TAMKANG JOURNAL OF MATHEMATICS
Volume 30, Number 4, Winter 1999

APPROXIMATION METHODS FOR THE SOLUTIONS OF IMPULSE DIFFERENTIAL EQUATIONS

A. GEORGIEVA AND S. KOSTADINOV

Abstract. In the present paper approximations for the solution of impulse differential equations by solutions of an appropriately constructed ordinary differential equations are found.

1. Introduction

The impulse differential equations are adequate models of processes which are undergoing salutatory changes in their evolutionary development in the form of impulses. These kinds of processes are commonly met in many branches of physics, chemistry, robotics, biotechnology etc. The quantitative theory of these equations has been intensively developing in the last 5 years (see [1]-[4]). In the present paper for a given impulse equation we construct an ordinary equation so that the solutions of these equations are "near" in various senses.

2. Statement of the Problem

We consider the following impulse equation

$$\frac{dx}{dt} = f(t,x) \quad (t \neq t_n) \tag{1}$$

$$x(t_n^+) = Q_n x(t_n) \quad (t = t_n), \tag{2}$$

where $F(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}(\mathbb{R}_+ = [0,\infty))$, $Q_n : \mathbb{R} \to \mathbb{R}$. The points of jump t_n satisfy the following conditions $0 = t_0 < t_n < t_{n+1}(n = 1,2,\cdots), \lim_{n\to\infty} t_n = \infty$.

The process simulated by (1), (2) evolves as follows: the point $P(t) = (t, x(t))$ which maps the trajectory having left the starting point (t_0, x_0), moves along the cruve $\{t, x(t)\}$ determined by the solution of the ordinary differential equation

$$\frac{dx}{dt} = F(t,x) \tag{3}$$

Received December 15, 1997; revised January 12, 1999.
1991 *Mathematics Subject Classification.* 34A37, 45L05
Key words and phrases. One-sided Hausdorff metric, L_p-approximation.

with initial value x_0 at point t_0. The motion along this curve goes on until the point $t_1 > t_0$ at which the point $(t, x(t))$ meets the hyperplane t_1. At the moment $t = t_1$ under the action of the impulse operator Q_1 a transition by a jump is performed from the state $(t_1, x(t_1))$ into the state $(t_1, x(t_1^+))$. Further on, for $t \in (t_1, t_2)$ the mapping point $P(t)$ moves along the curve $\{t, x(t)\}$, where $x(t)$ is a solution of equation (3) with initail valued $x(t_1^+)$. At the moment t_2 new transition by a jump is carried out from the state $(t_2, x(t_2))$ into the state $(t_2, x(t_2^+))$, etc. The motion on the left of the starting point is carried out similarly.

Definition 1. The function $\varphi(t)$ is said to be a solution of (1)-(2) if for $t \neq t_n (n = 1, 2, \cdots)$ it satisfies the ordinary differential equation (1) and for $t = t_n$ the condition of jump (2).

Further we assume that the solutions x of the impulse equation are continuous on the left.

We consider the Cauchy problem for (1), (2)

$$x(t_0) = x_0 \tag{4}$$

It is not hard to check, that the impulse Cauchy problem (1), (2), (4) has for any $x_0 \in \mathbb{R}$ a unique solution off the ordinary Cauchy problem (1), (4) has for any $x_0 \in \mathbb{R}$ a unique solution.

Lemma 1. *The solution of the problem* (1), (2), (4) *for* $t \in (t_n, t_{n+1}]$ *is given by the formula*

$$x(t) = Q_n x(t_n) + \int_{t_n}^{t} F(s, x(s))ds \tag{5}$$

Lemma 1 is proved by straightforward verification.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Let $I(\overline{\mathbb{R}})$ be the set of all intervals and $I(\mathbb{R})$ the set of all bounded intervals. For $\Omega \subset \overline{\mathbb{R}}$ we denote

$$\mathbb{A}_\Omega = \{f : \Omega \to I(\mathbb{R})\}, \qquad A_\Omega = \{f : \Omega \to \mathbb{R}\}$$

