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GROUPS WITH SMALL CONJUGACY CLASSES

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Abstract. A group satisfies property $(*)$ iff every conjugacy class has size not greater than 2. This paper proves properties of this type of group and conclude that it is a central product of an abelian group with 2-groups that are "almost" extra special.

I. Introduction

All groups considered are finite groups. All characters are over the complex number field.

Let us say that a group satisfies property $(*)$ iff every conjugacy class of G has size not greater than 2. This is obviously equivalent to the condition that $\forall x \in G$, $[G : C_G(x)] \leq 2$. This paper is to characterize all groups with property $(*)$.

In Section II, the problem of characterizing all groups with property $(*)$ is reduced to characterizing all such 2-groups, and some examples are given.

In Section III, we show the strong similarity between 2-groups satisfying property $(*)$ and the extra special 2-groups. Our conclusion is:

A group satisfying property $(*)$ is the direct product of an abelian group with an "almost" extra special 2-group.

II.

We reduce the problem of characterizing all groups with property $(*)$ to characterizing all such 2-groups. Two lemmas are needed.

Lemma 1. *If G satisfies property $(*)$, then*

(1) *All subgroups of G also satisfy property $(*)$*
(2) *All Sylow subgroups of odd order of G are abelian*

Proof. Let $H \leq G$, $\forall x \in H$, $[G : C_G(x)] \leq 2$ as G satisfies property $(*)$. If $H \subseteq C_G(x)$, then $[H : C_H(x)$ is clearly 1. If $H \not\subseteq C_G(x$ then $HC_G(x) = G$ and clearly in this case $[H : C_H(x)] \leq [G : C_G(x)] \leq 2$. This proves (1).

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Sylow subgroup P of G satisfies property $(*)$ by (1). Thus $\forall x \in P$, $[P : C_P(x)] \leq 2$. If P has odd order, the only possibility is $[P : C_P(x)] = 1$.This proves (2)

Lemma 2. *If G satisfies property $(*)$, then G is nilpotent of class two.*

Proof. $\forall x \in G$, $[G : C_G(x)] \leq 2$ implies that $G' \subseteq Z(G)$. Thus G is nilpotent of class 2.

Lemma 2 asserts that G is a direct product of all its Sylow subgroups. By Lemma 1, all Sylow subgroups of odd orders are abelian. So we conclude that G is a direct product of an abelian group with a 2-group satisfying property $(*)$. Thus, this reduces the problem to characterizing 2-group with property $(*)$.

Theorem 1. *Let G be a non-abelian 2-group. G satisfies property $(*)$ iff $|G'| = 2$.*

Proof. Assume one conjugacy class of G has size greater than 2. Let x, y, z be its distinct elements. Then there exists $g_1, g_2 \in G$ such that $y = g_1^{-1}xg_1$, $z = g_2^{-1}xg_2$. These give rise to two distinct nonidentity elements of G', namely, $yx^{-1} = g_1^{-1}xg_1x^{-1}$ and $zx^{-1} = g_2^{-1}xg_2x^{-1}$. We have prove that $|G'| = 2$ implies G satisfies property $(*)$.

Conversely, assume G satisfies property $(*)$. Let $|G| = 2^n$, $|G'| = 2^\alpha$, $|Z(G)| = 2^\beta$. Each element in $Z(G)$ forms a conjugacy class, and every two elements outside $Z(G)$ form a conjugacy class. Thus G has $\frac{|Z(G)+|G|}{2}$ conjugacy class, and $\frac{|Z(G)+|G|}{2}$ irreducible complex characters. Using the equality for character degree $\sum_{\chi \in Irr(G)} \chi(1)^2 = |G|$, and the fact that G has $[G : G']$ linear characters, we get

$$[G : G'] + \Big(\frac{|G| + |Z(G)|}{2} - [G : G']\Big)4 \leq |G|$$

$$\text{i.e.} \qquad 2^{n-\alpha} + \Big[\frac{2^n + 2^\beta}{2} - 2^{n-\alpha}\Big]4 \leq 2^n$$

This forces $\alpha = 1$, and so $|G'| = 2$.

By Theorem 1, all extra special 2-groups satisfy property $(*)$, but the converse is not true, since any abelian group obviously satisfies property $(*)$. Even if we assume G to be directly indecomposable, the converse is still not true. In fact, take $A =< x >$, the cyclic group of order 2^n, and extend it by a 2-cycle σ where $x^\sigma = x^{2^{n-1}+1}$. The group obtained has order 2^{n+1} with commutator subgoup of size 2. Thus it has property $(*)$, yet it is directly indecomposable and not extra special.

