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ON SOBOLEV-VIŠIK-DUBINSKIĬ TYPE INEQUALITIES

B. G. PACHPATTE

Abstract. In the present paper we establish some new integral inequalities of the Sobolev-Višik-Dubinskiĭ type involving functions of several independent variables and their first order partial derivatives. The analysis used in the proofs is elementary and the obtained results provide new estimates on these types of inequalities.

1. Introduction

One of the most interesting and useful variant of the well known Sobolev's inequality (see, [6, p.101]) given by Dubinskiĭ [5, p.168] can be stated as follows.

Let $-\infty < \alpha_0 < +\infty$, $\alpha_1 \geq 1$; $u(x)$, $|u(x)|^{\alpha_0+\alpha_1} \in C^1(G)$. Then the following inequality is valid

$$\int_G |u|^{\alpha_0+\alpha_1} dx \leq K[\int_{\partial G} |u|^{\alpha_0+\alpha_1} d\gamma + \int_G |u|^{\alpha_0} |\frac{\partial u}{\partial x_i}|^{\alpha_1} dx], \tag{1}$$

for $i = 1, \ldots, n$, where G is a bounded region in E_n (the n-dimensional Euclidean space) with the boundary ∂_G, $C^1(G)$ is the space of functions $u(x)$ with bounded first order derivatives in $\overline{G}$ (the closure of G). The constant K depends on α_0, α_1 and G.

The specail version of the inequality (1) was obtained earlier by M. I. Višik [16] when α_0, α_1 are even and the function u vanishes on the boundary ∂G. A number of interesting integral inequalities of the type (1) which can be used in the study of partial differential equations are also given by Dubinskiĭ in [5]. In particular, in [5] divergence theorem and the Young's inequality are used to establish the inequality (1) and its various extensions and the constants involved therein do not provide explicit information. In the present paper we establish some new integral inequalities involving functions of several independent variables and their first order partial derivatives which claim their origin to the special version of the inequality (1) given by Višik in [16]. The analysis used in the proofs is elementary and our results provide new estimates and precise information about the constants involved in these types of inequalities. An excellent account on this subject can be found in [1-16] and the references given therein.

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2. Statement of Results

In what follows, we denote by R^n the n-dimensional Euclidean space. Let S be the set of all real-valued functions $u(x)$, $x = (x_1, \dots, x_n)$, which are continuous on a subset $B = \prod_{i=1}^{n}[a_i, b_i]$ of R^n, which satisfy $u(x)|_{x_i=a_i} = u(x)|_{x_i=b_i}$ for each $i \in \{1, \dots, n\}$, and for which the partial derivatives $\frac{\delta}{\delta x_i}u(x)$, $i = 1, \dots, n$, exist. For $u \in S$, we denote by $\int_B u(x)dx$ the n-fold integral $\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} u(x_1, \dots, x_n)dx_1 \dots dx_n$. The notation $\int_{a_i}^{b_i} u(x_1, \dots, t_i, \dots, x_n)dt_1$ for $i = 1, \dots, n$, we mean for $i = 1$, it is $\int_{a_1}^{b_1} u(t_1, x_2, \dots, x_n)dt_1$ and so on and for $i = n$, it is $\int_{a_n}^{b_n} u(x_1, \dots, x_{n-1}, t_n)dt_n$. Further for $\mu > 0$ and $u \in S$, we define $\|\operatorname{grad} u(x)\|_\mu = (\sum_{i=1}^{n} |\frac{\delta}{\delta x_i}u(x)|^\mu)^{1/\mu}$.

Our main results are given in the following theorems.

