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NOTE ON 1-DIMENSIONAL INTEGRALLY CLOSED MORI SEMIGROUPS

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Abstract. We show that a 1-dimensional integrally closed quasi-local Mori semi-group need not be a valuation semigroup, which is a negative answer to a semigroup version of the question of Querré.

1. Introduction

In [4], Querré states that a 1-dimensional integrally closed quasi-local Mori domain is a discrete valuation domain. As a semigroup version of this statement, the second author posed the question [3, (3.17)]: Is a 1-dimensional integrally closed quasi-local Mori semigroup a discrete valuation semigroup of rank 1? For integral domains, V. Barruci [1] showed that a 1-dimensional integrally closed quasi-local Mori domain need not be a discrete valuation domain. Our aim of this short note is to answer the question in the negative.

Notation: Let $S \ni 0$ be a subsemigroup of a torsion-free abelian group, with the binary operation $+$. Let $q(S)$ denote the quotient group of S. A subsemigroup of $q(S)$ containing S is called an *oversemigroup* of S. A mapping v from a torsion-free abelian group G onto a totally ordered additive group Γ is called a *Γ-valued valuation* on G if $v(x+y) = v(x)+v(y)$ are satisfied for all $x, y \in G$. The subsemigroup $\{x \in G : v(x) \geq 0\}$ of G is called the *valuation semigroup* of G associated with v. A $\mathbb{Z}$-valued valuation is called a *discrete valuation of rank 1*. A *discrete valuation semigroup of rank 1* is the valuation semigroup associated with a discrete valuation of rank 1. An element x of an extension semigroup T of S is called *integral* over S if $nx \in S$ for some $n \in \mathbb{N}$. Let $\overline{S}$ be the set of all integral element of $q(S)$ over S. Then it is readily seen that $\overline{S}$ is an oversemigroup of S. We call $\overline{S}$ the *intrgral closure* of S. If $S = \overline{S}$, then S is called *integrally closed.* The supremum of the length of the strict chains of proper prime ideals $P_1 \subset P_2 \subset \cdots \subset P_n$ of S is called the *dimension* of S, which is denoted by $\dim(S)$. For a non-empty subset A of $q(S)$, we put $A^{-1} = \{x \in q(S) : x + A \subseteq S\}$, and $(A^{-1})^{-1} = A^v$. An ideal I of S is called *divisorial* if $I = I^v$. S is called a *Mori* semigroup if every ascending chain $I_1 \subset I_2 \subset \cdots$ of ideals of S, which are divisorial, is stationary.

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For other unexplained notation, our general reference is [2].

Let F be a torsion-free abelian group, and H a proper subgroup of F. We assume that H is integrally closed in F. Observe that $\mathbb{Z} \oplus \{0\} \subset \mathbb{Z} \oplus \mathbb{Z}$ is such an extension of semigroups. Let V be a discrete valuation semigroup of rank 1, of the form $F \cup M$, where M is the maximal ideal of V. Let v be the valuation associated with V. Here is an example satisfying all the properties above: Let $G = F + \mathbb{Z}X$ with a symbol X. It is readily seen that G is a torsion-free abelian group. Now define a mapping v from G to $\mathbb{Z}$ by $v(\alpha + nX) = n$ for $\alpha \in F$ and $n \in \mathbb{Z}$. It then follows that v is a discrete valuation on G, and $V = F + \mathbb{Z}_+X$ is the valuation semigroup associated with v, where $\mathbb{Z}_+$ is the non-negative integers. Further the maximal ideal of V is $M = F + \mathbb{N}X$, and hence $V = F \cup M$. Consider the subsemigroup $S = H \cup M$ of V. We have $q(S) = q(V) = G$, and M is the unique maximal ideal of S.

Under these preparation, we can prove that S *is a 1-dimensional integrally closed quasi-local Mori semigroup that is neither a valuation nor a Noetherian semigroup.* We will complete the proof in six steps. In the steps 1 and 2, similar arguments to those of [1] will be used.

