TAMKANG JOURNAL OF MATHEMATICS
Volume 30, Number 4, Winter 1999

OPTIMAL CONTROL FOR SYSTEMS GOVERNED BY DISCONTINUOUS NONLINEARITY

NGUYEN BUONG

Abstract. The aim of this paper is to present an existence theorem of optimal control for systems descrided by the operator equation of Hammerstein type $x + KF(u,x) = 0$ with the discontinuous monotone nonlinear operator F in x. Then, the theoretical result is applied to investigate an optimal control problem for system, where the state is written in the form of nonlinear integral equations in $L_p(\Omega)$.

1. Introduction

More and more technical and physical problems have been recently formulated in the form of equations of Hammerstein type, see, e.g., [2]-[4], [6]-[8], [10]-[18], [21], [22]. We often need to control such systems optimally. For the case when the nonlinear operator F is smooth, such control problems are considered in [1], [9], [14] and [23]. From the optimal control point of view, the main difficulty consists of the fact that the nonlinear operator F is generally discontinuous in x. This situation usually arises in optimal control for system with variable structure (see [11]). In this paper, for the last case, by using the method of monotone operators we shall prove an existence theorem of optimal control and give one application of the result.

Let X be a real Banach space and X^* be its dual which are uniformly convex. For the sake of simplicity, the norms of X and X^* will be denoted by one symbol $\|\cdot\|$. We write $\langle x^*, x\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let $F : \mathcal{U}_{ad} \times X \to X^*$ be a discontinuous operator in x, where $\mathcal{U}_{ad} \subset U$, U is another reflexive Banach space of controls, and $\mathcal{U}_{ad}$ is a convex, closed and nonempty set. Let $K : X^* \to X$ be a bounded (i.e. image of any bounded subset is bounded), nonegative and linear operator.

Consider the problem of optimal control: find $u_0 \in \mathcal{U}_{ad}$ such that

$$\begin{aligned} J(u_0) &= \min J(u), \quad u \in \mathcal{U}_{ad} \\ J(u) &= J(u, x(u)), \end{aligned} \tag{1.1}$$

where $J : U \to R^1$ is a cost function having the following properties:

Received October 14, 1998.
1991 *Mathematics Subject Classification.* 49J20.
Key words and phrases. Monotone operators, uniform convex Banach spaces.

• $J(u,x)$ is l.s.c. (lower semicontinuous) weakly in u and in x, i.e. $u_n \rightharpoonup u$, $x_n \rightharpoonup x \Rightarrow \liminf J(u_n, x_n) \geq J(u,x)$, and

• $J(u,x)$ is coercive in $u \in \mathcal{U}_{ad}$, i.e. $\liminf J(u,x) = +\infty$ as $\|u\| \to +\infty$, $u \in \mathcal{U}_{ad}$ uniformly with respect to x,

the symbols $\to$ and $\rightharpoonup$ denote convergence in norm and weak convergence, respectively, and the state of the considered system $x(u)$ is described by the operator equation

$$x + KF(u,x) = 0 \tag{1.2}$$

where $F(u,x)$ possesses the following properties:

i) $F : \mathcal{U}_{ad} \times X \to X^*$ is bounded,
ii) for every fixed $x \in X$, $F(\cdot, x)$ is weakly continuous in u, and
iii) for every fixed $u \in \mathcal{U}_{ad}$, $F(u,\cdot)$ is monotone in x, i.e.

$$\langle F(u,x) - F(u,y), x - y\rangle \geq 0, \qquad \forall_u \in \mathcal{U}_{ad}, \quad x, y \in X.$$

2. Main Result

Integral equations of Hammerstein type with the nonlinear discontinuous operator F was investigated in [8], [12], [20] by introducing a new concept of solution. But, throughout this paper the word 'solution' is meant in the classical sense.

Definition 1 (see [19]). A point $x \in X$ is called *a point of h-continuity of the operator* $G : X \to X^*$ if

$$\forall l \in X \lim_{t \to 0_+} \langle G(x + tl), l\rangle = \langle G(x), l\rangle.$$

A point $x \in X$ is called *a point of discontinuity*, if x does not satisfy the condition in Definition 1.