Theorem 1. *Let $p \geq 0$, $q \geq 1$, $r \geq 1$, $\mu > 0$ be constants and $u \in S$. Then*

$$\int_B |u(x)|^{r(p+q)}dx \leq M^q \int_B |u(x)|^{rp} \|\mathrm{grad} u(x)\|_\mu^{rq} dx, \tag{2}$$

$$\int_B |u(x)|^{r(p+q)}dx \leq M^{p+q} \int_B \|\mathrm{grad} u(x)\|_\mu^{r(p+q)} dx, \tag{3}$$

where

$$M = [(p+q)^r n^{-\min(1,r/\mu)} (\prod_{i=1}^{n}(b_i - a_i)^{r/n}) I(r)], \tag{4}$$

in which

$$I(r) = \int_0^1 [t^{1-r} + (1-t)^{1-r}]^{-1} dt. \tag{5}$$

Theorem 2. *Let $p \geq 0$, $q \geq 1$, $m \geq 0$, $r \geq 1$, $\mu > 0$ be constants and $u \in S$. Then*

$$\int_B |u(x)|^{r(p+q)} \|\mathrm{grad} u(x)\|_\mu^{rm} dx \leq L^q \int_B |u(x)|^{rp} \|\mathrm{grad} u(x)\|_\mu^{r(q+m)} dx, \tag{6}$$

$$\int_B |u(x)|^{r(p+q)} \|\mathrm{grad} u(x)\|_\mu^{rm} dx \leq L^{p+q} \int_B \|\mathrm{grad} u(x)\|_\mu^{r(p+q+m)} dx, \tag{7}$$

where

$$L = [(p+q+m)^r n^{-\min(1,r/\mu)} (\prod_{i=1}^{n}(b_i - a_i)^{r/n}) I(r)], \tag{8}$$

and $I(r)$ is definde by (5).

Remark 1. By using the Schwarz inequality on the right side of (2) we get the following indequality

$$\int_B |u(x)|^{r(p+q)}dx \leq M^q (\int_B |u(x)|^{2rp}dx)^{1/2} (\int_B \|\operatorname{grad} u(x)\|_\mu^{2rq} dx)^{1/2}. \tag{2'}$$

We note that the inequalities of the forms (3) and (2′) are known in the literature as Poincaré and Sobolev tyep inequalities respectively and the inequalities given in (6) and (7) are motivated by the similar inequalities given by Dubinskiĭ in [5].

3. Proof of Theorem 1

From the hypotheses we have the following identities

$$u^{p+q}(x) = (p+q)\int_{a_i}^{x_i} u^{p+q-1}(x_i\cdots,t_i,\cdots,x_n)\times\frac{\partial}{\partial t_i}u(x_1,\cdots,t_i,\cdots,x_n)dt_i, \tag{9}$$

$$u^{p+q}(x) = -(p+q)\int_{x_i}^{b_i} u^{p+q-1}(x_i\cdots,t_i,\cdots,x_n)\times\frac{\partial}{\partial t_i}u(x_1,\cdots,t_i,\cdots,x_n)dt_i, \tag{10}$$

for $i=1,\ldots,n$. From (9) and (10) and using the Hölder's inequality with indices r, $r/(r-1)$ we observe that

$$\begin{aligned}|u(x)|^{r(p+q)} \le (p+q)^r(x_i-a_i)^{r-1}&\int_{a_i}^{x_i}|u(x_1,\cdots,t_i,\cdots x_n)|^{r(p+q-1)}\\ &\times|\frac{\partial}{\partial t_i}u(x_1,\cdots,t_i,\cdots,x_n)|^r dt_i,\end{aligned} \tag{11}$$

$$\begin{aligned}|u(x)|^{r(p+q)} \le (p+q)^r(b_i-x_i)^{r-1}&\int_{x_i}^{b_i}|u(x_1,\cdots,t_i,\cdots x_n)|^{r(p+q-1)}\\ &\times|\frac{\partial}{\partial t_i}u(x_1,\cdots,t_i,\cdots,x_n)|^r dt_i,\end{aligned} \tag{12}$$

for $i=1,\ldots,n$. From (11) and (12), we obtain for $x_i\in(a_i,b_i)$

$$\begin{aligned}[(x_i-a_i)^{1-r}+(b_i-x_i)^{1-r}]|u(x)|^{r(p+q)} \le (p+q)^r&\int_{a_i}^{b_i}|u(x_1,\cdots,t_i,\cdots x_n)|^{r(p+q-1)}\\ &\times|\frac{\partial}{\partial t_i}u(x_1,\cdots,t_i,\cdots,x_n)|^r dt_i.\end{aligned} \tag{13}$$

