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CAUCHY PROBLEM AND REDUCTION OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The Cauchy problem for systems of homogeneous and also nonlinear partial differential equations is considered. If the compatibility conditions are satisfied, the solution is represented as functional series. The algorithm for reduction of a system of partial differential equations with linear homogeneous algebraic constraints is considered. It is proved that the compatibility conditions for the reduced system are identically satisfied.

1. Introduction

In the paper vector notations are used, where the vectors and matrices are in bold. For this notation we assume that the differential (row) operator $\frac{\partial}{\partial \mathbf{x}}$ denotes $[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$. For example, if φ is a scalar function, then

$$\frac{\partial \varphi}{\partial \mathbf{x}} = [\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n}],$$

and for any vector function $\mathbf{h}(x_1, \ldots, x_n) = \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix}$ we use the notation

$$\frac{\partial \mathbf{h}}{\partial \mathbf{x}} = \begin{bmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 & \cdots \partial h_1/\partial x_n \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 & \cdots \partial h_2/\partial x_n \\ \vdots & \vdots & \ldots \vdots \\ \partial h_m/\partial x_1 & \partial h_m/\partial x_2 & \cdots \partial h_m/\partial x_n \end{bmatrix}.$$

This paper is based on the papers [5,6,7,8] and represents a further generalization of them. So we will give a brief overview of these papers.

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In [6] a Charpit system of partial differential equations (PDEs)

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \cdot \mathbf{h}_i(\mathbf{x}, \mathbf{y}) + \mathbf{g}_i(\mathbf{x}, \mathbf{y}) = 0, \quad i = 1, \ldots, p, \tag{1.1}$$

is considered together with the algebraic constraints

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{z}) = \mathbf{0}, \tag{1.2}$$

where the unknown vector function $\mathbf{y}$ is m-dimensional and depends on the n-dimensional variable $\mathbf{x}$, while $\mathbf{h}_i$, $\mathbf{g}_i$, $\mathbf{f}$ are given vector functions. Then it is expanded up to the equivalent system with respect to the unknown function $F(\mathbf{x}, \mathbf{y}) = F(\mathbf{z})$:

$$\frac{\partial F}{\partial \mathbf{z}} \cdot \mathbf{w}_i(\mathbf{z}) = \mathbf{0}, \quad i = 1, \ldots, q, \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \tag{1.3}$$

where $\mathbf{w}_i(\mathbf{z}) = \begin{bmatrix} \mathbf{h}_i \\ -\mathbf{g}_i \end{bmatrix}$, for some $q \geq p$, so that (1.3) identically satisfies the compatiblity conditions, and it is proven [6] that if the solution of (1.1) satisfies the algebraic equations (1.2), then it also satisfies the algebraic equations

$$\frac{\partial \mathbf{f}}{\partial \mathbf{z}} \cdot \mathbf{w}_i(\mathbf{z}) = \mathbf{0}, \quad i = 1, \ldots, q. \tag{1.4}$$

We generate other algebraic equations, besides the equations (1.2). Let us put $\mathbf{f}_0(\mathbf{z}) = \mathbf{f}_0(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{z}) = \mathbf{0}$ and let us assume inductively that we have obtained the vector equation $\mathbf{f}_j(\mathbf{z}) = \mathbf{0}$. Let us sort the components of $\mathbf{f}_j$ into $\bar{\mathbf{f}}_j$ and $\tilde{\mathbf{f}}_j$ so that the rows of the matrix $\partial \bar{\mathbf{f}}_j / \partial \mathbf{z}$ be linearly independent and the rows of the matrix $\partial \tilde{\mathbf{f}}_j / \partial \mathbf{z}$ be linearly dependent on the rows of the matrix $\partial \bar{\mathbf{f}}_j / \partial \mathbf{z}$. While generating the vector functions $\bar{\mathbf{f}}_j$ let us assume that the components of $\mathbf{f}_j$ are ordered so the the first components of $\bar{\mathbf{f}}_j$ are the components of $\bar{\mathbf{f}}_{j-1}$. Applying the previous assertion to the equations $\bar{\mathbf{f}}_j(\mathbf{z}) = \mathbf{0}$, new equations can be obtained. Let us put

$$\mathbf{f}_{j+1} = \begin{bmatrix} \bar{\mathbf{f}}_j \\ (\partial \bar{\mathbf{f}}_j / \partial \mathbf{z}) \cdot \mathbf{w}_1 \\ \cdots \\ (\partial \bar{\mathbf{f}}_j / \partial \mathbf{z}) \cdot \mathbf{w}_q \end{bmatrix} = \mathbf{0}. \tag{1.5}$$

