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GLOBAL ATTRACTIVITY IN A FOUR-TERM RECURRENCE RELATION

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Abstract. Positive solutions of the four-term recurrence relation $x_{n+1} = f(x_n)g(x_{n-1}, x_{n-2})$ are shown to converge to its positive equalibrium points under relative mild conditions.

1. Introduction

The asymptotic behavior of sequences defined by recurrence relations has long been discussed in various branch of sciences (see e.g. Kocic and Ladas [1]). In this note, we are concerned with the asymptotic behavior of the set Ω of real sequences $\{x_n\}_{n=-2}^{\infty}$ defined by $x_{-2} = \alpha > 0$, $x_{-1} = \beta > 0$, $x_0 = \gamma > 0$ and

$$x_{n+1} = f(x_n)g(x_{n-1}, x_{n-2}), \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where
(H1) $f : (0, \infty) \to R$ and $g : [0, \infty) \times [0, \infty) \to R$ are positive functions; and
(H2) f is nondecreasing and g is nonincreasing in each of its independent variables.

A positive fixed point x^* that satisfies $x = f(x)g(x, x)$ is also called a positive equilibrium point of equation (1.1). Our objective of this note is to show that under mild conditions on the functions f and g, every real sequence in Ω tends to one of the positive equilibrium points of (1.1).

Similar results have been obtained for a number of recureence relations, see e.g. Kocic and Ladas [1], Camouzis et al. [2], Li et al. [3], and Li [4]. Indeed, this note is motivated by a concern raised in Kocic and Ladas [1, p.46] related to the stability of recurrence relations.

2. Stability Criteria

We need to establish a boundedness criteria as a preparatory result for proving our assertion. Its proof is motivated by that of Theorem 4.1 in [2], and involves novel use of a polynomial equation.

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Theorem 1. *Assume that* (H1) *and* (H2)*hold. Suppose further that there are positive constants* A, B, L, p, q *and* r *such that*

$$f(x) \leq Ax^p, \quad x \geq L > 0,$$
$$g(x,y) \leq \frac{B}{x^q y^r}, \quad x, y \geq L > 0,$$

and

$$\lambda^3 - p\lambda^2 + q\lambda + r = 0 \tag{2.1}$$

does not have any positive roots. Then every sequence in Ω *is bounded.*

Proof. Let $\{\lambda_n\}_{n=0}^{\infty}$ be defined by $\lambda_0 = p$ and

$$\lambda_n = p - \frac{\lambda_{n-1}q + r}{\lambda_{n-1}^2}, \quad n = 1, 2, 3, \ldots. \tag{2.2}$$

We assert that there is some positive integer N such that $\lambda_0, \lambda_1, \ldots, \lambda_{N-1} > 0$ and $\lambda_N \leq 0$. Indeed, if $p^3 \leq pq + r$, then

$$\lambda_1 = p - \frac{pq + r}{p^2} = \frac{p^3 - (pq + r)}{p^2} \leq 0.$$

If $p^3 > pq + r$, then

$$\lambda_1 = \frac{p^3 - (pq + r)}{p^2} > 0,$$

and

$$\lambda_1 - \lambda_0 = -\frac{pq + r}{p^2} < 0.$$

If $\lambda_n > 0$ for all n, then since

$$\lambda_{n+1} - \lambda_n = (\lambda_n - \lambda_{n-1}) \frac{\lambda_n \lambda_{n-1} q + \lambda_n r + \lambda_{n-1} r}{\lambda_n^2 \lambda_{n-1}^2}$$

for $n \geq 1$, we easily see by induction that $\{\lambda_n\}$ decreases to a nonnegative limit λ. By taking limits on both sides of (2.2), we see that λ is a nonnegative root of (2.1), furthermore, it must be positive, since 0 is not a root of (2.1). However, by our assumptions, the above equation does not have any nonnegative solution. This is a contradiction. In other words, there is some positive integer N such that $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{N-1} > 0$ and $\lambda_N \leq 0$.

