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POSITIVE SOLUTIONS IN AN ANNULUS FOR NONLINEAR DIFFERENTIAL EQUATIONS ON A MEASURE CHAIN

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Abstract. We study the existence of positive solutions of the second order differential equation in an annulus on a measure chain, $u^{\Delta\Delta}(t) + f(u(\sigma(t))) = 0$, $t \in [0, 1]$, satisfying the boundary conditions, $\alpha y(0) - \beta y^{\Delta}(0) = 0$ and $\gamma y(\sigma(1)) + \delta y^{\Delta}(\sigma(1)) = 0$, where f is a positive function and $f(x)$ is sublinear (respectively superlinear) at $x = 0$ and is superlinear (respectively sublinear) at $x = \infty$. The methods involve applications of a fixed point theorem for operators on a cone in a Banach space.

1. Introduction

In this paper, we study the existence of positive solutions of a certain second order differential equation in an annulus on a measure chain. Much recent attention has been given to differential equations on measure chains, and we refer the reader to [4], [9] and [15] for some historical works as well as to the more recent papers [1], [1], [2], [5], [6], [10], [11] and the book [19] for excellent references on these types of equations. Before introducing the problems of interest for this paper, we present some definitions and notation which are common to the recent literature. Our sources for this background material are the two papers by Erbe and Peterson [10], [11].

Definitions 1.1. Let T be a closed subset of R, and let T have the subspace topology inherited from the Euclidean topology on R. For $t < \sup T$ and $r > \inf T$, define the *forward jump operator*, σ, and the *backward jump operator*, ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in T \mid \tau > t\} \in T,$$
$$\rho(r) = \sup\{\tau \in T \mid \tau < r\} \in T,$$

for all $t, r \in T$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense.

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Definitions 1.2. For $x : T \to R$ and $t \in T$ (if $t = \sup T$, assume t is not left scattered), define the *delta derivative* of $x(t)$, $x^{\Delta}(t)$, to be the number (when it exists), with the property that, for any $\epsilon > 0$, there is a neighborhood, U, of t such that

$$\Big|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\Big| \le \epsilon\Big|\sigma(t) - s\Big|,$$

for all $s \in U$. The second delta derivative of $x(t)$ is defined by $x^{\Delta\Delta}(t) = (x^{\Delta})^{\Delta}(t)$.

If $F^{\Delta}(t) = h(t)$, then define the *integral* by

$$\int_a^t h(s)\Delta s = F(t) - F(a).$$

Throughout this paper, we assume T is a closed subset of R with $0, 1 \in T$.

Definition 1.3. Define the closed interval, $[0,1]$, in T by

$$[0,1] = \{t \in T \mid 0 \le t \le 1\}.$$

Other closed, open, and half-open intervals in T are similarly defined.

In this paper, we consider the existence of positive solutions of the differential equation on a measure chain,

$$u^{\Delta\Delta}(t) + f(u(\sigma(t))) = 0, \quad t \in [0,1], \tag{1}$$

satisfying

$$\alpha y(0) - \beta y^{\Delta}(0) = 0 \qquad \gamma y(\sigma(1)) + \delta y^{\Delta}(\sigma(1)) = 0, \tag{2}$$

where $f : R^+ \to R^+$ is continuous and $d = \gamma\beta + \alpha\delta + \alpha\gamma(\sigma(1)) > 0$.

We remark that by a solution, u, of (1), (2) , we mean $u : [0, \sigma^2(1)] \to R$, u satisfies (1) on $[0,1]$, and u satisfies the boundary conditions (2).