This completes the description of the algorithm.

A necessary condition for existence of a solution of (1.1) and (1.2) is the existence of points $\mathbf{z}_j$ such that

$$\bar{\mathbf{f}}_j(\mathbf{z}_j) = \mathbf{0}, \quad \tilde{\mathbf{f}}_j(\mathbf{z}_j) = \mathbf{0}, \quad j = 0, 1, \ldots \tag{1.6}$$

As a result of the algorithm we obtain that the number of components of the vector function $\bar{\mathbf{f}}_j$ permanently increases and its first components are the components of the previous function $\bar{\mathbf{f}}_{j-1}$. There exists a minimal positive integer k such that $\bar{\mathbf{f}}_{k+1} = \bar{\mathbf{f}}_k$, because the maximal number of components of $\bar{\mathbf{f}}_j$, $j = 0, 1, 2, \ldots$ is bounded above. A condition for the existence of a solution of the problem (1.1) and (1.2) is $\dim \bar{\mathbf{f}}_k \leq \dim \mathbf{y}$.

If there exists at least one solution $\mathbf{z}_k$ of the equation $\bar{\mathbf{f}}_k(\mathbf{z}_k) = \mathbf{0}$ which is a necessary condition, then from the equation $\bar{\mathbf{f}}_k(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ by the implicit function theorem $\mathbf{y}$ can be expressed as a function of $\mathbf{x}$. This can be done as follows. The vector $\mathbf{y}$ can be partitioned as $\mathbf{y} = \begin{bmatrix} \mathbf{y}' \\ \mathbf{y}'' \end{bmatrix}$ so that the matrix $\partial \bar{\mathbf{f}}_k / \partial \mathbf{y}'$ is nonsingular. Thus the equation $\bar{\mathbf{f}}_k(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ can be solved with respect to $\mathbf{y}'$, $\mathbf{y}' = \varphi(\mathbf{x}, \mathbf{y}'')$. Let us partition the PDE (1.1) in accordance with the partitioning of the vector $\mathbf{y}$, i.e.

$$\frac{\partial \mathbf{y}'}{\partial \mathbf{x}} \cdot \mathbf{h}_i(\mathbf{x}, \mathbf{y}) + \mathbf{g}_i'(\mathbf{x}, \mathbf{y}) = 0, \quad \frac{\partial \mathbf{y}''}{\partial \mathbf{x}} \cdot \mathbf{h}_i(\mathbf{x}, \mathbf{y}) + \mathbf{g}_i''(\mathbf{x}, \mathbf{y}) = 0, \quad i = 1, \ldots, q, \tag{1.7}$$

where

$$[\mathbf{g}_1, \ldots, \mathbf{g}_q] = \begin{bmatrix} \mathbf{g}_1' \cdots \mathbf{g}_q' \\ \mathbf{g}_1'' \cdots \mathbf{g}_q'' \end{bmatrix},$$

and obtain a new system of PDEs with respect to $\mathbf{y}''$

$$\frac{\partial \mathbf{y}''}{\partial \mathbf{x}} \cdot \mathbf{h}_i(\mathbf{x}, \varphi(\mathbf{x}, \mathbf{y}''), \mathbf{y}'') + \mathbf{g}_i''(\mathbf{x}, \varphi(\mathbf{x}, \mathbf{y}''), \mathbf{y}'') = 0, \quad i = 1, \ldots, q, \tag{1.8}$$

and a system of algebraic equations

$$\frac{\partial \varphi}{\partial \mathbf{x}} \cdot \mathbf{h}_i - \frac{\partial \varphi}{\partial \mathbf{y}''} \cdot \mathbf{g}_i'' + \mathbf{g}_i' = 0, \quad i = 1, \ldots, q, \tag{1.9}$$

which does not give us anything new in solving of the problem (1.1) and (1.2) because it is equivalent to the equation $\bar{\mathbf{f}}_{k+1} = \bar{\mathbf{f}}_k$, i.e. $\tilde{\mathbf{f}}_{k+1} = \mathbf{0}$.

