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A NOTE ON A THEOREM OF BOSANQUET

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Abstract. In this short note we have proved a general theorem on $|C,1|_k$ summability methods which generalizes a result of Bosanquet [3].

1. Introduction

Let Σa_n be given infinite series with partial sums (s_n). By z_n and t_n we denote the n-th $(C,1)$ means of the sequences (s_n) and (na_n), respectively. The series Σa_n is said to be summable $|C,1|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} |z_n - z_{n-1}|^k < \infty. \tag{1}$$

But since $t_n = n(z_n - z_{n-1})$ (see [6]) condition (1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n|^k < \infty. \tag{2}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty, \quad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{3}$$

The sequence-to-sequence transformation

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{4}$$

defines the sequence (u_n) of $(\bar{N}, p_n)$ means of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [5]).

The series Σa_n is said to be summable $|\bar{N}, p_n|_k$ $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |u_n - u_{n-1}|^k < \infty. \tag{5}$$

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In the special case when $p_n = 1$ for all values of n, $|\bar{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

The following theorem is known.

Theorem A([2]). *Let $k \geq 1$. Suppose (p_n) and (q_n) are positive sequences such that $P_n \to \infty$ as $n \to \infty$ and $Q_n \to \infty$ as $n \to \infty$. If $\Sigma a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, whenever Σa_n is summable $|\bar{N}, q_n|_k$, then*

$$\lambda_n = O\{\frac{q_n P_n}{p_n Q_n}\}^{\frac{1}{k}} \tag{6}$$

and

$$\Delta\lambda_n = O\{\frac{q_n}{Q_n}\}^{\frac{1}{k}} \tag{7}$$

2.

One of the simplest and most basic results on absolute Cesaro summability factors is the following theorem due to Bosanquet [3].

Theorem B. *Necessary and sufficient conditions for $\Sigma a_n \lambda_n$ to be summable $|C, 1|$, whenever Σa_n is summable $|C, 1|$ are*

$$\lambda_n = O(1) \tag{8}$$

and

$$\Delta\lambda_n = O(\frac{1}{n}) \text{ as } n \to \infty. \tag{9}$$

It may be noted that these two conditions are sufficient for $\Sigma a_n \lambda_n$ to be summable $|C, 1|_k$, whenever Σa_n is summable $|C, 1|_k$, $k \geq 1$. In fact, let (T_n) be the $(C, 1)$ mean of the sequence $(n a_n \lambda_n)$. hence

$$\begin{aligned} T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v &= \frac{1}{n+1} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{r=t}^{v} r a_r + \frac{\lambda_n}{n+1} \sum_{v=1}^{n} v a_v \\ &= \frac{1}{n+1} \sum_{v=1}^{n-1} (v+1) \Delta\lambda_v t_v + \lambda_n t_n \\ &= T_{n,1} + T_{n,2}, \text{ say.} \end{aligned}$$

Since $|T_{n,1} + T_{n,2}|^k < 2^k(|T_{n,1}|^k + |T_{n,2}|^k)$, to complete the proof of this result it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2. \tag{10}$$

Since $\Delta\lambda_n = O(\frac{1}{n})$, by (9), firstly by Hölder's inequality when $k > 1$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n}|T_{n,1}|^k &= \sum_{n=2}^{m+1} \frac{1}{n}\Big|\frac{1}{n+1}\sum_{v=1}^{n-1}(v+1)t_v\Delta\lambda_v\Big|^k \\
&\le \sum_{n=2}^{m+1} \frac{1}{n^{k+1}}\{\sum_{v=1}^{n-1}(v+1)|t_v||\Delta\lambda_v|\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{n^{k+1}}\{\sum_{v=1}^{n-1}\frac{v+1}{v}|t_v|\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{n^{k+1}}\{\sum_{v=1}^{n-1}|t_v|\}^k \\
&= O(1)\sum_{n=2}^{m+1} \frac{1}{n^2}\sum_{v=1}^{n-1}|t_v|^k \times \{\frac{1}{n}\sum_{v=1}^{n-1}1\}^{k-1} \\
&= O(1)\sum_{v=1}^{m}|t_v|^k \sum_{n=v+1}^{m}\frac{1}{n^2} \\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}|t_v|^k \\
&= O(1) \text{ as } m\to\infty, \text{ by (2).}
\end{aligned}$$

Finally, since $\lambda_n = O(1)$, by (8), we have

$$\sum_{n=1}^{m}\frac{1}{n}|T_{n,2}|^k = \sum_{n=1}^{m}\frac{1}{n}|\lambda_n t_n|^k = O(1)\sum_{n=1}^{m}\frac{1}{n}|t_n|^k = O(1) \text{ as } m\to\infty, \quad \text{by (2).}$$

Therefore we get that

$$\sum_{n=1}^{\infty}\frac{1}{n}|T_{n,r}|^k = O(1) \text{ as } m\to\infty \text{ for } r=1,2.$$

This completes the proof of the result.

So it is natural to ask whether they are also necessary when $k > 1$. We show they are not. In fact, taking $k > 1$ we show, a special case, that $\Sigma a_n\lambda_n$ is summable $|C,1|_k$, whenever Σa_n is summable $|C,1|_k$, in the case in which

$$\Delta\lambda_n = \begin{cases} 2^{-i/k} & (n = 2^i,\ i = 0,1,2,\ldots) \\ 0 & \text{otherwise} \end{cases}$$

Note that since $\Sigma\Delta\lambda_n$ converges, this implies that λ_n is bounded. Thus (8) holds. However (9) does not; thus (9) is not necessary when $k > 1$. Now, we give the following main theorem.

Theorem. *Let $k > 1$. If $\Sigma a_n \lambda_n$ is summable $|C,1|_k$, whenever Σa_n is summable $|C,1|_k$, then $\lambda_n = O(1)$ and $\Delta\lambda_n = O(1/n)^{1/k}$.*

Proof. If we take $p_n = q_n = 1$ for all values of n in Theorem A, then we get the result.

References

[1] H. Bor, *On two summability methods*, Math. Proc. Cambridge Phil. Soc. **97**(1985), 147-149.

[2] H. Bor, *On absolute weighted mean summability methods*, Bull. London Math. Soc. **25**(1993), 265-268.

[3] L. S. Bosanquet, *Note on convergence and summability factors*, J. London Math. Soc. **20**(1945), 39-48.

[4] T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and paley*, Proc. London Math. Soc. **7**(1957), 113-141.

[5] G. H. Hardy, Divergent Series, Oxford University press, 1949.

[6] E. Kogbetliantz, *Sur les series absolument sommables par la methode des moyannes arit- metiques*, Bull. Sci. Math. **49**(1925), 234-256.

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