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ON A CLASS OF FUNCTIONS OF COMPLEX ORDER

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Abstract. Denote by $\mathcal{R}(b)$, the class of normalized analytic functions f which satisfies $\operatorname{Re}(1+\frac{1}{b}(f'(z)-1))>0$, for $z\in D=\{z:|z|<1\}$ and b a non-zero complex number. In this paper, some results concerning functions belonging to this class are being considered.

1. Introduction

Denote by A the class of functions f which are normalized such that $f(0)=f'(0)-1=0$ and analytic in the unit disc $D=\{z:|z|<1\}$. Also denote by S, the subclass of A consisting of all univalent functions in D.

Let P be the class of functions $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$ which are analytic and also have a positive real part in D. A function f is said to be starlike if and only if, for $z\in D$, $\operatorname{Re}\frac{zf'(z)}{f(z)}>0$. The well-known class of starlike functions which we shall denote by S^* is a subclass of S and have been extensively studied, see [3], [4] and [14]. In [13], Nasr and Aouf intorduced the class consisting of functions which are starlike of complex order. Functions f belonging to this class, say $S^*(b)$ are those which satisfied the following condition:

$$\operatorname{Re}\Big(1+\frac{1}{b}\Big(\frac{zf'(z)}{f(z)}-1\Big)\Big)>0,\quad z\in D \text{ and } b\neq 0,\ b\in\mathbb{C}.$$

J. W. Alexander in [1] intorduced the class $\mathcal{R}$ consisting of normalized analytic functions whose derivative has a positive real part, and proved that for $z\in D$, if f belongs to $\mathcal{R}$ then f is univalent, i.e. $\mathcal{R}\subset S$. This class $\mathcal{R}$ has also been considered quite extensively by Macgregor (see [8]). We now introduce an extension of this class via the following definition.

Definition 1.1. A function $f\in A$ is said to belong to the class $\mathcal{R}(b)$, if and only if, for $z\in D$

$$\operatorname{Re}\Big(1+\frac{1}{b}(f'(z)-1)\Big)>0 \tag{1}$$

where b is a non-zero complex number.

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Note that $\mathcal{R}(1) \equiv \mathcal{R}$.

In this paper, the author gives some results concerning functions in $\mathcal{R}(b)$.

2. Preliminary Results

Lemma 2.1 [6]. *For any complex number μ and $p \in P$ given by $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$,*

$$|c_2 - \mu c_1^2| \le 2 \max\{1, |1 - 2\mu|\}.$$

Lemma 2.2 [9]. *Suppose that the function $\psi : \mathbb{C}^2 \times D \to \mathbb{C}$ satisfies the condition*

$$\operatorname{Re} \psi(ix, y, z) \le 0$$

for all $x, y \le -(1 + x^2)/2$ and all $z \in D$.

If the function $p(z)$ is analytic in D, with $p(0) = 1$ and if

$$\operatorname{Re} \psi(p(z), zp'(z), z) > 0 \quad \text{for } z \in D,$$

then Re $p(z) > 0$ in D.

3. Some Properties of $\mathcal{R}$

Theorem 3.1. *Let $f \in \mathcal{R}(b)$. If Re $b \ge |b|^2$ then f is univalent.*

Proof. First, we write $b = |b|e^{i\sigma}$. Now, since $\operatorname{Re} b \ge |b|^2$, it follows that $|b| \le 1$ and $|\sigma| < \frac{\pi}{2}$. Next, inequality (1) implies that

$$\operatorname{Re} \frac{f'(z)}{b} > \operatorname{Re}\left(\frac{1}{b}\right) - 1 \ge 0,$$

provided $\operatorname{Re} b \ge |b|^2$.

Thus, we have

$$\operatorname{Re}\, e^{-i\sigma} f'(z) > 0$$

and hence f is univalent.

Remark 3.1. The question of whether the converse of Theorem 1 is true or not remains open. On the other hand, one can construct a non-univalent function belonging to $\mathcal{R}(b)$ for which b is outside this disc.

Our next theorem looks at distortion results for $f \in \mathcal{R}(b)$.

Theorem 3.2. *Let $f \in \mathcal{R}(b)$. Then for $z = re^{i\theta} \in D$,*

(i) $2|b|(\log(1 + r) - r) + r \frac{\operatorname{Re} b}{|b|} \le |f(z)| \le (1 - 2|b|)r - 2|b| \log(1 - r)$,

(ii) $\frac{\text{Re}b}{|b|} - \frac{2r|b|}{1+r} \leq |f'(z)| \leq (1-2|b|) + \frac{2|b|}{1-r}$.