Let $\{x_n\}_{n=-2}^{\infty}$ be a real sequence in Ω. If f is bounded by M, then clearly

$$x_{n+1} = f(x_n) g(x_{n-1}, x_{n-2}) \leq M g(0, 0)$$

for all $n \geq 0$. If f is not bounded, then $\lim_{n\to\infty} f(x_n) = \infty$ since f is nondecreasing. If $\{x_n\}$ is not bounded above, then there is a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ such that $\lim_{i\to\infty} x_{n_i} =$

∞. Without loss of any generality, we may also assume that $n_{i+1} - n_i > 2N + 3$. Since $g(x, y)$ is decreasing in x and in y, thus

$$x_{n_i} = f(x_{n_i-1})g(x_{n_i-2}, x_{n_i-3}) \leq f(x_{n_i})g(0,0), \quad i = 0, 1, 2, \ldots,$$

which shows that $\lim_{i\to\infty} f(x_{n_i-1}) = \infty$. But then $\lim_{i\to\infty} x_{n_i-1} = \infty$. Similarly, we may show that

$$\lim_{i\to\infty} x_{n_i-k} = \infty, \quad k = 1, 2, \ldots, 2N + 3.$$

As a consequence,

$$\lim_{i\to\infty} \frac{x_{n_i}}{x_{n_i-1}^{\lambda_N}} = \infty. \tag{2.3}$$

On the other hand, let M be a positive integer such that

$$x_{n_i-k} \geq L, \quad k = 0, 1, 2, \ldots, 2N + 3,$$

for $i \geq M$. Since

$$x_{n_i-k} = f(x_{n_i-k-1})g(x_{n_i-k-2}, x_{n_i-k-3}) \leq AB \frac{x_{n_i-k-1}^p}{x_{n_i-k-2}^q x_{n_i-k-3}^r},$$

thus

$$\lim_{i\to\infty} \frac{x_{n_i-k}}{x_{n_i-k-1}^p} = 0 \tag{2.4}$$

for $k = 0, 1, 2, \ldots, 2N$ and

$$\frac{x_{n_i-k}}{x_{n_i-k-1}^{\lambda_j}} \leq AB \frac{x_{n_i-k-1}^{p-\lambda_j}}{x_{n_i-k-2}^q x_{n_i-k-3}^r} = AB \left(\frac{x_{n_i-k-1}}{x_{n_i-k-2}^{\lambda_j-1}}\right)^{p-\lambda_j} \left(\frac{x_{n_i-k-2}}{x_{n_i-k-3}^{\lambda_j-1}}\right)^{r/\lambda_j-1}$$

for $k = 0, 1, 2, \ldots, 2N$, $j = 0, 1, 2, \ldots, N-1$ and $i \geq M$, where we have used the fact that $\lambda_{j-1}q = \lambda_{j-1}^2 p - \lambda_{i-1}^2 \lambda_j - r$ in obtaining the last equality. But then

$$\begin{aligned}
\frac{x_{n_i}}{x_{n_i-1}^{\lambda_N}} &\leq AB \left(\frac{x_{n_i-1}}{x_{n_i-2}^{\lambda_{N-1}}}\right)^{p-\lambda_N} \left(\frac{x_{n_i-2}}{x_{n_i-3}^{\lambda_{N-1}}}\right)^{r/\lambda_{N-1}} \\
&\leq (AB)^{1+p-\lambda_N+r/\lambda_{N-1}} \left(\frac{x_{n_i-2}}{x_{n_i-3}^{\lambda_{N-2}}}\right)^{(p-\lambda_N)(p-\lambda_{N-1})} \left(\frac{x_{n_i-3}}{x_{n_i-4}^{\lambda_{N-2}}}\right)^{(p-\lambda_N)(r/\lambda_{N-2})} \\
&\quad \times \left(\frac{x_{n_i-3}}{x_{n_i-4}^{\lambda_{N-2}}}\right)^{(p-\lambda_{N-1})(r/\lambda_{N-1})} \left(\frac{x_{n_i-4}}{x_{n_i-5}^{\lambda_{N-2}}}\right)^{(r/\lambda_{N-1})(r/\lambda_{N-2})} \\
&\leq \cdots \\
&\leq A^m B^m \left(\frac{x_{n_i-N}}{x_{n_i-N-1}^p}\right)^{s_1} \cdots \left(\frac{x_{n_i-2N}}{x_{n_i-2N-1}^p}\right)^{s_{2N}}
\end{aligned}$$