III.

Some basic character theory are needed here,

Theorem A. (Isaac) [3] *Theorem* 6.18

Let G be a group, K/L be an elemetary abelian chief factor of G. Let $\chi \in Irr(K)$ and suppose that χ is G-invariant. Then one of the following holds:

(1) $\chi_L \in Irr(L)$

(2) $\chi_L = e\theta$ *where* $\theta \in Irr(L)$ *and* $e^2 = [K : L]$
(3) $\chi_L = \sum_{i=1}^{t} \theta_i$ *where* $t = [K : L]$ *and the* $\theta_i \in Irr(L)$ *are distinct.*

Theroem B. (Itô) [1] *Theorem* 6.15
Let A *be any abelian normal subgroup of* G. *Then* $\forall \chi \in Irr(G)$, $\chi(1) | [G : A]$.

Theorem C. [3] *Theorem* 6.11
Let $H \leq G$ *and* θ *be an irreducible character of* H *which is invariant in* G. *Let* $I_G(\theta)$ *be the inertial subgroup of* θ. *Then* $\exists \zeta \in Irr(I_G(\theta)$ *such that* $\zeta^G \in Irr(G)$ *and* $\zeta_{1_{G(\theta)}} = \theta$.

Lemma 3. *Let* A *be an abelian normal subgroup of* G *such that* G/A *is abelian. Then any nonlinear irreducible character of* G *is induced from some linear character of some normal subgroup* N *of* G *with* $A \leq N \leq G$.

Proof. Since G/A is abelian, $G' \leq A$.
Let $G = G_0 > G_1 > G_2 > \cdots > A$ be a chief series of G through A. Each chief factor G_i/G_{i+1} has prime order. Let χ be any nonlinear irreducible character of G. χ_A reduces as A is abelian, so we choose k such that χ_{G_K} is irreducible but $\chi_{G_{k+1}}$ reduces. Clearly χ_{G_K} is invariant in G. By Theorem A, $\chi_{G_{k+1}} = \sum_{i=1}^{P} \theta_i$ where $P = [G_k : G_{k+1}$ is a prime, and θ_i are distinct irreducible character of G_{k+1}. Thus the inertia group $T_1 = I_G(\theta_1$ is a proper subgroup of G. By Lemma C, $\exists \gamma \in Irr(T_1$ such that $\gamma^G = \chi$ and $\gamma_{G_{k+1}} = \theta_1$.
Note that $T_1 \triangleleft G$ as $T_1 \geq A \geq G'$, so if γ is linear, the proof is complete. If γ is nonlinear, then consider the situation where A is an abelian normal subgroup of T_1, and T_1/A is abelian. By induction on $|G|$, every nonlinear irreducible character of T_1 is induced from some linear character of some normal subgroup N of T_1 with $A \leq N \leq T_1$. In particular, $\exists \theta \in Irr(N)$, $\theta(1) = 1$ and $\theta^{T_1} = \gamma$. Thus $\theta^G = \gamma^G = \chi$. Note that $N \triangleleft G$ as $G' \leq N$. This complete the proof.

Corollary 4. *Let* G *be a metabelian group. Then any nonlinear irreducible character of* G *is induced from a linear character of some normal subgroup* N *of* G *with* $G' < N$.

Proof. Take $A = G'$ in Lemma 3.

Theorem 2. *Let* G *be a nonabelian 2-group with property* $(*)$. *Then*

(1) *All nonlinear irreducible characters of* G *have the same degree equals* $[G : Z(G)]^{\frac{1}{2}}$.
(2) *All maximal abelian normal subgroups of* G *have the same order. Fix any maximal abelian normal subgroup* A, *then any nonlinear irreducible character of* G *is induced from some linear character of* A.
(3) $\Phi(G) \leq Z(G)$, *where* $\Phi(G)$ *is the Frattini subgroup of* G.