This paper constitutes an extension of the recent work by Erbe and Peterson [11] in which they obtained positive solutions of both (1), (2) for the cases when either $f(x)$ is superlinear (at $x = 0$ and $x = \infty$), or $f(x)$ is sublinear (at $x = 0$ and $x = \infty$). The solutions obtained in [11] were found to belong to the intersection of a cone with an annular type region. In a more recent paper, Erbe and Peterson [12] applied index theory in a cone to obtain multiple positive solutions of (1), (2). Our goal is to extend to differential equations on measure chains the works of Erbe, Hu, and Wang [13], Erbe and Tang [14], and Kaufmann [17], to obtain two positive solutions of (1), (2), when f is superlinear at one endpoint (zero or infinity) and sublinear at the other (respectively, infinity or zero); that is when either $(i) f_0 = \infty$ and $f_\infty = \infty$, or $(ii) f_0 = 0$ and $f_\infty = 0$, where,

$$f_0 = \lim_{x \to 0^+} \frac{f(x)}{x} \quad \text{and} \quad f_\infty = \lim_{x \to +\infty} \frac{f(x)}{x}.$$

We will make our assumptions on f more precise in Section 3. The results herein are also related to those by Atici [3] and Eloe and Henderson [8].

A Krasnosel'skii fixed point theorem [20] is applied to yield positive solutions of (1), (2) in our arguments. These methods have been used successfully, in the cases when (1) is either an ordinary differential equation or a finite difference equation; see [17], [18], [22].

In Section 2, we present some properties of a Green's function associated with (1), (2) that are used in defining a positive operator. We also state the Krasnosel'skii fixed point theorem. In Section 3, we give an appropriate Banach space and construct a cone on which we apply the fixed point theorem yielding twin positive solutions of (1), (2).

2. Some Preliminaries

In this section, we state the above mentioned Krasnosel'skii fixed point theorem. We will apply this fixed point theorem in the next section to a completely continuous integral operator whose kernel, $G(t,s)$, is the Green's function for

$$-y^{\Delta\Delta}(t) = 0, \tag{3}$$

$$\alpha y(0) - \beta y^{\Delta}(0) = 0 \qquad \gamma y(\sigma(1)) + \delta y^{\Delta}(\sigma(1)) = 0. \tag{4}$$

Erbe and Peterson [11] have found

$$G(t,s) = \begin{cases} \frac{1}{d}\{\alpha t + \beta\}\{\gamma(\sigma(1) - \sigma(s)) + \delta\}, t \le s, \\ \frac{1}{d}\{\alpha\sigma(s) + \beta\}\{\gamma(\sigma(1) - t) + \delta\}, \sigma(s) \le t, \end{cases} \tag{5}$$

where

$$d = \gamma\beta + \alpha\delta + \alpha\gamma(\sigma(1)) > 0,$$

from which

$$G(t,s) > 0, \quad (t,s) \in (0,\sigma(1)) \times (0,1), \tag{6}$$

$$\frac{G(t,s)}{G(\sigma(s),s)} = \begin{cases} \frac{\alpha t+\beta}{\alpha\sigma(s)+\beta}, & t \le s, \\ \frac{\gamma(\sigma(1)-t)+\delta}{\gamma(\sigma(1)-\sigma(s))+\delta}, & \sigma(s) \le t. \end{cases} \tag{7}$$

This implies

$$G(t,s) \le G(\sigma(s),s) \quad for \quad t \in [0,\sigma(1)],\ s \in [0,1],$$

and it is also shown in [11] that

$$G(t,s) \ge kG(\sigma(s),s), \quad t \in [\frac{\sigma(1)}{4}, \frac{3\sigma(1)}{4}],\ s \in [0,1], \tag{8}$$

where $k = \min\{\frac{\alpha\sigma(1)+4\beta}{4\alpha\sigma(1)+4\beta}, \frac{\gamma\sigma(1)+4\delta}{4\gamma(\sigma(1)-\sigma(0))+4\delta}\}$.

We will apply the following fixed point theorem which can be found in the book by Krasnosel'skii [20] as well as in the book by Deimling [7].

Theorem 2.1. *Let B be a Banach space, and let $P \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$$

be a completely continuous operator such that either,

(i) $||Tu|| \leq ||u||$, $u \in P \cap \partial\Omega_1$, *and* $||Tu|| \geq ||u||$, $u \in P \cap \partial\Omega_2$, *or*
(ii) $||Tu|| \geq ||u||$, $u \in P \cap \partial\Omega_1$, *and* $||Tu|| \leq ||u||$, $u \in P \cap \partial\Omega_2$.