for some positive numbers $m, s_1, \ldots, s_{2N}$ where $s_1, \ldots, s_{2N}$ are rational expressions of $\lambda_1, \lambda_2, \ldots, \lambda_{N-1}$. It follows from (2.4) that

$$\lim_{i\to\infty} \frac{x_{n_i}}{x_{n_i-1}^{\lambda_N}} = 0,$$

which is contrary to (2.3). The proof is complete.

We remark that the polynomial equation (2.1) does not have any positive roots if, and only if, the difference equation

$$x_{n+3} - px_{n+2} + qx_{n+1} + r = 0, \quad n = 0, 1, 2, \ldots$$

is oscillatory (see e.g. [1, Theorem 1.2.1]). Furthermore, oscillatory criteria for such difference equations can be found in a number of recent papers.

As an application, let us consider the recurrence relation

$$y_{n+1} = \frac{e + y_n^2}{d + y_{n-1}y_{n-2}}, \quad n = 0, 1, 2, \ldots \tag{2.5}$$

where $d, e > 0$. Here $f(x) = e + x^2 \leq Ax^2$ for all large x, and $g(x, y) = 1/(d + xy) \leq 1/(xy)$ for all large x and y. Furtheremore, the polynomial

$$h(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 1$$

does not have any positive roots since $h(0) = 1$, $h(\infty) = \infty$ and $\min_{\lambda>0} h(\lambda) = 13 > 0$. Hence all the assumptions in Theorem 1 are satisfied. This shows that every solution of (2.5) defined by positive initial conditions is bounded. We assert further that the sequence $\{y_n\}$ defined by positive initial conditions and (2.5) converges when $d > 1/4$. Indeed, by Theorem 1, the sequence $\{y_n\}$ is bounded between 0 and some positive number M. Note that

$$y_{n+1} = \frac{e + y_n^2}{d + y_{n-1}y_{n-2}} \geq \frac{e}{d + y_{n-1}y_{n-2}} \geq \frac{e}{d + M^2},$$

thus

$$\frac{1}{\sqrt{y_k}} \leq \sqrt{\frac{d + M^2}{e}} \equiv H, \quad k \geq -2.$$

On the other hand,

$$\begin{aligned} y_{n+1} &= \frac{e + y_n^2}{d + y_{n-1}y_{n-2}} \\ &= \frac{e}{d + y_{n-1}y_{n-2}} + \frac{y_n y_n}{d + y_{n-1}y_{n-2}} \\ &= \frac{e}{d + y_{n-1}y_{n-2}} + \frac{y_n(e + y_{n-1}^2)}{(d + y_{n-1}y_{n-2})(d + y_{n-2}y_{n-3})} \\ &= \frac{e}{d + y_{n-1}y_{n-2}} + \frac{y_n e}{(d + y_{n-1}y_{n-2})(d + y_{n-2}y_{n-3})} \end{aligned}$$

$$
\begin{aligned}
&+\frac{y_n y_{n-1}(e+y_{n-2}^2)}{(d+y_{n-1}y_{n-2})(d+y_{n-2}y_{n-3})(d+y_{n-3}y_{n-4})}\\
&=\cdots\\
&=\frac{e}{d+y_{n-1}y_{n-2}}+\frac{y_n e}{(d+y_{n-1}y_{n-2})(d+y_{n-2}y_{n-3})}\\
&+\cdots+\frac{y_n y_{n-1}\cdots y_1 e}{(d+y_{n-1}y_{n-2})\cdots(d+y_0y_{-1})(d+y_{-1}y_{-2})}\\
&+\frac{y_n y_{n-1}\cdots y_1 y_0^2}{(d+y_{n-1}y_{n-2})\cdots(d+y_0y_{-1})(d+y_{-1}y_{-2})}.
\end{aligned}
$$