Theorem 1.1. *Sufficient conditions for the existence of a solution for the problem* (1.1) *and* (1.2) *are*

i) $\dim \bar{\mathbf{f}}_k \leq \dim \mathbf{y}$,
ii) *there exists a solution* $\mathbf{z}_k$ *of* $\bar{\mathbf{f}}_k(\mathbf{z}) = \mathbf{0}$,
iii) $\tilde{\mathbf{f}}_{k+1}(\mathbf{z}_k) = \mathbf{0}$,
iv) *the new system of PDEs* (1.8) *is satisfied.*

A solution of the problem is given by $\mathbf{y}_k'$ *and* $\mathbf{y}_k''$, *where the components of the vector* $\mathbf{y}$ *are partitioned (and eventually permuted) into the vectors* $\mathbf{y}'$ *and* $\mathbf{y}''$ *so that the matrix* $\partial \bar{\mathbf{f}}_k / \partial \mathbf{y}'$ *is nonsingular. Thus* $\mathbf{y}_k''$ *is a function which satisfies the system of PDEs* (1.8) *and* $\mathbf{y}_k'$ *is a solution of* $\bar{\mathbf{f}}_k(\mathbf{x}, \mathbf{y}_k) = \bar{\mathbf{f}}_k(\mathbf{x}, \mathbf{y}_k', \mathbf{y}_k'') = \mathbf{0}$.

According to this theorem we obtain again a system of PDEs (1.8) with dimension less than the former one and without algebraic constraints. Moreover, the following proposition holds.

Proposition 1.2. *If the system of PDEs* (1.3) *identically satisfies the compatibility conditions, then the system of PDEs* (1.8) *also identically satisfies the compatibility conditions.*

The paper [5] is a special case of the previous results such that $\mathbf{f}$ is a linear function of $\mathbf{y}$, $\mathbf{h}_i$ are functions of $\mathbf{x}$ only and $\mathbf{g}_i = \mathbf{0}$.

In the papers [7] and [8] solutions of systems of PDEs are given assuming that the systems are analytical. Let us consider the following system

$$\frac{\partial \mathbf{y}}{\partial x_u} + \mathbf{G}_u \cdot \mathbf{y} = \mathbf{0} \quad (1 \le u \le n), \tag{1.10}$$

where $\mathbf{y} = (y_1, \dots, y_m)^T$ are unknown functions of $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{G}_u$ are analytical matrix functions of $\mathbf{x}$. Then the compatibility conditions for (1.10) are

$$\mathbf{R}_{ij} \equiv \mathbf{0} \tag{1.11}$$

where

$$\mathbf{R}_{ij} = \partial \mathbf{G}_j / \partial x_i - \partial \mathbf{G}_i / \partial x_j + \mathbf{G}_i \cdot \mathbf{G}_j - \mathbf{G}_j \cdot \mathbf{G}_i \quad (1 \le i, j \le n).$$

In [7] the following theorem is proved.