Step 1. We show that if I is a non-principal divisorial ideal of S, then I is an ideal of V. Let I be a non-principal divisorial ideal of S. Let $y \in I^{-1}$. It is readily seen that $y + I$ is a proper subset of S, since I is not principal. Therefore $y + I$ is contained in the maximal ideal M. It then follows that $y + I + V \subseteq M + V = M \subseteq S$. Hence $I^{-1} + I \subseteq S$, or $I + V \subseteq I^v$. Since I is divisorial, it follows that I is an ideal of V, completing Step 1.

Step 2. In order to see that S is a Mori semigroup, let $I_1 \subset I_2 \subset \cdots$ be an ascending chain of ideals of S, which are divisorial. We must show that the chain is stationary. Now Step 1 reduces it to the following two cases: (1) each I_n is an ideal of V, and (2) each I_n is a principal ideal of S.

If case (1) happens, then we see that $k_n = \min\{v(x) : x \in I_n\}$ form a descending sequence $k_1 \geq k_2 \geq \cdots \geq 0$. Hence we find $r \in \mathbb{N}$ such that $k_r = k_n$ for each $n \geq r$. Since each I_n is an ideal of V, we have that $I_r = I_n$ for each $n \geq r$.

If case (2) happens, then each I_n is of the form $S + a_n$ for some $a_n \in S$. Then the chain $V + a_1 \subseteq V + a_2 \subseteq \cdots$ must be stationary. Let $r \in \mathbb{N}$ be such that $V + a_r = V + a_n$ for each $n \geq r$. Let $a_n = v_n + a_{n+1}$ with $v_n \in V$. Then we have $v_n = a_n - a_{n+1}$ is in $S \backslash M$ for each $n \geq r$. Hence $I_r = I_n$ for each $n \geq r$.

Thus S is a Mori semigroup.

Step 3. We will show that S is not a valuation semigroup. To this end, suppose that S be a valuation semigroup. Choose $\alpha \in F\backslash H$. Then $q(S) = G$ implies that either α or $-\alpha$ is contained in S. This in turn shows $\alpha \in H$, a contradiction. Hence S is not a valuation semigroup.

Step 4. In order to see $\dim(S) = 1$, let P be a proper prime ideal of S. Let $x \in M$. Take $y \in P$. We can find $n \in \mathbb{N}$ such that $v(nx) > v(y)$. It then follows that $nx - y \in M$, and so $nx \in P$. Hence $x \in P$. Thus $P = M$.

Step 5. To show that S is integrally closed, let $\alpha \in G$ be intrgral over S. Since V is a valuation semigroup on G, we have $\alpha \in V$. If $\alpha \in F$, then we have $\alpha \in H$, since H is integrally closed in F. If $\alpha \notin F$, then $\alpha \in M$. Thus S is integraly closed.

Step 6. It remains to show that S is not Noetherian. To this end we show that M is not finitely generated. For a contradiction we suppose that M is finitely generated. Let $x_1, \ldots, x_n$ be a minimal generator of M so that $M = (S + x_1) \cup \cdots \cup (S + x_n)$. Note that each $v(x_i) = 1$. Now take $\alpha \in F \backslash H$. It is readily seen that $\alpha + x_1 \in M$, and further $\alpha + x_1 \notin S + x_1$. Hence we have $\alpha + x_1 \in H + x_2$, say. Let $k > 1$. We assume that we can arrange the $x_i's$ in such a way that $\alpha + x_i \notin \bigcup_{j=1}^{i}(S + x_j)$, and $\alpha + x_i \in H + x_{i+1}$ for each $1 \leq i \leq k-1$. We want to show that $\alpha + x_k \notin \bigcup_{j=1}^{k}(S + x_j)$. If otherwise, then we would find $1 \leq j \leq k$ such that $\alpha + x_k \in H + x_j$. Then we see that $\alpha + x_j \in H + x_{j+1}, \ldots, \alpha + x_{k-1} \in H + x_k, \alpha + x_k \in H + x_j$ yield $(k - j + 1)\alpha \in H$. This implies that $\alpha \in H$, since H is integrally closed in F. This contradiction completes $\alpha + x_k \notin \bigcup_{j=1}^{k}(S + x_j)$. Thus in particular $\alpha + x_n \notin M$, the required contradiction.

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