Next, we multiply both sides of (13) by $[(x_i-a_i)^{1-r}+(b_i-x_i)^{1-r}]^{-1}$ and integrate over B. Then we have

$$\begin{aligned}\int_B|u(x)|^{r(p+q)}dx \le (p+q)^r&\int_{a_i}^{b_i}[(x_i-a_i)^{1-r}+(b_i-x_i)^{1-r}]^{-1}dx_i\\ &\times\int_B|u(x)|^{r(p+q-1)}|\frac{\partial}{\partial t_i}u(x)|^r dx.\end{aligned} \tag{14}$$

Now, by taking $i = 1, \ldots, n.$ in (14) and multiplying both sides of the resulting inequalities and applying the arithmetic mean-geometric mean inequality we obtain

$$\int_B |u(x)|^{r(p+q)} dx \leq (p+q)^r \prod_{i=1}^{n} (\int_{a_i}^{b_i} [(x_i - a_i)^{1-r} + (b_i - x_i)^{1-r}]^{-1} dx_i)^{1/n}$$
$$\times \prod_{i=1}^{n} (\int_B |u(x)|^{r(p+q-1)} |\frac{\partial}{\partial t_i} u(x)|^r dx)^{1/n}$$
$$\leq \frac{1}{n}(p+q)^r \prod_{i=1}^{n} (\int_{a_i}^{b_i} [(x_i - a_i)^{1-r} + (b_i - x_i)^{1-r}]^{-1} dx_i)^{1/n}$$
$$\times \int_B \sum_{i=1}^{n} |u(x)|^{r(p+q-1)} |\frac{\partial}{\partial x_i} u(x)|^r dx$$
$$= \frac{1}{n}(p+q)^r (\prod_{i=1}^{n} (b_i - a_i)^{r/n} \int_0^1 [t^{1-r} + (1-t)^{1-r}]^{-1} dt$$
$$\times \int_B |u(x)|^{r(p+q-1)} \| \text{ grad } u(x)\|_r^r dx. \tag{15}$$

We now use the elementary inequality

$$\sum_{i=1}^{n} c_i^{\alpha} \leq n^{1-\min(1,\alpha)} (\sum_{i=1}^{n} c_i)^{\alpha}, \quad c_i \geq 0, \quad i = 1, \cdots, n, \quad \alpha > 0$$

(see [4, pp.143, 159]), to obtain

$$\| \text{ grad } u(x)\|_r^r = \sum_{i=1}^{n} |\frac{\partial}{\partial x_i} u(x)|^r = \sum_{i=1}^{n} (|\frac{\partial}{\partial x_i} u(x)|^{\mu})^{r/\mu}$$
$$\leq n^{1-\min(1,r/\mu)} (\sum_{i=1}^{n} |\frac{\partial}{\partial x_i} u(x)|^{\mu})^{r/\mu}$$
$$= n^{1-\min(1,r/\mu)} \| \text{ grad } u(x)\|_{\mu}^r, \tag{16}$$

so that (15) and (16) imply

$$\int_B |u(x)|^{r(p+q)} dx \leq (p+q)^r n^{-\min(1,r/\mu)} (\prod_{i=1}^{n} (b_i - a_1)^{r/n}) I(r)$$
$$\times \int_B |u(x)|^{r(p+q-1)} \| \text{ grad } u(x)\|_{\mu}^r dx$$
$$= M \int_B [|u(x)|^{rp/q} \| \text{ grad } u(x)\|_{\mu}^{\frac{r}{q}}] \times [|u(x)|^{r(p+q-1)-rp/q}] dx. \tag{17}$$

Using the Hölder's inequality with indices q, $q/(q-1)$ on the right side of (17) we have

$$\int_B |u(x)|^{r(p+q)} dx \leq M [\int_B |u(x)|^{rp} \| \text{ grad } u(x)\|_{\mu}^{rq} dx]^{1/q} \times [\int_B |u(x)|^{r(p+q)} dx]^{(q-1)/q}. \tag{18}$$

If $\int_B |u(x)|^{r(p+q)}dx = 0$, then (18) is trivially true, otherwise we divide both sides of (18) by $[\int_B |u(x)|^{r(p+q)}dx]^{(q-1)/q}$ and then raise both sides to the power q to get the required inequality in (2).