Theorem 1.3. *Let the system* (1.10) *be given and let the compatibility conditions* (1.11) *be satisfied. Then there exist* $m \times m$ *matrix functions* $\mathbf{P}^{<v_1,\dots,v_n>}(\mathbf{x}), v_1, \dots, v_n \in \mathbf{N}_0$ *such that*

$$\mathbf{P}^{<0,\dots,0>} = \mathbf{I}; \tag{1.12}$$

$$\mathbf{P}^{<v_1,\dots,v_i+1,\dots,v_n>} = \frac{\partial}{\partial x_i}\mathbf{P}^{<v_1,\dots,v_n>} + \mathbf{G}_i \cdot \mathbf{P}^{<v_1,\dots,v_n>} \tag{1.13}$$

and the solution of (1.10) *is given by*

$$\mathbf{y} = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} \cdots \sum_{v_n=0}^{\infty} \frac{(-x_1)^{v_1}}{v_1!} \frac{(-x_2)^{v_2}}{v_2!} \cdots \frac{(-x_n)^{v_n}}{v_n!} \cdot \mathbf{P}^{<v_1,\dots,v_n>} \cdot \mathbf{c}, \tag{1.14}$$

where $\mathbf{c} = (c_1, \dots, c_m)$, $c_s = y_s(\mathbf{0})$.

Further in [7] the nonlinear system of PDEs

$$\frac{\partial \mathbf{y}}{\partial x_u} + \sum_{i_1,\dots,i_m \in \mathbf{Z}} \mathbf{f}_{i_1 \cdots i_m u}(\mathbf{x}) y_1^{i_1} y_2^{i_2} \cdot y_m^{i_m} = \mathbf{0} \quad (1 \le u \le n) \tag{1.15}$$

is considered, where $\mathbf{f}_{i_1 \dots i_m u}$ are analytical functions. Moreover, suppose that there exists a neighbourhood U of $\mathbf{0}$ such that all vector functions $\mathbf{f}_{i_1 \cdots i_m u}$ are regular in U. Let W be such that the Laurent series in (1.15) converge for $\mathbf{y} \in W$ and $\mathbf{x} \in U$. If $f_{ri_1 \cdots i_m u}(\mathbf{x})$

$(1 \le r \le m, 1 \le u \le n, i_1, \dots, i_m \in \mathbf{Z})$ are the coordinates of $\mathbf{f}_{i_1 \cdots i_m u}$, we define new functions

$$h_{i_1 \cdots i_m j_1 \cdot j_m u} \text{ and } R_{i_1 \cdots i_m j_1 \cdot j_m uv} \quad (i_1, \dots, i_m, j_1, \dots, j_m \in \mathbf{Z},\ 1 \le u, v \le n)$$

by

$$h_{i_1 \cdot i_m j_1 \cdots j_m u} = \sum_{s=1}^{m} i_s f_s (j_1 - i_1) \cdots (j_s - i_s + 1) \cdots (j_m - i_m) u \tag{1.16}$$

and

$$R_{i_1 \cdots j_m j_1 \cdots j_m uv} = \frac{\partial}{\partial x_u} h_{i_1 \cdots i_m j_1 \cdots j_m v} - \frac{\partial}{\partial x_v} h_{i_1 \cdots i_m j_1 \cdots j_m u} + \sum_{t_1, \dots, t_m \in \mathbf{Z}} h_{t_1 \cdots t_m j_1 \cdots j_m v} h_{i_1 \dots i_m t_1 \cdots t_m u} - \sum_{t_1, \dots, t_m \in \mathbf{Z}} h_{t_1 \cdots t_m j_1 \cdots j_m u} h_{i_1 \cdots i_m t_1 \cdots t_m v}. \tag{1.17}$$

Note that the series

$$\sum_{j_1, \dots, j_m \in \mathbf{Z}} h_{i_1 \dots i_m j_1 \cdots j_m u} y_1^{j_1} \cdots y_m^{j_m} \text{ and } \sum_{j_1, \dots, j_m \in \mathbf{Z}} R_{i_1 \cdots i_m j_1 \cdots j_m uv} y_1^{j_1} \cdots y_m^{j_m}$$

converge for $\mathbf{y} \in W$ and $\mathbf{x} \in U$. In order to simplify the notations, sometimes we will denote by the Greek indices $\alpha, \beta, \gamma, \dots$ a set of m integer indices $i_1 \cdots i_m; j_1 \cdots j_m; \cdots$ We will denote by $\{r\}$ the set of m indices $0 \dots 010 \dots 0$ where the unit appears at the r-th place. Now $\alpha + \beta$ and $\alpha - \beta$ are defined by $i_1 \cdots i_m \pm j_1 \cdots j_m = (i_1 \pm j_1)(i_2 \pm j_2) \cdots (i_m \pm j_m)$. The following properties are proved in [7]:

