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A NOTE ON CERTAIN CLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. The object of the present paper is to derive distortion inequalities for fractional integral operator of functions in the class $K(n,\alpha,\beta)$ consisting of analytic and univalent functions with negative coefficients.

1. Introduction

Let $A(n)$ denote the class of functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k \quad (a_k\geq 0, n\in N=\{1,2,3,\cdots,\}) \tag{1.1}$$

that are analytic in the unit disk $U=\{z,|z|<1\}$. A function $f(z)\in A(n)$ is said to be in the class $K(n,\alpha,\beta)$ if and only if there exists a function $\varphi(z)\in A(n)$ such that

$$\left|\frac{\frac{zf'(z)}{\varphi(z)}-1}{\frac{zf'(z)}{\varphi(z)}+(1-2\alpha)}\right|<\beta \tag{1.2}$$

for some $\alpha(0\leq\alpha<1),\beta(0<\beta\leq 1)$ and for all $z\in U$. We note that $K(1,\alpha,\beta)\equiv K(\alpha,\beta)$, the class $K(\alpha,\beta)$ was studied by Gupta [1].

A function $f(z)\in A(n)$ is said to be in the class $C^*(n,\alpha,\beta)$ if and only if there exists a convex function $\varphi\in A(n)$ such that

$$\left|\frac{\frac{(zf'(z))'}{\varphi'(z)}-1}{\frac{(zf'(z))'}{\varphi'(z)}+(1-2\alpha)}\right|<\beta \tag{1.3}$$

for some $\alpha(0\leq\alpha<1),\beta(0<\beta\leq 1)$ and for all $z\in N$.

It follows immediately that $f(z)\in C^*(n,\alpha,\beta)$ if and only if $zf'\in K(n,\alpha,\beta)$.

The following coefficient theorem for the class $K(\alpha,\beta)$ will be required in our investigation.

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Lemma 1. [1]. If $f \in K(\alpha,\beta)$ then

$$\sum_{k=2}^{\infty}(1+\beta)ka_k-(1-\beta+2\alpha\beta)b_k \leq 2\beta(1-\alpha). \tag{1.4}$$

2. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastave, Saigo and Owa [5]

Definition. Let

$$F(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k(1)_k}z^k, \tag{2.1}$$

where

$$(\nu)_k=\frac{\Gamma(\nu+k)}{\Gamma(\nu)}\begin{cases}1, & k=0\\ \nu(\nu+1)\cdots(\nu+k-1), & k\in N=\{1,2,3,\cdots\}\end{cases} \tag{2.2}$$

For real number $\rho>0,\delta,\eta$, and $\epsilon>\max\{0,\delta-\eta\}-1$, the fractional integral operator $I_{0,z}^{\rho,\delta,\eta}$ is defined by

$$I_{0,z}^{\rho,\delta,\eta}f(z)=\frac{z^{-\rho-\delta}}{\Gamma(\rho)}\int_0^z(z-t)^{\rho-1}\Big(\rho,-\eta;\rho;1-\frac{t}{z}\Big)f(t)dt. \tag{2.3}$$

where $f(z)$ is a function analytic in a simply connected region of the z-plane containing the origin, satisfying

$$f(z)=O(|z|)^{\epsilon},\quad z\to 0,$$

and the multiplicity of $(z-t)^{\rho-1}=\exp((\rho-1)\log(z-t))$ where $\log(z-t)$ is supposed to be real when $z-t>0$.

Remark. For $\delta=-\rho$, we note that

$$I_{0,z}^{\rho,-\rho,-\eta}f(z)=D_z^{-\rho}f(z),$$

where $D_z^{-\rho}f(z)$ is the fractional integral of order ρ of $f(z)$ which was introduced by Owa ([2], [3]).

In order to prove our results for the fractional operator, we have to recall here the following lemma due to Srivastava, Saigo and Owa [6].