Thus, in view of the inequality $a^2+b^2 \geq 2ab$, we see that

$$
\begin{aligned}
y_{n+1} &\leq \frac{e}{2\sqrt{dy_{n-1}y_{n-2}}}+\frac{y_n e}{4d\sqrt{y_{n-1}y_{n-2}^2y_{n-3}}}\\
&+\frac{y_n y_{n-1} e}{8d\sqrt{y_{n-1}y_{n-2}^2y_{n-3}^2y_{n-4}d}}+\frac{y_n y_{n-1}y_{n-2} e}{16d^2\sqrt{y_{n-1}y_{n-2}^2y_{n-3}^2y_{n-4}^3y_{n-5}}}\\
&+\cdots+\frac{y_n y_{n-1}\cdots y_1 e}{2^{n+1}d^{(n+1)/2}\sqrt{y_{n-1}y_{n-2}^2y_{n-3}^2\cdots y_0^2y_{-1}^2y_{-2}}}\\
&+\frac{y_n y_{n-1}\cdots y_1 y_0^2}{2^{n+1}d^{(n+1)/2}\sqrt{y_{n-1}y_{n-2}^2\cdots y_0^2y_{-1}^2y_{-2}}}\\
&\leq \frac{eH^2}{2\sqrt{d}}+\frac{M^2eH^4}{4d}+\frac{M^2eH^6}{8d\sqrt{d}}+\frac{M^2eH^6}{16d^2}\\
&+\cdots+\frac{M^2eH^6}{2^{n+1}d^{(n+1)/2}}+\frac{M^3H^4}{2^{n+1}d^{(n+1)/2}}.
\end{aligned}
$$

Since the infinite series

$$\sum_{n=1}^{\infty}\frac{1}{2^n d^{n/2}}$$

is convergent when $d > 1/4$ in view of the ratio test, we see that $\lim_{n\to\infty} y_n = L \in [0,\infty)$ as desired. Taking limits on both sides of (2.5), we see that

$$L=\frac{e+L^2}{d+L^2},$$

or

$$L^3-L^2+dL-e=0.$$

It is possible to find necessary and sufficient conditions such that this polynomial equation has a unique positive root. This can be done since third order polynomial equation can be solved exactly. Here, however, we will only give an example. For instance, when $d=1$ and $e=1$, the corresponding polynomial equation has the roots $1, i, -i$. Thus the corresponding solution sequence $\{y_n\}$ converges to 1 irrespective of the values of the positive initial conditions.

We remark that the rational recursive sequence defined by

$$x_{n+1} = \frac{a + bx_n^2}{c + x_{n-1}x_{n-2}}, \quad a, b, c > 0;\ n = 0, 1, 2, \ldots,$$

can be transformed into (2.5) by writing it in the form

$$\frac{x_{n+1}}{b} = \frac{\frac{a}{b^3} + \frac{x^2}{b^2}}{\frac{c}{b^2} + \frac{x_{n-1}}{b}\frac{x_{n-2}}{b}}, \quad n = 0, 1, 2, \ldots.$$

The same idea can be empolyed for obtaining a different result.

Theorem 2. *In addition to the assumptions in Theorem 1, suppose further that*

$$f(x) = a + xh(x), \quad 0 < x < \infty$$
$$g(x, y) \leq u(x)v(x), \quad 0 \leq x, y < \infty$$

and

$$h(x)u(x)v(x) \leq \delta < 1, \quad 0 \leq x < \infty,$$

for some positive nondecreasing function h defined on $[0, \infty)$, and positive nonincreasing functions u and v defined on $[0, \infty)$. Then a real sequence in Ω tends to one of the positive equilibrium points of (1.1).