$$h_{(\alpha+\beta)\gamma u} = h_{\alpha(\gamma-\beta)u} + h_{\beta(\gamma_\alpha)u}, \tag{1.18}$$

$$R_{(\alpha+\beta)\gamma uv} = R_{\alpha(\gamma-\beta)uv} + R_{\beta(\gamma-\alpha)uv}, \tag{1.19}$$

$$h_{\alpha\beta u} = \sum_{s=1}^{m} i_s h_{\{s\}(\beta-\alpha+\{s\})u}, \tag{1.20}$$

$$R_{\alpha\beta uv} = \sum_{s=1}^{m} i_s R_{\{s\}(\beta-\alpha+\{s\})uv}, \text{ where } \alpha = i_1 \cdots i_m. \tag{1.21}$$

Theorem 1.4. (i) *The compatibility conditions for the system* (1.15) *for arbitrary initial conditions* $\mathbf{y}(\mathbf{0}) = \mathbf{c}$ *are*

$$R_{\alpha\beta uv} \equiv 0, \quad i.e. \quad R_{\{r\}\beta uv} \equiv 0. \tag{1.22}$$

(ii) *If the compatibility conditions* (1.22) *are satisfied, then there exist functions*

$$P_{i_1 \dots i_m j_1 \cdots j_m}^{<w_1, \dots, w_n>}(\mathbf{x}), \quad w_1, \dots, w_n \in \mathbf{N}_0, \quad i_1, \dots, i_m, j_1, \dots, j_m \in \mathbf{Z},$$

in a neighbourhood of $\mathbf{0}$ *such that*

$$P^{<0,\ldots,0>}_{i_1\cdots i_m j_1\cdots j_m} = \delta_{i_1 j_1}\delta_{i_2 j_2}\cdots\delta_{i_m j_m}, \tag{1.23}$$

$$P^{<w_1,\ldots,w_u+1,\ldots,w_n>}_{i_1\cdots i_m j_1\cdots j_m} = \frac{\partial}{\partial x_u} P^{<w_1,\ldots,w_n>}_{i_1\cdots i_m j_1\cdots j_m}$$

$$+ \sum_{t_1,\ldots,t_m\in\mathbf{Z}} \Big(\sum_{s=1}^{m} i_s f_{s(t_1-i_1)\cdots(t_s-i_s+1)\cdots(t_m-i_m)u}\Big) P^{<w_1,\ldots,w_n>}_{t_1\cdots t_m j_1\cdots j_m}. \tag{1.24}$$

If $\mathbf{c}\in W$, *then the solution of* (1.15) *in a neighbourhood of* $\mathbf{0}$ *is given by*

$$y_r = \sum_{w_1,\ldots,w_n\in\mathbf{N}_0} \Big[\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_n)^{w_n}}{w_n!} \sum_{j_1,\ldots,j_m\in\mathbf{Z}} P^{<w_1,\ldots,w_n>}_{\{r\}j_1 j_2\cdots j_m} c_1^{j_1}\cdot c_2^{j_2}\cdots c_m^{j_m}\Big] \tag{1.25}$$

for $1\le r\le m$. *This solution is unique with the given initial conditions in a neighbourhood of* $\mathbf{0}$.

The paper [8] generalizes the paper [7] so that for the corresponding system the compatibility conditions are not assumed identically satisfied and they depend on the choice of the initial conditions. If the initial conditions are given, then the necessary and sufficient conditions for the existence of a solution are given, and if they are satisfied, the solution is given in a functional series as in Theorems 1.3 and 1.4.