Lemma 2. If $\rho>0$ and $k>\delta-\eta-1$, then

$$I_{0,z}^{\rho,\delta,\eta}z^k=\frac{\Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\rho+\eta+1)}z^{k-\delta}. \tag{2.4}$$

With the aid of Lemma 2, we have

Theorem 1. *Let $\rho > 0, \delta < 2, \rho + \eta > -2, \delta - \eta < 2,$ and $\delta(\rho + \eta) \leq 3\rho$. If $f(z) \in K(n, \alpha, \beta)$, then*

$$|I_{0,z}^{\rho,\delta,\eta} f(z)| \geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}\left\{1 - \frac{(1+(1+2n)\beta - 2n\alpha\beta)(2-\delta+\eta)_n(2)_n|z|^n}{(1+\beta)(n+1)^2(2-\delta)_n(2+\rho+\eta)_n}\right\} \tag{2.5}$$

and

$$|I_{0,z}^{\rho,\delta,\eta} f(z)| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}\left\{1 + \frac{(1+(1+2n)\beta - 2n\alpha\beta)(2-\delta+\eta)_n(2)_n|z|^n}{(1+\beta)(n+1)^2(2-\delta)_n(2+\rho+\eta)_n}\right\} \tag{2.6}$$

for $z \in U_0$, where

$$U_0 = \begin{cases} U & (\delta \leq 1) \\ U - \{0\} & (\delta > 1) \end{cases}$$

The result is sharp.

Proof. By using Lemma 2, we have

$$I_{0,z}^{\rho,\delta,\eta} f(z) = \frac{\Gamma(2-\delta+\eta)}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)} z^{1-\delta} - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\delta+\eta+1)}{\Gamma(k-\delta+1)\Gamma(k+\rho+\eta+1)} a_k z^{k-\delta} \tag{2.7}$$

letting

$$\begin{aligned} H(z) &= \frac{\Gamma(2-\delta)\Gamma(2-\rho+\eta)}{\Gamma(2+\delta+\eta)} z^{\delta} I_{0,z}^{\rho,\delta,\eta} f(z) \\ &= z - \sum_{k=n+1}^{\infty} h(k) a_k z^k, \end{aligned} \tag{2.8}$$

where

$$h(k) = \frac{(2-\delta+\eta)_{k-1}(1)_k}{(2-\delta)_{k-1}(2+\rho+\eta)_{k-1}}, \quad (k \geq n+1). \tag{2.9}$$

We can see that $h(k)$ is non-increasing for integers $k \geq n+1$, and we have

$$0 < h(k) \leq h(n+1) = \frac{(2-\delta+\eta)_n(2)_n}{(2-\delta)_n(2+\rho+\eta)_n}. \tag{2.10}$$

Since $f(z) \in K(n, \alpha, \beta)$, lemma 1 implies that $\sum_{k=n+1}^{\infty} a_k \leq \frac{(1+(1+2n)\beta - 2n\alpha\beta)}{(1+\beta)(n+1)^2}$. Therefore, by using (2.10), we have

$$\begin{aligned} |H(z)| &= |z| - h(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\geq |z| - \frac{\{1+(1+2n)\beta - 2n\alpha\beta\}(2)_n(2-\delta+\eta)_n}{(1+\beta)(n+1)^2(2-\delta)_n(2+\rho+\eta)_n}|z|^{n+1} \end{aligned} \tag{2.11}$$

and

$$|H(z)| = |z| + h(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k$$
$$\geq |z| + \frac{\{1+(1+2n)\beta - 2n\alpha\beta\}(2-\delta+\eta)_n(2)_n|z|^{n+1}}{(1+\beta)(n+1)^2(2-\delta)_n(2+\rho+\eta)_n} \tag{2.12}$$

Sharpness follows if we take the function

$$f(z) = z - \frac{\{1+(1+2n)\beta - 2n\alpha\beta\}}{(1+\beta)(n+1)^2}|z|^{n+1}. \tag{2.13}$$

Similarly by applying Corollary 1 to the function $f(z)$ belonging to the class $C^*(n, \alpha, \beta)$, we can derive

Theorem 2. *Let $\rho > 0$, $\delta < 2$, $\rho + \eta > -2$, $\delta - \eta < 2$ and $\delta(\rho+\eta) \leq 3\rho$. If $f(z) \in C^*(n, \alpha, \beta)$, then*

$$\left|I_{0,z}^{\rho,\delta,\eta} f(z)\right| \geq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}$$
$$\left\{1 - \frac{(1+(1+2n)\beta - 2n\alpha\beta)(2-\delta+\eta)_n(2)_n|z|^n}{(1+\beta)(n+1)^3(2-\delta)_n(2+\rho+\eta)_n}\right\} \tag{2.14}$$

and

$$\left|I_{0,z}^{\rho,\delta,\eta} f(z)\right| \leq \frac{\Gamma(2-\delta+\eta)|z|^{1-\delta}}{\Gamma(2-\delta)\Gamma(2+\rho+\eta)}$$
$$\times \left\{1 + \frac{[1+(1+2n)\beta - 2n\alpha\beta](2-\delta+\eta)_n(2)_n|z|^n}{(1+\beta)(n+1)^3(2-\delta)_n(2+\rho+\eta)_n}\right\} \tag{2.15}$$