Proof. From (1), we write

$$f'(z) = b(p(z)-1)+1 \tag{2}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$. Thus for $z = re^{i\theta} \in D$,

$$\begin{aligned} |p(z)| - 1 &\leq \sum_{n=1}^{\infty} |c_n z^n| \\ &\leq 2\sum_{n=1}^{\infty} r^n \\ &= -1 + \frac{1+r}{1-r}. \end{aligned} \tag{3}$$

Here, we use the known result that $|c_n| \leq 2$, (2) and (3) then gives

$$\begin{aligned} |f(z)| &= \left|z + b\int_0^z (p(t)-1)dt\right| \\ &\leq r - 2|b|(r + \log(1-r)), \end{aligned}$$

which is the upper bound in (i).
To prove the lower bound in (i), write

$$\begin{aligned} \left|\frac{f(z)}{z}\right| &= \left|\frac{1}{z}\int_0^z f'(t)dt\right| \\ &= |b|\left|\frac{1}{z}\int_0^z \left(p(t) - 1 + \frac{1}{b}\right)dt\right| \\ &\geq |b|\text{Re}\left\{\frac{1}{z}\int_0^z \left(p(t) - 1 + \frac{1}{b}\right)dt\right\}. \end{aligned}$$

Next, write $t = \rho e^{i\theta}$, we have

$$\begin{aligned} |f(z)| &\geq r|b|\left\{\frac{1}{r}\int_0^r \text{Re}\left(p(\rho e^{i\theta}) - 1 + \frac{1}{b}\right)d\rho\right\} \\ &\geq |b|\left\{\int_0^r \left(\frac{1-\rho}{1+\rho}\right)d\rho - r + r\text{Re}\frac{1}{b}\right\} \\ &= -2r|b| + 2|b|\log(1+r) + \frac{r}{|b|}\text{Re } b. \end{aligned}$$

The upper bound in (ii) follows at once from (2) and (3).
Finally, for the lower bound of (ii), write

$$|f'(z)| = |b|\left|p(z) - 1 + \frac{1}{b}\right|$$

$$\geq |b|\text{Re}\Big(p(z) - 1 + \frac{1}{b}\Big)$$
$$\geq |b|\Big(\frac{1-r}{1+r} - 1 + \frac{\text{Re } b}{|b|^2}\Big)$$

which gives the result.

Thoerem 3.3. *Let $f \in \mathcal{R}(b)$. Then for $z = re^{i\theta} \in D$,*

(i) $\text{Re}f'(z) \geq 1 - \frac{2r}{1-r^2}(|b| - r \text{ Re } b)$,

(ii) $\text{Re}\frac{f(z)}{z} \geq 1 - 2\text{Re } b + \frac{\log(1=r}{r}(|b| + \text{Re } b) + \frac{\log(1-r)}{r}(|b| + \text{Re } b)$.

We note that for $b = 1$, (ii) reduces to a result proved by Hallenbeck [5].

Proof. in [12], Nasr and Aouf showed that if $p \in P$, then

$$\text{Re}[b(p(z) - 1) + 1] \geq 1 - \frac{2r}{1-r^2}(|b| - \text{Re } b).$$

hence, using (2), (i) follow trivially.
Next,

$$\text{Re}\frac{f(z)}{z} = \text{Re}\Big\{\frac{1}{z}\int_0^z [bp(t) + 1 - b]dt\Big\}.$$

Now write $t = \rho e^{i\theta}$, we have

$$\text{Re}\frac{f(z)}{z} \geq \frac{1}{r}\int_0^z \Big(1 - \frac{2|b|\rho}{1-\rho^2} + \frac{2\text{Re}b\rho^2}{1-\rho^2}\Big)d\rho$$
$$= 1 + \frac{|b|}{r}\log(1-r^2) + 2\frac{\text{Re}b}{r}\Big(\frac{\log(1+r)}{2} - \frac{\log(1-r)}{2} - r\Big),$$

which on simplication completes the proof.

4. Coefficient Results

We next consider coefficient estimates.

Theorem 4.1. *Let $f \in \mathcal{R}(b)$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then*

$$\text{(i) } |a_n| \leq \frac{2|b|}{n}, \quad \text{for } n \geq 2, \tag{4}$$

$$\text{and (ii) } |a_3 - \mu a_2^2| \leq \frac{2|b|}{3}\max\Big\{1, \Big|1 - \frac{3}{2}\mu b\Big|\Big\}, \tag{5}$$

where μ is any complex number. The inequalities are sharp.