2. Solution of the Cauchy Problem

In this section we will give the solution of the Cauchy problem for linear and nonlinear systems of PDEs. Let $p\le n$ and let us consider the following system

$$\partial\mathbf{y}/\partial x_u + \mathbf{G}_u\cdot\mathbf{y} = \mathbf{0} \quad (u=1,\ldots,p) \tag{2.1}$$

with the initial condition

$$\mathbf{y}(0,\ldots,0,x_{p+1},\ldots,x_n) = \psi(x_{p+1},\ldots,x_n) \tag{2.2}$$

with respect to the unknown vector function $\mathbf{y}=(y_1,\ldots,y_m)$ of the vector variable $\mathbf{x}=(x_1,\ldots,x_n)$, where $\mathbf{G}_u$ are given analytical matrix functions of $\mathbf{x}$ regular in a neighbourhood of $(0,\ldots,0,z_{p+1},\ldots,z_n)$ and ψ is a given vector function. The compatibility conditions for the system (2.1) with arbitrary initial conditions (2.2) are

$$\mathbf{R}_{ij}\equiv 0 \quad (i,j=1,\ldots,p), \tag{2.3}$$

where $\mathbf{R}_{ij}$ were defined in Section 1. Now we will prove the following theorem.

Theorem 2.1. *Let the system* (2.1) *with the initial conditions* (2.2) *be given and the compatibility conditions* (2.3) *be satisfied. Then there exist* $m\times m$ *matrix functions*

$\mathbf{P}^{<w_1,\dots,w_p>}(\mathbf{x})$, $w_1,\dots,w_p \in \mathbf{N}_0$ *such that*

$$\mathbf{P}^{<0,\dots,0>} = \mathbf{I} \tag{2.4a}$$

$$\mathbf{P}^{<w_1,\dots,w_u+1,\dots,w_p>} = \frac{\partial}{\partial x_u}\mathbf{P}^{<w_1,\dots,w_p>} + \mathbf{G}_u \cdot \mathbf{P}^{<w_1,\dots,w_u,\dots,w_p>}, \quad u = 1,\dots,p, \tag{2.4b}$$

and the solution of (2.1) *with* (2.2) *in a neighbourhood of* $(0,\dots,0,z_{p+1},\dots,z_n)$ *is given by*

$$\mathbf{y} = \sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty}\cdots\sum_{w_p=0}^{\infty}\frac{(-x_1)^{w_1}}{w_1!}\frac{(-x_2)^{w_2}}{w_2!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\times\mathbf{P}^{<w_1,\dots,w_p>}\cdot\psi(x_{p+1}\cdot\dots,x_k). \tag{2.5}$$

This solution is unique with the given initial conditions in a neighbourhood of $(0,\dots,0,$ $z_{p+1},\dots,z_n)$.

Proof. Suppose that the system (2.1) is given and the compatibility conditions (2.3) are satisfied. In order to prove that there exist matrix functions

$$\mathbf{P}^{<w_1,\dots,w_p>}(\mathbf{x}) \quad (w_1,\dots,w_p \in \mathbf{N}_0)$$

such that (2.4a) and (2.4b) are satisfied, it is sufficient to prove that

$$\mathbf{P}^{<w_1,\dots,w_u^{(2)}+1,\dots,w_v^{(1)}+1,\dots,w_p>} = \mathbf{P}^{<w_1,\dots,w_u^{(1)}+1,\dots,w_v^{(2)}+1,\dots,w_p>} \tag{2.6}$$

for each $u,\ v \in \{1,\dots,p\}$, $u < v$, where $w_u^{(1)} = w_v^{(1)}$, $w_u^{(2)} = w_v^{(2)}$. Indeed, using the definition of the matrix functions $\mathbf{P}$ and the compatibility condition (2.3), we obtain

$$\begin{aligned}&\mathbf{P}^{<w_1,\dots w_u^{(2)}+1,\dots w_v^{(1)}+1,\dots w_p>} - \mathbf{P}^{<w_1,\dots,w_u^{(1)}+1,\dots w_v^{(2)}+1,\dots w_p>}\\ &= \mathbf{R}_{uv}\cdot\mathbf{P}^{<w_1,\dots,w_p>} \equiv \mathbf{0}.\end{aligned}$$

