$K(t,s,\lambda) = \begin{cases} K^a(t,s,\lambda) \quad \text{on } (a,c] \\ K^b(t,s,\lambda) \quad \text{on } [c,b). \end{cases}$$

and $K^{(\cdot)}(t,s,\lambda)$ can be obtained as in (4.10). Similarly,

$$R_\lambda^* g = \int_a^b K^+(s,t,\overline{\lambda})g(s)w(s)ds.$$

Form (4.10) and (4.14) we have that,

$$\int_a^b |K(t,s,\lambda)|^2 w(t)dt < \infty, \quad \int_a^b |K^+(s,t,\overline{\lambda})|^2 w(s)ds < \infty, \quad a < s,t < b$$

and (4.17) implies that,

$$\begin{aligned} \int_a^b |K(t,s,\lambda)|^2 w(s)ds &= \int_a^b |K^+(s,t,\overline{\lambda})|^2 w(s)ds < \infty, \\ \int_a^b |K^+(s,t,\overline{\lambda})|^2 w(t)dt &= \int_a^b |K(t,s,\lambda)|^2 w(t)dt < \infty. \end{aligned}$$

The rest of the proof is entirely similar to the corresponding part of the proof of Theorem 4.2. We refer to [1], [8] and [12] for more details.

Corollary 5.3. *Let* $\lambda \in \Pi[T_0(M), T_0(M^+)]$ *with*

$$\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = n.$$

Then,

$$\sigma_{ek}(S) = \emptyset, \quad (k = 1, 2, 3), \tag{5.9}$$

of all regularly solvable extensions S *with respect to the compatible adjoint pair* $T_0(M)$ *and* $T_0(M^+)$.

Proof. Since,

$$\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = n \text{ for all } \lambda \in \Pi[T_0(M), T_0(M^+)]$$

Then we have from [2, Theorem III.3,5] that,

$$\begin{aligned} \dim\{D(S)/D_0(M)\} &= \operatorname{def}[T_0(M) - \lambda I] = n, \\ \dim\{D(S^*)/D_0(M^+)\} &= \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = n. \end{aligned}$$

thus S is an n-dimensional extension of $T_0(M)$ and so by [2, Corollary IX.4.2],

$$\sigma_{ek}(S) = \sigma_{ek}[T_0(M)], \quad (k = 1, 2, 3), \tag{5.10}$$

From Lemmas 3.7 and 3.8 we get,

$$\sigma_{ek}[T_0(M)] = \emptyset, \quad (k = 1, 2, 3)$$

Hence, by (5.10)we have that

$$\sigma_{ek}(S) = \emptyset, \quad (k = 1, 2, 3).$$

Remark 5.4. If S is well-posed (say the Visik extension) we get from (4.18) and (5.10) that

$$\sigma_{ek}[T_0(M)] = \emptyset, \quad (k = 1, 2, 3).$$

On applying (5.10) again to any regularly solvable operators S under consideration, hence (5.9).

Corollary 5.5. *If for some* $\lambda_0 \in \mathbb{C}$, *there are* n *linearly independent solutions of* $M[u] - \lambda_0 wu = 0$ *and* $M^+[v] - \overline{\lambda}_0 wv = 0$ *in* $L^2_w(a, b)$, *then* $\lambda_0 \in \Pi[T_0(M), T_0(M^+)]$, *and hence* $\Pi[T_0(M), T_0(M^+)] = \mathbb{C}$ *and* $\sigma_{ek}[T_0(M), T_0(M^+)] = \emptyset$, $k = 1, 2, 3$, *where* $\sigma_{ek}[T_0(M), T_0(M^+)]$ *is the joint essential spectra of* $T_0(M)$ *and* $T_0(M^+)$ *defined as* $\Pi[T_0(M), T_0(M^+)]$.

Proof. Since all solutions of $M[u] - \lambda_0 wu = 0$ and $M^+[v] - \overline{\lambda}_0 wv = 0$ are in $L^2_w(a,b)$ for some $\lambda_0 \in \mathbb{C}$, then

$$\operatorname{def}[T_0(M) - \lambda_0 I] + \operatorname{def}[T_0(M^+) - \overline{\lambda}_0 I] = 2n \text{ for some } \lambda_0 \in \Pi[T_0(M), T_0(M^+)].$$

From Lemma 3.7, we have that $T_0(M)$ has no eigenvalues and so $[T_0(M) - \lambda_0 I]^{-1}$ exists and its domain $R[T_0(M) - \lambda_0 I]$ is a closed subspace of $L^2_w(a,b)$. Hence, since $T_0(M)$ is a closed operator, then $[T_0(M) - \lambda_0 I]^{-1}$ is bounded and hence $\Pi[T_0(M) = \mathbb{C}$. Similarly $\Pi[T_0(M^+)] = \mathbb{C}$. Therefore $\Pi[T_0(M), T_0(M^+)] = \mathbb{C}$ and hence,

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = 2n \text{ for all } \lambda \in \Pi[T_0(M), T_0(M^+)].$$

Form Corollary 5.3 we have for any regularly solvable extension S of $T_0(M)$ that $\sigma_{ek}(S) = \emptyset$ and by (5.10) we get $\sigma_{ek}[T_0(M)] = \emptyset$, $k = 1,2,3$. Similarly $\sigma_{ek}[T_0(M^+)] = \emptyset$. Hence, $\sigma_{ek}[T_0(M), T_0(M^+)] = \emptyset$, $k = 1,2,3$.

Remark 5.4. If there are n linearly independent solutions of the equations $M[u] - \lambda wu = 0$ and $M^+[v] - \overline{\lambda} wv = 0$ in $L^2_w(a,b)$ for some $\lambda_0 \in \mathbb{C}$, then the complex plane can be divided into two disjoint sets:

$$\mathbb{C} = \Pi[T_0(M), T_0(M^+)] \cup \sigma_{ek}[T_0(M), T_0(M^+)], \quad k = 1,2,3.$$

We refer to [13] and [14] for more details.

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