Proof. Since $f \in \mathcal{R}(b)$, there exists $p \in P$ such that for $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ (2) holds. Thus

$$1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = 1 + \sum_{n=1}^{\infty} b c_n z^n. \tag{6}$$

Equating the coefficients in (6) gives

$$na_n = bc_{n-1}$$

and so (i) follows immediately, since $|c_{n-1}| \leq 2$ for $n \geq 2$. Finally using Lemma 2.1, it follows that

$$\begin{aligned} |a_3 - \mu a_2^2| &= \frac{|b|}{3}\left|c_2 - \frac{3\mu}{4}bc_1^2\right| \\ &\leq \frac{2|b|}{3}\max\left\{1, \left|1 - \frac{3}{2}\mu b\right|\right\}. \end{aligned}$$

(4) and (5) are sharp for $f_1(z) = z - 2b(z + \log(1-z))$ in the case $|2 - 3\mu b| > 2$. For $|2 - 3\mu b| \leq 2$, inequality (5) is sharp for f_2 given by

$$f_2'(z) = b\left\{\frac{1+z^2}{1-z^2} - 1\right\} + 1.$$

5. Integral Operators

In this final section, we look at integral operator which preserves the class $\mathcal{R}(b)$.

Definition 5.1. Suppose that $g \in A$ with $g(z) \neq 0$ for $z \in D$. for $z \in D$, define the integral operator

$$F(z) = \frac{1+c}{g(z)^c}\int_0^z f(t)[g(t)]^{c-1}g'(t)dt, \tag{7}$$

where $c > -1$.

Putting $g(z) \equiv z$ and $c = 1$ in (7) gives the operator first introduced by Libera in [7]. This was then followed by Bernardi [2] who generalised Libera's operator by introducing the constant c for $c \in \mathbb{N}$. Since then, many people have extended the results obtained for Libera and Bernardi integral operators. See for example [15] and [16]. Some authors considered other similar forms of integral operators such as that given in (7) for various subclasses of S. This include Mocanu [11], Miller et. al [10] and Selinger [17].

We now proceed to state our result.

Theorem 5.1. *Let $g \in S^*$ with $Q(z) = \frac{g(z)}{zg'(z)}$, for $z \in D$. If $f \in \mathcal{R}(b)$ and*

$$\begin{aligned} &\mathrm{Re}Q(z) \cdot \mathrm{Re}\left[Q(z) + \frac{2}{b}(b-1)(zQ'(z) + Q(z) - 1)\right] \\ &\geq \{Im(zQ'(z) + Q(z) + c\}^2, \end{aligned} \tag{8}$$

for $z \in D$, then the integral operator F given by (7) also belongs to $\mathcal{R}(b)$.

We note that, in the case $b = 1$, this result reduces to the one given by Selinger [17]. To prove this theorem, we require Lemma 2.2.

Proof. For F given by (7), it can easily be deduced that

$$zQ(z)F'(z) + cF(z) = (1+c)f(z). \tag{9}$$

Differentiating (9) again, gives

$$zQ(z)F''(z) + F'(z)[zQ'(z) + Q(z) + c] = (1+c)f'(z).$$

Now, since $f \in \mathcal{R}(b)$ and $c > -1$, thus

$$\text{Re}\Big\{zQ(z)h'(z) + (zQ'(z) + Q(z) + c)h(z) + \frac{1}{b}(1-b)(zQ'(z) + Q(z) - 1)\Big\} > 0, \tag{10}$$

where $h(z) = 1 + \frac{1}{b}(F'(z) - 1)$.
For convenience, we introduce functions B and D, and rewrite (10) as follows: $\text{Re}\{zQ(z)h'(z) + B(z)h(z) + D(z)\} > 0$.

We next show that for $g \in S^*$ with inequality (8) true, $h \in P$ and this in turn implies that $F \in \mathcal{R}(b)$. We do this by using Lemma 2.2. First, define

$$\psi(ix, y, z) = yQ(z) + ixB(z) + D(z).$$

In order to use Lemma 2.2, we need to verify that $\forall x \in \mathbb{R}$, $y \leq \frac{-(1+x^2)}{2}$ and $z \in D$, $\text{Re}\psi(ix, y, z) \leq 0$.

$$\begin{aligned}\text{Re}\psi(ix, y, z) &= y\text{Re}Q(z) - x\text{Im}B(z) + \text{Re}D(z)\\ &\leq \frac{-(1+z^2)}{z}\text{Re}Q(z) - x\text{Im}B(z) + \text{Re}D(z);\end{aligned}$$

for $y \leq \frac{-(1+x^2)}{2}$ and $g \in S^*$.
Therefore,

$$\text{Re}\psi(ix, y, z) \leq -x^2\frac{\text{Re}Q(z)}{2} - x\text{Im}B(z) + Re\Big(D(z) - \frac{Q(z)}{z}\Big).$$

Since, inequality (8) is true, i.e. Re $Q(z)\cdot\text{Re}[Q(z) - 2D(z)] \geq [\text{Im}B(z)]^2$, thus we conclude that Re $\psi(ix, y, z) \leq 0$. Furthermore, since (10) implies that $\text{Re}(h(z), zh'(z), z) > 0$, for $z \in D$, thus by Lemma 2.2, $h \in P$ and this completes our proof.

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