Definition 2. [5] The low and upper boundary of Baire of $f \in \mathbb{A}_\Omega$ in the point x we call the number

$$I(f; x) = \lim_{\delta \to 0_+} I(\delta, f; x) \text{ and } S(f; x) = \lim_{\delta \to 0_+} S(\delta, f; x)$$

respectively, where

$$I(\delta, f; x) = \inf\{y : y \in f(t), t \in [x - \delta, x + \delta,] \cap \Omega\}$$
$$S(\delta, f; x) = \sup\{y : y \in f(t), t \in [x - \delta, x + \delta,] \cap \Omega\}$$

By $gr(h)$ we denote the graph of the function $h \in \mathbb{A}_\Omega$, i.e.

$$gr(h) = \{(x,y) : x \in \Omega, y \in f(x)\}.$$

Definition 3. [5] A completion of the graph of the function $f \in \mathbb{A}_\Omega$ we call the intervalued function $F : \mathbb{A}_\Omega \to \mathbb{A}_{\overline{\Omega}}$ defined by the formula

$$F(f;x) = [I(f;x), S(f;x)].$$

Let $\mathbb{F}_\Omega$ be the set of all intervalued functions which coindide with the completion of his graph, i.e.

$$\mathbb{F}_\Omega = \{f \in \mathbb{A}_\Omega : F(f;x) = f(x), x \in \Omega\}.$$

Let $f, g \in \mathbb{F}_\Omega$.

Definition 4. [5] The number

$$h(f,g) = \sup_{(x,y)\in gr(f)} \inf_{(\xi,\eta)\in gr(g)} \max[|x-\xi|, |y-\eta|]$$

is called *one-sided Hausdorff metric.*

We shall find approximations for the solution of the problem (1), (2), (4). For this aim we construct an ordinary differential equation, such that the solution of the two equations are nearly w.r. to the Hausdorff metric.

3. Main Results

We introduce the following condition:

$(H1)$
$$\Delta = \inf_n (t_{n+1} - t_n) > 0.$$

Theorem 1. *We assume that the impulse Cauchy problem* (1), (2), (4) *has for any* $x_0 \in \mathbb{R}$ *a unique solution* $x(t)$ *and that the condition* $(H1)$ *holds. Then for any* $\varepsilon \in (0, \Delta)$ *there exists a function* $G_\varepsilon(t,y)(t \geq 0, y \in \mathbb{R})$ *such that the Cauchy problem*

$$\frac{dy}{dt} = G_\varepsilon(t,y) \tag{6}$$

$$y(t_0) = x_0 \tag{7}$$

has a unique solution $y(t)$ *with* $h(y,x) < \varepsilon$.

Proof. Let $\varepsilon > 0$ be arbitrary choosen. We set

$$G_\varepsilon(t,y) = \begin{cases} F(t,y) + \frac{1}{\varepsilon}(Q_n - I)x(t_n), & t \in (t_n, t_n + \varepsilon] \\ F(t,y), & t \in (t_n + \varepsilon, t_{n+1}] \end{cases} \tag{8}$$

$n = 0, 1, 2, \cdots, Q_0 = I$.

The solution of the ordinary equation (6), (7) is given by the formula.

$$y(t) = y(t_n) + \int_{t_n}^{t} G_\varepsilon(s, y(s))ds \tag{9}$$

It is not diffcult to check that for $n = 1, 2, \cdots$

$$y(t_n) = x(t_n)$$

holds. Indeed from (5), (8) and (9) it follows immediately that

$$x(t_1) = y(t_1)$$

We suppose that $y(t_{n-1}) = x(t_{n-1})$ and we shall prove that $y(t_n) = x(t_n)$.

