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THE BEST CONSTANT IN AN INEQUALITY OF OSTROWSKI TYPE

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Abstract. We prove the constant $\frac{1}{2}$ in Dragomir-Wang's inequality [2] is best.

1. Introduction

The classical inequality of Ostrowski, [1, p. 469] is

Theorem 1.1. *Let I be an interval in $\mathbf{R}$, I° the interior of I, $f : I \to \mathbf{R}$ be differentiable on I°. Let $a, b \in I^\circ$ with $a < b$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)| < \infty$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a, b]$.

The constant $\frac{1}{4}$ in (1.1) is the best possible.

For, suppose that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[k + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.2}$$

for all $x \in [a, b]$. Taking $f(x) = x$, gives $\|f'\|_\infty = 1$ and (1.2) becomes

$$\left|x - \frac{a+b}{2}\right| \le \left[k + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)$$

for all $x \in [a, b]$. With $x = a$ this becomes

$$\frac{b-a}{2} \le \left(k + \frac{1}{4}\right)(b-a)$$

giving $k \ge \frac{1}{4}$.

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2. The Results

In [2], Dragomir and Wang obtained a related inequality:

Theorem 2.1. *Let I, f, a, b be as above and $f' \in L_1[a,b]$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] \|f'\|_1 \tag{2.1}$$

for all $x \in [a,b]$,

but did not prove that the constant $\frac{1}{2}$ is the best possible one.

In [3], S. S. Dragomir gave an extension of Theorem 2.1 for mappings with bounded variation, i.e., he proved the result:

Theorem 2.2. *Let $f : [a,b] \to \mathbf{R}$ be a mapping with bounded variation on $[a,b]$. Then for all $x \in [a,b[$, we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] \bigvee_a^b (f) \tag{2.2}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

The constant $\frac{1}{2}$ is the best possible one.

For the sake of completeness and as the paper [3] is not published yet, we give here a short proof of Theorem 2.2.

Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\int_a^b p(x,t)df(t) = f(x)(b-a) - \int_a^b f(t)dt \tag{2.3}$$

where

$$p(x,t) := \begin{cases} t - a \; if t \in [a,x) \\ t - b \; if t \in [x,b]. \end{cases}$$

for all $x, t \in [a,b]$.

It is well known that if $p : [a,b] \to \mathbf{R}$ is continuous on $[a,b]$ and $v : [a,b] \to \mathbf{R}$ is with bounded variation on $[a,b]$, then

$$\left| \int_a^b p(x)dv(x) \right| \le \sup_{x\in[a,b]} |p(x)| \bigvee_a^b (v). \tag{2.4}$$

Applying the inequality (2.4) for $p(x,\cdot)$ and f, we get

$$\left| \int_a^b p(x,t)df(t) \right| \le \sup_{t\in[a,b]} |p(x,t)| \bigvee_a^b (f)$$

$$= \max\{x-a, b-x\} \bigvee_a^b (f) = \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f).$$

Using the identity (2.3), we deduce the desired result (2.2).

To prove the sharpness of the constant $\frac{1}{2}$ in the class of mappings with bounded variation, assume that the inequality (2.2) holds with a constant $C > 0$, i.e.,

$$\left|\int_a^b f(t)dt - f(x)(b-a)\right| \leq \left[C(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f), \tag{2.5}$$

for all $x \in [a, b]$.

Consider the mapping $f : [a, b] \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 \text{ if } x \in [a,b] \backslash \{\frac{a+b}{2}\} \\ 1 \text{ if } x = \frac{a+b}{2} \end{cases}$$

in (2.5). Then f is with bounded variation on $[a, b]$ and

$$\bigvee_a^b (f) = 2, \quad \int_a^b f(t)dt = 0$$

and for $x = \frac{a+b}{2}$ we get in (2.5), $1 \leq 2C$; which implies that $C \geq \frac{1}{2}$ and the theorem is completely proved.

Now, it is clear that if f is differentiable on (a, b) and $f' \in L_1[a,b]$, then f is with bounded variation on $[a, b]$ and applying Theorem 2.2 we get Theorem 2.1. But we are not sure that the constant $\frac{1}{2}$ is best in the class of differentiable mappings whose derivatives are in $L_1(a, b)$. We give an example showing that the constant $\frac{1}{2}$ remains best for this class of mappings, too.

Suppose that

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[k + \frac{|x - \frac{a+b}{2}|}{b-a}\right] \|f'\|_1, \quad x \in [a,b]. \tag{2.6}$$

Let C be any positive real and let

$$f(x) = \frac{C}{C^2 + x^2} - \tan^{-1}\left(\frac{1}{C}\right)$$

with $a = -1$ and $b = 1$.

Direct calculation shows that $\int_a^b f(t)dt = 0$.

Also, since $f'(x) \leq 0$ for all $x \geq 0$,

$$\begin{aligned} \|f'\|_1 &= 2\int_0^1 |f'(t)|dt = -2\int_0^1 f'(t)dt = 2[f(0) - f(1)] \\ &= 2\left[\frac{1}{C} - \frac{C}{C^2+1}\right] = \frac{2}{C(C^2+1)}. \end{aligned}$$

Substituting these into (2.6) and taking $x = 0$ then gives

$$\left|\frac{1}{C} - tan^{-1}\left(\frac{1}{C}\right)\right| \leq k\frac{2}{C(C^2+1)}$$

so that

$$k \geq \frac{C^2+1}{2}\left[1 - C\tan^{-1}\left(\frac{1}{C}\right)\right].$$

Since the right side tends to $\frac{1}{2}$ as $C \to 0+$, we get $k \geq \frac{1}{2}$, which shows that the constant $\frac{1}{2}$ is the best possible in Theorem 2.1.

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