By using the Hölder's inequality with indices $(p+q)/p$, $(p+q)/q$ to the right side of (2) we get

$$\int_B |u(x)|^{r(p+q)}dx \le M^q[\int_B |u(x)|^{r(p+q)}dx]^{p/(p+q)} \times [\int_B \| \text{ grad } u(x)\|_\mu^{r(p+q)}dx]^{q/(p+q)} \quad (19)$$

If $\int_B |u(x)|^{r(p+q)}dx = 0$, then (19) is trivially true, otherwise we devide both sides of (19) by $[\int_B |u(x)|^{r(p+q)}dx]^{p/(p+q)}$ and then raise both sides to the power $(p+q)/q$ to get the required inequality in (3). The proof is complete.

4. Proof of Theorem 2

By rewriting the integral on the left side of (6) and using the Hölder's inequality with indices $(q+m)/m$, $(q+m)/q$ and the inequality (2) we observe that

$$\begin{aligned}
&\int_B |u(x)|^{r(p+q)}\| \text{ grad } u(x)\|_\mu^{rm}dx \\
&= \int_B [|u(x)|^{r(pm/(q+m))}\| \text{ grad } u(x)\|_\mu^{rm}] \times [|u(x)|^{r(p+q)-r(pm/(q+m))}]dx \\
&\le [\int_B |u(x)|^{rp}\| \text{ grad } u(x)\|_\mu^{r(q+m)}dx]^{m/(q+m)} \times [\int_B |u(x)|^{r(p+q+m)}dx]^{q/(q+m)} \\
&\le [\int_B |u(x)|^{rp}\| \text{ grad } u(x)\|_\mu^{r(q+m)}dx]^{m/(q+m)} \times [[(p+q+m)^r n^{-\min(1,r/\mu)} \\
&\qquad (\prod_{i=1}^{n}(b_i-a_i)^{r/n})I(r)]^{(q+m)} \times \int_B |u(x)|^{rp}\| \text{ grad } u(x)\|_\mu^{r(q+m)}dx]^{q/(q+m)} \\
&= L^q \int_B |u(x)|^{rp}\| \text{ grad } u(x)\|_\mu^{r(q+m)}dx.
\end{aligned}$$

This is the required inequality in (6).

By rewriting the inequality (6) and using the Hölder's inequality with indices $(p+q)/p$, $(p+q)/q$ we observe that

$$\begin{aligned}
&\int_B |u(x)|^{r(p+q)}\| \text{ grad } u(x)\|_\mu^{rm}dx \\
&\le L^q \int_B [|u(x)|^{rp}\| \text{ grad } u(x)\|_\mu^{r(mp/(p+q))}] \times [\| \text{ grad } u(x)\|_\mu^{r(q+m)-r(mp/(p+q))}]dx \\
&\le L^q[\int_B |u(x)|^{r(p+q)}\| \text{ grad } u(x)\|_\mu^{rm}dx]^{p/(p+q)} \times [\int_B \| \text{ grad } u(x)\|_\mu^{r(p+q+m)}dx]^{q/(p+q)}
\end{aligned}$$

Now by following the arguments as in the last part of the proof of inequality (3) with suitable modifications, we get the required inequality in (8). The proof is complete.

Remark 2. We note that in [5] Dubinskiĭ has given the inequalities of the type established in Theorem 1 and 2 by using different method and the constants involved therein are not exact and may in general be sufficiently large. Here we note that our proofs are quite elementary and the constants involved in the inequalities (2), (3), (6), (7) provide explicit information. For various forms of such inequalities, see [2,3,5,10,13,14] and the references given therein.

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Department of Mathematics, Marathwada University, Aurangabad 431 004 (Maharashtra) India.