Definition 2. A discontinuous point x of G is called regular, if

$$\exists l \in X : \lim_{t \to 0_+} \langle G(x + tl), l\rangle < 0.$$

Theorem 2.1. *Assume that $K : X^* \to X$ is a linear, bounded and nonegative operator, conditions $i) - iii)$ hold, all the discontinuous points of F are regular, and that there exists a positive constant r such that*

$$\langle F(u,x), x\rangle > 0, \quad if \quad \|x\| > r, \quad \forall u \in \mathcal{U}_{ad}.$$

Then, Eq. (1.2) has a solution $x(u)$, for each $u \in \mathcal{U}_{ad}$.

Proof. As in [6], consider the regularized equation

$$x + B_n F(u,x) = 0, \qquad B_n = B + \alpha_n V, \tag{2.1}$$

where V is the standard dual mapping of X^*, i.e.,

$$\langle V(x^*), x^*\rangle = \|V(x^*)\|\|x^*\| = \|x^*\|^2, \qquad \forall x^* \in X^*,$$

and α_n is the sequence of positive real numbers such that $\alpha_n \to 0$, as $n \to +\infty$. Then, $R(B_n) = X$, $B_n^{-1}(0) = 0$, B_n^{-1} is an one-to-one mapping, and B_n^{-1} is continuous (see [5]). Therefore, all the discontinuous points of F in x are the discontinuous points of $\tilde{B}_n + F$, and inversely, all the discontinuous points of $\tilde{B}_n + F$ are the discontinuous points of F, where $\tilde{B}_n(x) = -B_n^{-1}(-x)$. Obviously, we can rewrite Eq. (2.1) in the form

$$\tilde{B}_n(x) + F(u, x) = 0. \tag{2.2}$$

By virtue of [19], Eq. (2.2) has a unique solution, henthforth denoted by $x_n(u)$. Moreover, $\|x_n(u)\| \leq r$, $\forall n$. As F is bounded, the sequence $\{F(u, x_n)\}$ is bounded, too. Without loss of generality, assume that

$$x_n \rightharpoonup x_0, \text{ and } F(u, x_n) \rightharpoonup y_0^*.$$

From (2.1) it implies that

$$x_0 + By_0^* = 0. \tag{2.3}$$

Now, we have to prove that $y_0^* = F(u, x_0)$. Since F is monotone, then

$$\langle F(u, x) - F(u, x_n), x - x_n\rangle \geq 0, \qquad \forall x \in X.$$

Hence,

$$\langle F(u, x), x - x_n\rangle - \langle F(u, x_n), x\rangle \geq \langle F(u, x_n), BF(u, x_n)\rangle + \alpha_n\langle F(u, x_n), VF(u, x_n)\rangle.$$

By passing $n \to +\infty$ in the last inequality, because of

$$\liminf\langle F(u, x_n), BF(u, x_n)\rangle \geq \langle y_0^*, By_0^*\rangle,$$
$$\lim_{n\to+\infty} \alpha_n\langle F(u, x_n), VF(u, x_n)\rangle = 0,$$

and (2.3) we have

$$\langle F(u, x) - y_0^*, x - x_0\rangle \geq 0. \tag{2.4}$$

Replacing, for any $l \in X$ and $t > 0$, x by $x_0 + tl$ in (2.4) we see that

$$\langle F(u, x_0 + tl) - y_0^*, l\rangle \geq 0, \qquad \forall l \in X.$$

Hence, x_0 is the point of h-continuity of F. Consequently, from (2.4) and Minty's lemma (see [22]) $y_0^* = F(u, x_0)$. Theorem is proved.

Remark. If one of the operators K and $F(u, \cdot)$ is strictly monotone, then the solution $x(u)$ is unique.

Theorem 2.2. *Assume that all the conditions in Theorem 2.1 hold, and that J is l.s.c. weakly in u, x and coercive in u, then Problem (1.1) has an optimal control u_0.*

Proof. Let u_n be a minimizing sequence of the functional $J(u)$. As $J(u)$ is coercive, the sequence $\{u_n\}$ is bounded. We denote by x_n the solution of Eq. (2.1) with $u = u_n$. We shall prove that the sequence $\{x_n\}$ also is bounded. To do this, we construct the Banach space $Z = X \times X^*$ with the norm $\|z\|^2 = \|x\|^2 + \|x^*\|^2$, $z = [x, x^*]$, $x \in X$ and $x^* \in X^*$. Consider the operator $\mathcal{A}(u,z) := [F(u,x), Kx^*] + [-x^*, x]$. Obviously, $z_n = [x_n, x_n^*]$ is a solution of the equation

$$\mathcal{A}(u_n, z_n) = 0$$

if and only if x_n is a solution of the equation (2.1) with $u = u_n$. The space Z has the dual space $Z^* = X^* \times X$ and the operator $\mathcal{A} : U \times Z \to Z^*$ is also a monotone operator in z and weakly continuous in u (see [6]). Certainly, $\mathcal{A}$ is discontinuous in z because of discontinuity of F in x.