for $z \in U_0$ where U_0 is defined in Theorem 1

The equalities in (2.14) and (2.15) are attained by the function

$$f(z) = z - \frac{(1+(1+2n)\beta - 2n\alpha\beta)}{(1+\beta)(1+n)^3} z^n.$$

3. Convolution Product

Let $f_i(z)$, $(i = 1, 2)$ be defined by

$$f_i(z) = z - \sum_{k=n+1}^{\infty} a_{i,k} z^k \qquad (a_{i,k} \leq 0). \tag{3.1}$$

We denote by $f_1 * f_2(z)$ the convolution product of the functions $f_1(z)$ and $f_2(z)$ defined by

$$f_1 * f_2(z) = z - \sum_{k=n+1}^{\infty} a_{1,k} a_{2,k} z^k, \tag{3.2}$$

In order to show our results, we need the following

Theorem 3. *A function $f(z) \in A(n)$ is in the class $K(n, \alpha, \beta)$ only if*

$$\sum_{k=n+1}^{\infty} \{(1+\beta)|ka_k - b_k| + 2(1-\alpha)\beta b_k\} \leq 2\beta(1-\alpha). \tag{3.3}$$

Proof. Let $|z| = r < 1$. Noting

$$|zf'(z) - \varphi(z)| < \sum_{k=n+1}^{\infty} |ka_k - b_k| r$$

and

$$|zf'(z) + (1-2\alpha)\varphi(z)| > r\Big\{2(1-\alpha) - \sum_{k=n+1}^{\infty} |ka_k - b_k| + 2(1-\alpha)b_k\Big\}.$$

We see that

$$\begin{aligned} &|zf'(z) - \varphi(z)| - \beta|zf'(z) + (1-2\alpha)\varphi(z)| \\ &\quad < r\left[\sum_{k=n+1}^{\infty} (|ka_k - b_k|(1+\beta) + 2(1-\alpha)\beta b_k) - 2(1-\alpha)\beta\right]. \end{aligned} \tag{3.4}$$

The right hand side of (3.4) is non-positive by (3.3) so that $f(z) \in K(n, \alpha, \beta)$.

Corollary 1. *Let the function $f(z)$ defined by (1.1) be analytic in U. Then $f(z)$ is in $C^*(n, \alpha, \beta,)$ if and only if*

$$\sum_{k=n+1}^{\infty} [(1+\beta)k|ka_k - b_k| + 2(1-\alpha)\beta k b_k] \leq 2(1-\alpha)\beta. \tag{3.5}$$

Theorem 4. *Let the functions $f_i(z)$, $(i = 1, 2)$ be in the class $K(n, \alpha, \beta)$. Then $f_1 * f_2$ belongs to the class $K(n, \gamma, \beta)$, where*

$$\gamma = \frac{[1 + (1+2n)\beta][(1+\beta)n + 2(1+2\alpha\beta)] - 4n\alpha^2\beta^2}{2\beta(1+\beta)(1+n)^2}. \tag{3.6}$$

The result if sharp.

Proof. Employing the technique used earlier by Schild and Silverman [4], we need to find the largest γ such that

$$\sum_{k=n+1}^{\infty} \frac{(1+\beta)(1+n)k}{1+(1+2n)\beta-2n\gamma\beta} a_{1,k}a_{2,k} \leq 1 \tag{3.7}$$

It follows from (1.1) and the Cauchy-Schwarz inequality that

$$\sum_{k=n+1}^{\infty} \frac{(1+\beta)(1+n)k}{1+(1+2n)\beta-2n\alpha\beta} \sqrt{a_{1,k}a_{2,k}} \leq 1 \tag{3.8}$$

Thus we need to find the largest γ such that

$$\frac{(1+\beta)k(n+1)}{1+(1+2n)\beta-2n\gamma\beta} a_{1,k}a_{2,k} \leq \frac{(1+\beta)k(n+1)}{1+(1+2n)\beta-2n\alpha\beta} \sqrt{a_{1,k}a_{2,k}}, \quad (k \geq n+1). \tag{3.9}$$

Or equaivlently, that

$$\sqrt{a_{1,k}a_{2,k}} \leq \frac{1+(1+2n)\beta-2n\gamma\beta}{1+(1+2n)\beta-2n\alpha\beta}. \tag{3.10}$$