Then T *has a fixed point in* $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Solutions of (1.1), (1.2) in a Cone

In this section, we apply Theorem 2.1 to the eigenvalue problem (1), (2). Throughout this section, we assume $\sigma(1)$ is right dense so that $G(t,s) \geq 0$, for $t \in [0, \sigma^2(1)]$, $s \in [0, \sigma(1)]$.

Assume also throughout this section that $[0, \sigma(1)]$ is such that

$$\xi = \min\{t \in T \mid t \geq \frac{\sigma(1)}{4}\}, \text{ and}$$
$$\omega = \max\{t \in T \mid t \leq \frac{3\sigma(1)}{4}\}$$

both exist and satisfy,

$$\frac{\sigma(1)}{4} \leq \xi < \omega \leq \frac{3\sigma(1)}{4},$$

and if $\sigma(\omega) = 1$, also assume $\sigma(\omega) < \sigma(1)$. Next, let $\tau \in [\xi, \omega]$ be defined by

$$\int_\xi^\omega G(\tau, s)\Delta s = \max_{t \in [\xi,\omega]} \int_\xi^\omega G(t,s)\Delta s. \tag{9}$$

Finally, we define

$$\ell = \min_{s \in [0,\sigma(1)]} \frac{G(\sigma(\omega), s)}{G(\sigma(s), s)}, \tag{10}$$

and set

$$m = \min\{k, \ell\}. \tag{11}$$

We note that $u(t)$ is a solution of (1), (2), if and only if,

$$u(t) = \int_0^{\sigma(1)} G(t,s) f(u(\sigma(s)))\Delta s, \quad t \in [0, \sigma^2(1)].$$

For our constructions, let $B = \{x : [0, \sigma^2(1)] \to R \mid x \text{ is bounded}\}$, with norm $||x|| = \sup\{|x(t)| \ : \ t \in [0, \sigma^2(1)]\}$. Then, define the cone $P \subset B$

$$P = \{x \in B \mid x(t) \geq 0 \text{ on } [0, \sigma^2(1)], \text{ and } x(t) \geq m||x||, \text{ for } t \in [\xi, \sigma(\omega)]\}.$$

Theorem 3.1. *Assume that* f *satisfies the following conditions:*

(A) $f_0 = \infty$ *and* $f_\infty = \infty$.

(B) *There exists a $p > 0$ such that, if $0 \le x \le p$, then,*

$$f(x) \le \eta p$$

where,

$$\eta = \Big(\int_0^{\sigma(1)} G(\sigma(s), s)\Delta s\Big)^{-1}$$

Then there exist at least two solutions u_1, u_2 of (1), (2) *which belong to P, such that*

$$0 \le ||u_1|| \le p \le ||u_2||.$$

Proof. Define an integral operator $T : P \to B$ by

$$Tu(t) = \int_0^{\sigma(1)} G(t,s)f(u(\sigma(s)))\Delta s, \quad u \in P. \tag{12}$$

We seek a fixed point of T which belongs to P, and so we first claim $T : P \to P$. We choose $u \in P$. Then from (7),

$$\begin{aligned} Tu(t) &= \int_0^{\sigma(1)} G(t,s)f(u(\sigma(s)))\Delta s \\ &\le \int_0^{\sigma(1)} G(\sigma(s),s)f(u(\sigma(s)))\Delta s, \end{aligned}$$

and so

$$||Tu|| \le \int_0^{\sigma(1)} G(\sigma(s),s)f(u(\sigma(s)))\Delta s. \tag{13}$$