From (5), (8)and (9) it follows that

$$x(t_n) = Q_{n-1}x(t_{n-1}) + \int_{t_{n-1}}^{t_n} F(s, x(s))ds$$

and

$$\begin{aligned} y(t_n) &= y(t_{n-1}) + \int_{t_{n-1}}^{t_n} G_\varepsilon(s, y(s))ds \\ &= x(t_{n-1}) + \int_{t_{n-1}}^{t_{n-1}+\varepsilon} F(s, y(s))ds + \int_{t_{n-1}}^{t_{n-1}+\varepsilon} \frac{1}{\varepsilon}(Q_{n-1}x(t_{n-1}) - x(t_{n-1}))ds \\ &\quad + \int_{t_{n-1}+\varepsilon}^{t_n} F(s, y(s))ds \\ &= Q_{n-1}x(t_{n-1}) + \int_{t_{n-1}}^{t_n} F(s, x(s))ds \end{aligned}$$

Hence

$$y(t_n) = x(t_n) \tag{10}$$

From (5), (8), (9) and (10) we obtain

$$x(t_n + \varepsilon) = Q_n x(t_n) + \int_{t_n}^{t_n+\varepsilon} F(s, x(s))ds$$

and

$$\begin{aligned} y(t_n + \varepsilon) &= y(t_n) + \int_{t_n}^{t_n+\varepsilon} G_\varepsilon(s, y(s))ds \\ &= x(t_n) + \int_{t_n}^{t_n+\varepsilon} F(s, y(s))ds + \int_{t_n}^{t_n+\varepsilon} \frac{1}{\varepsilon}(Q_n x(t_n) - x(t_n))ds \\ &= Q_n x(t_n) + \int_{t_n}^{t_n+\varepsilon} F(s, x(s))ds \end{aligned}$$

Hence

$$y(t_n + \varepsilon) = x(t_n + \varepsilon) \tag{11}$$

From (8), (10) and (11) we obtain

$$y(t) = x(t), \qquad t \in [t_n + \varepsilon, t_{n+1}] \tag{12}$$

and therefore $h(y, x) < \varepsilon$.

Remark 1. The function $G_\varepsilon(t, y)$ can be defined on the following way too

$$G_\varepsilon(t,y) = \begin{cases} F(t,y) + P_n(t), & t \in (t_n, t_n + \varepsilon] \\ F(t,y), & t \in (t_n + \varepsilon, t_{n+1}] \end{cases}$$

for $n = 0, 1, 2, \cdots$, where $P_0(t) \equiv 0$,

$$P_n(t) = \begin{cases} a_n(t - t_n)(t - (t_n + \varepsilon)), & t \in (t_n, t_{n+\varepsilon}) \\ 0, & t \notin (t_n, t_{n+\varepsilon}) \end{cases}$$

for $n = 1, 2, \cdots$ and the numbers a_n is given by

$$a_n = (Q_n x(t_n) - x(t_n))\Big(\int_{t_n}^{t_n+\varepsilon} (t - t_n)(t - (t_{n+\varepsilon}))dt\Big)^{-1}.$$

Theorem 2. *Let the following conditions are fulfilled:*

1. *The impulse Cauchy problem* (1), (2), (4) *has a unique solution* $x(t)$ *for any* $x_0 \in \mathbb{R}$.
2. *The condition* $(H1)$ *holds.*

Then for any sufficiently small $\varepsilon > 0$ *there exists a function* $G_\varepsilon(t, y), (t \geq 0, y \in \mathbb{R})$ *such that the Cauchy problem* (6), (7) *has a unique solution* $y(t)$ *so that*

$$||y - x||_p = (\int_0^\infty |y(t) - x(t)|^p dt)^{\frac{1}{p}} \leq \varepsilon \qquad (1 \leq p < \infty).$$

Proof. Let $\varepsilon > 0$ be so small that

$$\varepsilon_n = \frac{\varepsilon^p}{|M_n + 1|^p}(\frac{p}{p+1})^{n-1} \leq \Delta \qquad (n = 1, 2 \cdots).$$

Here $M_n := Qx(t_n) - x(t_n)$.
We set

$$G_\varepsilon(t,y) = \begin{cases} F(t,y) + \frac{1}{\varepsilon_n}(Q_n - I)x(t_n), & t \in (t_n, t_n + \varepsilon_n] \\ F(t,y), & t \in (t_n + \varepsilon_n, t_{n+1}] \end{cases}$$

$n = 0, 1, 2, \cdots, Q_0 = I$.