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a real sequence in Ω. Then by Theorem 1, $\{x_n\}$ is a sequence bounded between 0 and a positive number M. For $n \geq 0$,

$$\begin{aligned} x_{n+1} &= f(x_n)g(x_{n-1}, x_{n-2}) \\ &= (a + x_n h(x_n))g(x_{n-1}, x_{n-2}) \\ &= ag(x_{n-1}, x_{n-2}) + (a + x_{n-1}h(x_{n-1}))g(x_{n-2}, x_{n-3})h(x_n)g(x_{n-1}, x_{n-2}) \\ &= ag(x_{n-1}, x_{n-2})(1 + h(x_n)g(x_{n-2}, x_{n-3})) \\ &\quad + ag(x_{n-3}, x_{n-4})h(x_n)h(x_{n-1})g(x_{n-1}, x_{n-2})g(x_{n-2}, x_{n-3}) \\ &\quad + \cdots + ah(x_n)h(x_{n-1} \cdots h(x_1)g(x_{-1}, x_{-2})g(x_0, x_{-1}) \cdots g(x_{n-1}, x_{n-2}) \\ &\quad + x_0 h(x_n)h(x_{n-1}) \cdots h(x_1)h(x_0)g(x_{-1}, x_{-2})g(x_0, x_{-1}) \cdots g(x_{n-1}, x_{n-2}). \end{aligned}$$

hence, in view of our assumptions,

$$\begin{aligned} x_{n-1} = {} & au(x_{n-1})v(x_{n-2}) + ah(x_n)u(x_{n-1})u(x_{n-2})v(x_{n-2})v(x_{n-3}) \\ & + ah(x_n)h(x_{n-1})u(x_{n-1})u(x_{n-2})u(x_{n-3})v(x_{n-2})v(x_{n-3})v(x_{n-4}) \\ & + ah(x_n)h(x_{n-1})h(x_{n-2})u(x_{n-1})u(x_{n-2}) \\ & \times u(x_{n-3})u(x_{n-4})v(x_{n-2})v(x_{n-3})v(x_{n-4})v(x_{n-5}) \\ & + \cdots \\ & + ah(x_n)h(x_{n-1}) \cdots h(x_1)u(x_{-1})v(x_{-2}) \cdots \end{aligned}$$

$$
\begin{aligned}
&u(x_0)u(x_{-1})v(x_{n-2})\cdots v(x_0)v(x_{-1})v(x_{-2})\\
&+x_0h(x_n)h(x_{n-1})\cdots h(x_0)u(x_{n-1})u(x_{n-2})\cdots\\
&\quad u(x_0)u(x_{-1})v(x_{n-2})\cdots v(x_{-1})v(x_{-2})\\
\le\ & au(x_{n-1})v(x_{n-2}) + ah(x_n)u(x_{n-1})u(x_{n-2})v(x_{n-2})v(x_{n-3})\\
&+ah(x_n)h(x_{n-1})u(x_{n-1})u(x_{n-2})u(x_{n-3})v(x_{n-2})v(x_{n-3})v(x_{n-4})\\
&+ah(x_n)h(x_{n-1})u(x_{n-1})u(x_{n-3})u(x_{n-4})v(x_{n-3})v(x_{n-4})v(x_{n-5})\delta\\
&+\cdots\\
&+ah(x_n) + h(x_{n-1})u(x_{n-1})u(x_{-1})v(x_{-1})v(x_{-2})u(x_0)v(x_0)\delta^{n-2}\\
&+x_0h(x_n)h(x_{n-1})u(x_{n-1})u(x_{-1})v(x_{-1})v(x_{-2})\delta^{n-1}.
\end{aligned}
$$

By means of the monotonicity assumptions on the funtions h, u and v, we see further that

$$
\begin{aligned}
x_{n+1} \le\ & au(0)v(0) + ah(M)u^2(0)v^2(0)\\
&+ah^2(M)u^3(0)v^3(0)\\
&+ah^2(M)u^3(0)v^3(0)\delta\\
&+\cdots\\
&+ah^2(M)u^3(0)v^3(0)\delta^{n-2}\\
&+x_0h^2(M)u^2(0)v^2(0)\delta^{n-1}
\end{aligned}
$$

Since $1 + \delta + \cdots + \delta^n + \cdots$ converges when $\delta < 1$, we see that $\{x_n\}$ converges to a nonnegative number. By taking limits on both sides of (1.1), we see that $\{x_n\}$ converges to one of the positive equilibrium points of (1.1). The proof is complete.

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