So, from (8) and (13), for $u \in P$,

$$\begin{aligned} \min_{t\in[\xi,\omega]} Tu(t) &= \min_{t\in[\xi,\omega]} \int_0^{\sigma(1)} G(t,s)f(u(\sigma(s)))\Delta s \\ &\ge \int_0^{\sigma(1)} kG(\sigma(s),s)f(u(\sigma(s)))\Delta s \\ &\ge k||Tu|| \\ &\ge m||Tu||, \end{aligned}$$

and from (10)

$$\begin{aligned} Tu(\sigma(\omega)) &= \int_0^{\sigma(1)} G(\sigma(\omega),s)f(u(\sigma(s)))\Delta s \\ &\ge \int_0^{\sigma(1)} \ell G(\sigma(s),s)f(u(\sigma(s)))\Delta s \\ &\ge m\int_0^{\sigma(1)} G(\sigma(s),s)f(u(\sigma(s)))\Delta s \\ &\ge m||Tu||. \end{aligned}$$

Thus $Tu \in P$, and we conclude $T : P \to P$. The standard arguments show T is completely continuous.

From $f_0 = \infty$ there is an $p > r > 0$ such that $f(u) \geq Mu$, for $0 \leq u \leq r$, where the constant $M > 0$ satisfies

$$Mm \int_{\xi}^{\omega} G(\tau, s)\Delta s \geq 1. \tag{14}$$

Let

$$\Omega_1 = \{x \in B \mid ||x|| < r\},$$

and then choose $u \in P$ with $||u|| = r$. Then

$$\begin{aligned} Tu(\tau) &= \int_0^{\sigma(1)} G(\tau, s) f(u(\sigma(s)))\Delta s \\ &\geq \int_{\xi}^{\omega} G(\tau, s) f(u(\sigma(s)))\Delta s \\ &\geq M \int_{\xi}^{\omega} G(\tau, s) u(\sigma(s))\Delta s \\ &\geq Mm \int_{\xi}^{\omega} G(\tau, s)\Delta s ||u|| \\ &\geq ||u||. \end{aligned}$$

Thus, $||Tu|| \geq ||u||$, and in particular,

$$||Tu|| \geq ||u||, \ \mathit{for}\ u \in P \cap \partial\Omega_1. \tag{15}$$

Now consider $u \in P$ with $||u|| = p$. From condition (B),

$$\begin{aligned} Tu(t) &= \int_0^{\sigma(1)} G(t, s) f(u(\sigma(s)))\Delta s \\ &\leq \int_0^{\sigma(1)} G(\sigma(s), s) f(u(\sigma(s)))\Delta s \\ &\leq \int_0^{\sigma(1)} G(\sigma(s), s)\Delta s p\eta \\ &= ||u||. \end{aligned}$$

If we define $\Omega_2 = \{u \in B : ||u|| < p\}$, then

$$||Tu|| \leq ||u||, \quad u \in P \cap \partial\Omega_2. \tag{16}$$

Theorem 2.1, together with (15) and (16), implies that there exists a fixed point, u_1, satisfying $r < ||u_1|| < p$.

Next, using condition (A) again, we know there exists an $H_1 > 0$ such that

$$f(u) \geq \varepsilon u, \tag{17}$$

for all $u \geq H_1$, where we choose $\varepsilon > 0$ such that,

$$\varepsilon m \int_{\xi}^{\omega} G(\tau, s)\Delta s \geq 1. \tag{18}$$

Let $H = max\{2p, \frac{H_1}{m}\}$ and pick $u \in P$ with $||u|| = H$. Since $u(t) \geq m||u|| \geq H_1$ on $t \in [\xi, \sigma(\omega)]$, we have

$$\begin{aligned} Tu(\tau) &= \int_0^{\sigma(1)} G(\tau, s) f(u(\sigma(s)))\Delta s \\ &\geq \varepsilon \int_{\xi}^{\omega} G(\tau, s) u(\sigma(s))\Delta s \\ &\geq \varepsilon m \int_{\xi}^{\omega} G(\tau, s)\Delta s ||u|| \\ &\geq ||u||. \end{aligned}$$

Set $\Omega_3 = \{u \in B : ||u|| < H\}$. Then

$$||Tu|| \geq ||u||, \quad u \in P \cap \partial\Omega_3. \tag{19}$$

Theorem 2.1, together with (16) and (19), implies that there exists a fixed point, u_2, of T such that $p < ||u_2|| < H$ and the proof is complete.