Let $y(t)$ be the solution of the problem (6), (7) and $x(t)$ the solution of the impulse problem (1), (2), (4). Then for $t \in (t_n, t_n + \varepsilon_n)$ we have

$$y(t) - x(t) = (Q_n x(t_n) - x(t_n))\frac{t - (t_n + \varepsilon_n)}{\varepsilon_n}$$

Because $y(t) = x(t), (t \in [t_n + \varepsilon_n, t_{n+1}])$ we have

$$\|y - x\|_p^p = \int_0^\infty |y(t) - x(t)|^p dt = \sum_{n=1}^\infty \int_{t_n}^{t_n+\varepsilon_n} |(Q_n x(t_n) - x(t_n))\frac{t - (t_n + \varepsilon_n)}{\varepsilon_n}|^p dt$$

$$\leq \sum_{n=1}^\infty \frac{|Q_n x(t_n) - x(t_n)|^p}{\varepsilon_n^p} \int_{t_n}^{t_n+\varepsilon_n} (t_n + \varepsilon_n - t)^p dt \leq \sum_{n=1}^\infty \frac{|M_n|^p}{\varepsilon_n^p}\frac{\varepsilon_n^{p+1}}{p+1} = \varepsilon^p$$

i.e. $\|x - y\|_p < \varepsilon$.

Remark 2. The function $G_\varepsilon(t, y)$ can be defined by the following way too.

$$G_\varepsilon(t, y) = \begin{cases} F(t, y) + P_n(t), & t \in (t_n, t_n + \varepsilon_n] \\ F(t, y), & t \in (t_n + \varepsilon_n, t_{n+1}] \end{cases} \tag{13}$$

for $n = 0, 1, 2, \cdots$, where $P_0(t) \equiv 0$,

$$P_n(t) = \begin{cases} a_n(t - t_n)(t - (t_n + \varepsilon_n)), & t \in (t_n, t_n + \varepsilon_n) \\ 0, & t \notin (t_n, t_n + \varepsilon_n) \end{cases}$$

for $n = 1, 2, \cdots$ and the number a_n is given by

$$a_n = (Q_n x(t_n) - x(t_n))\Big(\int_{t_n}^{t_n+\varepsilon_n} (t - t_n)(t - (t_n + \varepsilon_n))dt\Big)^{-1},$$

i.e.

$$a_n = -\frac{6M_n}{\varepsilon_n^3}.$$

Let $\varepsilon > 0$ is so small that

$$\varepsilon_n = (\frac{2}{3})^p \frac{\varepsilon^p}{|M_n + 1|^p}(\frac{p}{p+1})^{n-1} \leq \Delta \qquad (n = 1, 2, \cdots).$$

Then for $t \in (t_n, t_n + \varepsilon_n)$ we have

$$y(t) - x(t) = x(t_n) - Q_n x(t_n) + \int_{t_n}^t P_n(s)ds = \int_{t_n+\varepsilon_n}^t P_n(s)ds$$

$$|y(t) - x(t)| \leq \int_t^{t_n+\varepsilon_n} |P_n(s)|ds \leq \max_t |P_n(t)|(t_n + \varepsilon_n - t)$$

$$= |P_n(t_n + \frac{\varepsilon_n}{2})|(t_n + \varepsilon_n - t) = \frac{6|M_n|}{\varepsilon_n^3}\frac{\varepsilon_n^2}{4}(t_n + \varepsilon_n - t) = \frac{3|M_n|}{2\varepsilon_n}(t_n + \varepsilon_n - t)$$

Hence

$$\begin{aligned}||y-x||_p^p &= \int_0^\infty |y(t)-x(t)|^p dt \\ &\le \sum_{n=1}^\infty \int_{t_n}^{t_n+\varepsilon_n} \left(\frac{3}{2}\right)^p \frac{|M_n|^p}{\varepsilon_n^p}(t_n+\varepsilon_n-t)^p dt = \left(\frac{3}{2}\right)^p \sum_{n=1}^\infty |M_n|^p \frac{\varepsilon_n}{p+1} = \varepsilon^p,\end{aligned}$$

i.e. $||y-x||_p < \varepsilon$.

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Faculty of Mathematics and Informatics, University of Plovdiv, Paissi Hilendarski, 4000 Plovdiv, Bulgaria.