On the other hand, from condition ii) we obtain

$$\Big(\gamma(\|x_n\|) - \|F(u_n, 0)\|\Big)\|x_n\| \le \langle \mathcal{A}(u_n, z_n) - \mathcal{A}(u_n, 0), z_n\rangle \le \|F(u_n, 0)\|\|x_n\|.$$

Therefore, the sequence $\{x_n\}$ also is bounded. Let $u_n \rightharpoonup u_0$, $z_n \rightharpoonup z_0 := [x_0, x_0^*]$. We shall prove that x_0 is a solution of (1.2) with $u = u_0$. Since $\langle \mathcal{A}(u_n, z_n), z - z_n\rangle = 0$, then from

$$\langle \mathcal{A}(u_n, z) - \mathcal{A}(u_n, z_n), z - z_n\rangle \ge 0, \qquad \forall z \in Z$$

it follows

$$\langle \mathcal{A}(u_0, z), z - z_0\rangle \ge 0, \qquad \forall z \in Z.$$

Thus, z_0 can only be the point of h-continuity of the operator $\mathcal{A}$. So, $\mathcal{A}(u_0, z_0) = 0$. In other words, $x_0 = x(u_0)$ is a solution of (1.2) with $u = u_0$. From the weak l.s.c. of J and $x_n \rightharpoonup x_0$ (see [7]) we have

$$J(u_0) \le \liminf J(u_n).$$

Theorem is proved.

3. Example

Consider a system described by the nonlinear integral equation

$$x(s) + \int_\Omega k(s,t) f(u, x(t))dt = 0, \tag{3.1}$$

where the kernel function $k(s,t)$ is such that the operator K defined by

$$(Kx)(s) = \int_\Omega k(s,t)x(t)dt$$

is nonnegative, and K acts from $L_q(\Omega)$ into $L_p(\Omega)$, $L_p(\Omega)$ denotes the space of p-summable functions in the σ-finite measure set $\Omega \subset R^n$, with $p^{-1} + q^{-1} = 1$, and the nonlinear function $f(u,t)$ satisfies the following conditions:

a) $f(u,t)t \geq a_0|t|^p + b_0|t|^\gamma + c_0, a_0 > 0, b_0 < 0, c_0 < 0, \gamma < p$,
b) $f(u,t)$ is not decreasing, and is right continuous in t, at the point of discontinuity t_0 $f(u, t_0 - 0)f(u, t_0) > 0$, for every fixed $u \in \mathcal{U}_{ad}$,
c) $|f(u,t)| \leq a_1 + b_1|t|^{p-1}$, $\forall t \in R^1$, $a_1 + b_1 > 0$, $a_1 \geq 0$, $b_1 \geq 0$, and
d) $f(\cdot, x(t)) : U \to L_q(\Omega)$ is, for every fixed $x(t) \in L_p(\Omega)$, compact.

Because of c) we define the operator $F : X = \mathcal{U}_{ad} \times L_p(\Omega) \to X^* = L_q(\Omega)$ as follows

$$F(u,x)(t) = f(u,x(t)), \qquad \forall x(t) \in L_p(\Omega),$$

where $\mathcal{U}_{ad}$ is a convex, closed subset in a real reflexive Banach space of controls U. Then, Eq. (3.1) can be rewritten in the form (1.1), where the defined operator $F(u,x)$ possesses all the properties from Section 1. Indeed, condition a) guarantees the existence of r in Theorem 2.1, the monotone property and the regularity of discontinuity points of $F(u,x)$ in x follow from condition b) (see [19]), and remaining properties are verified on base of conditions c) and d). Therefore, for each fixed $u \in \mathcal{U}_{ad}$, Eq. (3.1) has only a solution, and the solution is unique if one of K and F is strictly monotone.