We now consider the case $f_0 = 0$ and $f_\infty = 0$.

Theorem 3.2. *Assume that f satisfies the following conditions:*
(A') $f_0 = 0$ and $f_\infty = 0$.
(B') There exists a $q > 0$ such that, if $mq \leq x \leq q$, then,

$$f(x) \geq \lambda\, q$$

where,

$$\lambda = \Big(\int_{\xi}^{\omega} G(\tau, s)\Delta s\Big)^{-1}.$$

Then there exist at least two solutions u_1, u_2 of (1), (2) *which belong to P, such that*

$$0 \leq ||u_1|| \leq q \leq ||u_2||.$$

Proof. From $f_0 = 0$ there is an $0 < r < q$ such that $f(u) \leq \eta u$ for $0 \leq u \leq r$ where the constant

$$\eta = \Big(\int_0^{\sigma(1)} G(\sigma(s), s)\Delta s\Big)^{-1}.$$

Let

$$\Omega_1 = \{x \in B \mid ||x|| < r\}.$$

Then

$$\begin{aligned} Tu(t) &= \int_0^{\sigma(1)} G(t,s)f(u(\sigma(s)))\Delta s \\ &\leq \int_0^{\sigma(1)} G(\sigma(s),s)\eta u(\sigma(s))\Delta s \\ &\leq \eta \int_0^{\sigma(1)} G(\sigma(s),s)\Delta s ||u|| \\ &\leq ||u||. \end{aligned}$$

If we define $\Omega_1 = \{u \in B : ||u|| < r\}$, then

$$||Tu|| \leq ||u||, \quad u \in P \cap \partial\Omega_1. \tag{20}$$

Now consider $u \in P$ with $||u|| = q$. Then, from (B'),

$$\begin{aligned} Tu(\tau) &= \int_0^{\sigma(1)} G(\tau,s)f(u(\sigma(s)))\Delta s \\ &\geq \int_\xi^{\omega} G(\tau,s)f(u(\sigma(s)))\Delta s \\ &\geq \lambda \int_\xi^{\omega} G(\tau,s)\Delta s q \\ &\geq ||u||. \end{aligned}$$

Thus, $||Tu|| \geq ||u||$, and in particular, if we define $\Omega_2 = \{u \in B : ||u|| < q\}$, then

$$||Tu|| \geq ||u||, \text{ for } u \in P \cap \partial\Omega_2. \tag{21}$$

Theorem 2.1, together with (20) and (21), implies that there exists a fixed point, u_1, satisfying $r < ||u_1|| < q$. Returning to condition (A'), we know that for any $\varepsilon > 0$, there exists an $M > 0$ such that,

$$f(u) \leq M + \varepsilon u \quad \text{for} \quad u \geq 0,\ 0 \leq t \leq 1.$$

And so,

$$\begin{aligned} Tu(t) &\leq \int_0^{\sigma(1)} G(t,s)[M + \varepsilon u(\sigma(s))]\Delta s \\ &\leq M \int_0^{\sigma(1)} G(\sigma(s),s)\Delta s + \varepsilon \int_0^{\sigma(1)} G(\sigma(s),s)u(\sigma(s))\Delta s. \end{aligned}$$

We choose $\varepsilon > 0$ sufficiently small and $L > \frac{M}{\eta}$ sufficiently large, so that for $u \in P \cap \partial\Omega_3$

$$||Tu|| \leq L = ||u||, \tag{22}$$

where

$$\Omega_3 = \{x \in B : ||u|| < L\}.$$

We obtain the desired result $0 \leq ||u_1|| \leq q \leq ||u_2||$ by appealing to (20), (21), (22) and Theorem 2.1. This completes the proof.

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