By virtue of (3.8) it is sufficient to find the largest γsuth that

$$\sqrt{a_{1,k}a_{2,k}} \leq \frac{1+(1+2n)\beta-2n\alpha\beta}{(1+\beta)k(n+1)}, \qquad (k \geq n+1). \tag{3.11}$$

or, equivalently, that

$$\gamma \leq \frac{k(1+n)(1+\beta)[1+(1+2n)\beta]-[1+(1+2n)\beta-2n\alpha\beta]^2}{2n\beta k(1+n)(1+\beta)}, \quad (k \geq n+1). \tag{3.12}$$

Since

$$\phi(k) = \frac{k(1+n)(1+\beta)[1+(1+2n)\beta]-[1+(1+2n)\beta-2n\alpha\beta]^2}{2n\beta k(1+n)(1+\beta)} \tag{3.13}$$

is an increasing function of k. Letting $k = n+1$ in (3.12) we obtain

$$\gamma \leq \phi(n+1) = \frac{[1+(1+2n)\beta][(1+\beta)n+2(1+2\alpha\beta)]-4n\alpha^2\beta^2}{2\beta(1+\beta)(1+n)^2} \tag{3.14}$$

which completes the proof of the theorem.

Finally, by taking the function given by

$$f_i(z) = z - \frac{1+(1+2n)\beta-2n\alpha\beta}{(1+n)^2(1+\beta)} z^{n+1}, \quad (i = 1, 2), \tag{3.15}$$

we can see the result is sharp.

Similarly, we have

Theorem 5. *Let the function* $f_i(z)$, $(i = 1,2)$ *defined by* (3.1) *be in the class* $C^*(n,\alpha,\beta)$. *Then* $f_1 * f_2(z)$ *belongs to the class* $C^*(n,\gamma,\beta)$, *where*

$$\gamma = \frac{(1+\beta)(1+n)^3[1+(1+2n)\beta] - [1+(1+2n)\beta - 2n\alpha\beta]^2}{2n\beta(1+\beta)(1+n)^3}. \tag{3.16}$$

The result is sharp for the functions given by

$$f_i(z) = z - \frac{1+(1+2n)\beta - 2n\alpha\beta}{(n+1)^3(1+\beta)} z^{n+1}, \quad (i=1,2). \tag{3.17}$$

Theorem 6. *Let the functions* $f_i(z)$, $(i = 1,2,\ldots,m)$ *defined by* (3.1) *be in the class* $K(n,\alpha,\beta)$. *Then the function*

$$h(z) = z - \sum_{k=n+1}^{\infty} (a_{1,k}^2 + a_{2,k}^2 + \cdots + a_{m,k}^2) z^k$$

belongs to the class $K(n,\gamma,\beta)$, *where*

$$\gamma = \frac{[1+(1+2n)\beta]\{(1+\beta)(1+n)^2 - m[1+\beta(1+2n(1-\alpha))]\} - 4mn^2\alpha^2\beta^2}{2n\beta(1+\beta)(1+n)^2}$$

The result is sharp for the functions $f_i(z)$, $(i = 1,\ldots,m)$ defined by (3.15).

Proof. By virtue of Lemma 1, we obtain

$$\sum_{k=n+1}^{\infty} \frac{(1+\beta)^2k^2(1+n)^2}{(1+(1+2n)\beta - 2n\alpha\beta)^2}(a_{1,k}^2 + a_{2,k}^2 + \cdots + a_{m,k}^2) \le m.$$

Therefore we need to find the largest γ such that

$$\frac{(1+\beta)(1+n)k}{1+(1+2n)\beta - 2n\gamma\beta} \le \frac{1}{m}\frac{(1+\beta)^2k^2(1+n)^2}{(1+(1+2n)\beta - 2n\alpha\beta)^2}, \quad (k \ge n+1)$$

that is,

$$\gamma \le \frac{(1+\beta)k(1+n)[1+(1+2n)\beta] - m[1+(1+2n)\beta - 2n\alpha\beta]^2}{2n\beta(1+\beta)k(1+n)}, \quad (k \ge n+1).$$

Just as in the proof of Theorem 4, we conclude that

$$\gamma \le \varphi(n+1) = \frac{[1+(1+2n)\beta]\{(1+\beta)(1+n)^2 - m[1+\beta(1+2n(1-\alpha))]\} - 4mn^2\alpha^2\beta^2}{2n\beta(1+\beta)(1+n)^2}$$

and Theroem 6 follows at once.

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