Moreover, we are given these additional data: W is a real Banach space (observation space), $C : L_p(\Omega) \to W$ a linear continuous operator, $w_d \in W$ a fixed element. Then, it is easy to see that the cost function

$$J(u) = J(u,x) = ||Cx - w_d||_W^\mu + A||u||_U^\theta, \tag{3.2}$$

where A, μ, θ are the positive constants, and $||\cdot||_W$ denotes the norm of W, is coercive in u uniformly with respect to x and l.s.c. weakly in x and in u. Therefore, Probl.(3.1) and (3.2) has an optimal control u_0.

References

[1] D. E. Akbarov, *Necessary condition of optimality in form of variational inequalities for objects described by equations of Hammerstein type*, Cybern. and System Analysis, **1**(1996), 146-152(in Russian).

[2] H. Amann, *Ein Existenz und Eindeutigkeitssatz fur die Hammersteinshe Gleichung in Banachraumen*, Math. Z., **111**(1969), 175-190.

[3] H. Amann, *Uber die naherungsweise Losung nichlinearer Integralgleichungen*, Numer. Math., **19** (1972), 29-45.

[4] H. Brezis and F. Browder, *Nonlinear integral equations and systems of Hammerstein's type*, Adv. Math., **10** (1975), 115-144.

[5] H. Brezis and M. Sibony, *Méthod d'Approximation et d'Itération pour les opérateurs monotones*, Arch. Rat. Mech. Anal., **28**(1968), 59-82.

[6] N. Buong, *On approximate solutions of Hammerstein equation in Banach spaces*, Ukrainian Math. J., **8**(1985), 1256-1260(in Russian).

[7] N. Buong, *On solutions of the equations of Hammerstein type in Banach spaces*, J. of Math. Computation & Math. Physics, **25**(1985), 1256-1280(in Russian).

[8] N. Buong, *On solution of the operator equation of Hammerstein type with semimonotone and discontinuous nonlinearity*, J. of Math., **12**(1984), 25-28(in Vietnamese).

[9] N. Buong, *Concerning optimal control of one class of nonlinear equation in the Banach space*, Ukrainian Math. J., **42**(1990), 539-541(in Russian).
[10] N. Buong, *On approximate solution for operator equation of Hammerstein type*, J. of Comp. and Applied Math., **75**(1996), 77-86.
[11] C. V. Emellianov, *Systems of automatic control with variable structure*, M. Nauka, 1967(in Russian).
[12] D. P. Gaidarov and P. K. Raguimkhanov, *On integral inclusion of Hammerstein*, Sibirian Math. J., **21**(1980), 19-24(in Russian).
[13] C. Gupta, *On nonlinear integral equations of Hammerstein type*, Functional Analysis, Lecture Notes in Math. Springer-Verlage, New-York, **384**(1973), 184-238.
[14] M. Joshi, *On the existence of optimal control in Banach space*, Bull. Austr. Math. Soc., **27**(1983), 395-401.
[15] H. Kaneko and Y. Xu, *Degenerate kernel method for Hammerstein equation*, Math. Comp., **56**(1991), 141-148.
[16] H. Kaneko, R. Noren and Y. Xu, *Numerical solutions for weakly singular Hammerstein equations and their supperconvergence*, J. Int. Eqs. Appl., **4**(1992), 391-407.
[17] S. Kumar, *Superconvergence of a collocation-type method for Hammerstein equations*, IMA Journal of Numerical Analysis, **7**(1987), 313-325.
[18] C. D. Panchal, *Existence theorems for equations of Hammerstein type*, The Quart. J. of Math., **35**(1984), 311-320.
[19] V. N. Pavlenko, *Nonlinear equations with discontinuous operators in Banach spaces*, Ukrainian Math. J., **31**(1979), 569-572(in Russian).
[20] P. K. Raguimkhanov, *Concerning an existence problem for solution of Hammerstein equation with discontinuous mappings*, Izvestia Vuzov, Math., **10**(1975), 62-70(in Russian).
[21] D. Vaclav, Monotone Operators and Applications in Control and Network Theory, Elsevier, New York, 1979.
[22] M. M. Vainberg, Variational Method and Method of Monotone Operators, Moscow, Nauka, 1972, (in Russian).
[23] L. V. Wolfersdorf, *Optimal control of a class of process described by general integral equations of Hammerstein type*, Math. Nachr., **71**(1976), 115-141.

National Centre for Natural Sciences and Technology of Vietnam, Institute of Information Technology, Nghia do, Tu liem, Ha noi, Vietnam 10000.

Fax: (84)4-345217

e-mail: nbuong@math-ioit.ac.vn