Proof. By Theorem 1, $|G'| = 2$, so G is metabelian. Let χ be any nonlinear irreducible character of G. Cor 4 asserts that χ is induced from some linear character θ of some normal subgroup N with $G' \leq N \leq G$. By Clifford's Theorem, $\chi_N = k \sum_{i=1}^{s} \theta_i$, where θ_i are conjugate to $\theta_1 = \theta$. Thus Ker $\chi \geq N'$. As $|G'| = 2$, either $N' = G'$ or

$N' = \{e.N' = G'$ means Ker $\chi \geq G'$ contradicting the nonlinearity of χ. So $N' = \{e$ and N is abelian. Clearly N is maximal abelian or else θ^G reduces.

If $\chi_1, \chi_2 \in Irr(G)$ are nonlinear. By the above argument, there exists maximal abelian normal subgroup N_1, N_2 of G such that $\chi_1 = \theta_1^G$, $\chi_2 = \theta_2^G$ where $\theta_i \in Irr(N_i$ $i = 1, 2$. By Theorem B, $\chi_1(1)|[G : N_2] = \chi_2(1)$ and $\chi_2(1)|G : N_1] = \chi_1(1)$ proving $\chi_1(1) = \chi_2(1)$. This proved that any two nonlinear irreducible character of G have the same degree. Using the equality $\sum_{\chi \in Irr(G)} \chi(1)^2 = |G|$, we get $[G : G'] + (\frac{|G|+|Z(G)|}{2} - [G; G'])\chi(1)^2 = |G|$ and this forces $X(1)^2 = [G : Z(G)$. This proved (1). To prove (2), let A be any maximal abelian normal subgroup of G and χ any nonlinear irreducible character of G. By Lemma 3, $\exists N \lhd G$ such that $A \leq N \leq G$ and $\chi = \theta^G$ with θ a linear character of N. Note that $G' \leq A \leq N$. By the same argument as in the first part of the proof, we see that N is abelian and so $N = A$ as A is maximal abelian normal in G. This in particular shows that $[G : A] = \chi(1)$, or $|A| = |G|/\chi(1)$. This proves (2).

$\forall x \in G$, if $x^2 \notin Z(G)$, then $\exists y \in G$ such that $x^2yx^{-2}y^{-1} \neq e$. So $xyx^{-1}y^{-1} \neq e$. As $|G'| = 2$, we must have $x^2yx^{-2}y^{-1} = xyx^{-1}y^{-1}$ and this forces $xy = yx$, a contradiction. Thus $G^2 \leq Z(G)$ and $\Phi(G) = G'G^2 \subseteq Z(G)$.

Theorem 2 shows strong similarity between nonabelian 2-group with property $(*)$ and extra special 2-group.

A group G is a central product of its subgroups $N_1, \ldots, N_k$ if $G = N_1N_2 \ldots N_k$ and for each $x \in N_i$, $y \in N_j$, $i \neq j$, then $x^{-1}y^{-1}xy = e$. See [1] pg 179.

Theorem 3. *If G is a nonabelian 2-group satisfying property $(*)$, then G is a central product of 2-generator groups and an abelian group.*

Proof. Without loss of generality, we assume that G is directly indecomposable. Let $x \in G \backslash Z(G)$, so $[G : C_G(x)] = 2$. Choose $y \in G \backslash C_G(x)$, and let $A = \langle x, y \rangle$. Obviously, $[G : C_G(y)] = 2$ and hence $C_G(A)$ has index 4 in G. As $[A : Z(A)] \geq 4$, we have

$$|AC_G(A)| = \frac{|A||C_G(A)|}{|Z(A)|} \geq |G|$$

So $AC_G(A) = G$. We conclude that G is a central product of A and $C_G(A)$. Now $C_G(A)$ being a subgroup of G also satisfies property $(*)$. If it is abelian, the proof is complete. Otherwise, use induction on $|G|$ to conclude that $C_G(A)$ is a central product of 2-generator gropus with an abelian group. $C_G(A) = A_2A_3 \cdots A_sL$ where A_i are 2-generator groups and L is abelian. So $G = AA_2A_3 \cdots A_sL$ is a central product of 2-generator groups and an abelian group as required. This complete the proof.

References

[1] L. Dornhoff, Group Representation Theory, Part A. Marcel Dekker, New York, 1971.

[2] G. A. How, *Special Classes of Monomial Groups III*, Chinese Journal of Mathematics **12** (1984), 199-211.

[3] I. M. Isaacs, Character Theory of Finite Groups, Academic Press, New York, San Franscisco, London, 1976.

[4] I. M. Isaacs and D. Passman, *Groups with Representations of bounded degree*, Canad. J. Math. **16** (1964), 299-309.

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