Now we should prove that the vector function $\mathbf{y}$ defined by (2.5) satisfies the system (2.1) and the initial condition (2.2). The uniform convergence of the right-hand side of (2.5) in a neighbourhood of $\mathbf{0}$ is proved in [7]. Further, for an arbitrary vector function $\psi(x_{p+1},\dots,x_n)$ according to (2.5) and (2.4b) we obtain

$$\begin{aligned}\partial\mathbf{y}/\partial x_u = &\sum_{w_1,\dots,w_p\in\mathbf{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots(-1)\frac{(-x_u)^{w_u-1}}{(w_u-1)!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\\ &\times\mathbf{P}^{<w_1,\dots,w_p>}\cdot\psi(x_{p+1},\dots,x_n)\\ &+\sum_{w_1,\dots,w_p\in\mathbf{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\cdot\frac{\partial}{\partial x_u}[\mathbf{P}^{<w_1,\dots,w_p>}\cdot\psi(x_{p+1},\dots,x_n)]\\ = -&\sum_{w_1,\dots,w_p\in\mathbf{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_u)^{w_u}}{w_u!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\\ &\times\mathbf{P}^{<w_1,\dots,w_u+1,\dots,w_p>}\cdot\psi(x_{p+1},\dots,x_n)\end{aligned}$$

$$+\sum_{w_1,\dots,w_p\in\mathbf{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\cdot\Big[\frac{\partial}{\partial x_u}\mathbf{P}^{<w_1,\dots,w_p>}\Big]\cdot\psi(x_{p+1},\dots,x_n)$$
$$=-\sum_{w_1,\dots,w_p\in\mathbf{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\cdot\Big[\mathbf{P}^{<w_1,\dots,w_u+1,\dots,w_p>}$$
$$-\frac{\partial}{\partial x_u}\mathbf{P}^{<w_1,\dots,w_p>}\Big]\cdot\psi(-x_{p+1},\dots,x_n)$$
$$=-\sum_{w_1,\dots,w_p\in\mathbf{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\cdot[\mathbf{G}_u\cdot\mathbf{P}^{<w_1,\dots,w_p>}]\cdot\psi(x_{p+1},\dots x_n)$$
$$=-\mathbf{G}_u\cdot\Big[\sum_{w_1,\dots,w_p\in\mathbf{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\cdot\mathbf{P}^{<w_1,\dots,w_p>}\cdot\psi(x_{p+1},\dots,x_n)\Big]$$
$$=-\mathbf{G}_u\cdot\mathbf{y},$$

and hence (2.1) is satisfied. Further, according to (2.4a) we obtain

$$\mathbf{y}(0)=\mathbf{P}^{<0,\dots,0>}\cdot\psi(x_{p+1},\dots,x_n)=\mathbf{I}\cdot\psi(x_{p+1},\dots,x_n)=\psi(x_{p+1},\dots,x_n).$$

The solution is unique with the given initial coditions. This is proved analogously to [7].

Let us consider the following nonlinear system of PDEs

$$\partial\mathbf{y}/\partial x_u+\sum_{i_1,\dots i_m\in\mathbf{Z}}\mathbf{f}_{i_1\cdots i_m u}(\mathbf{x})y_1^{i_1}y_2^{i_2}\cdots y_m^{i_m}=0,\quad u=1,\dots,p\quad(p\le n)\tag{2.7}$$

with the initial condition

$$\mathbf{y}(0,\dots,0,x_{p+1},\dots,x_n)=\psi(x_{p+1},\dots,x_n),\tag{2.8}$$

where $\mathbf{f}_{i_1\cdots i_m u}=(f_{1i_1\cdots i_m u},\dots,f_{mi_1\cdots i_m u})$ are analytical functions and suppose that there exists a neighbourhood U of $(0,\dots,0,z_{p+1},\dots,z_n)$ such that all functions $f_{ri_1,\dots i_m u}$ are regular in U. Let W be such that the Laurent series in (2.7) converge for $\mathbf{y}\in W$ and $\mathbf{x}\in U$, analogously as in Section 1. We define the quantities $R_{\alpha\beta uv}$ in the same way as in Section 1 and now we have the following theorem.

Theorem 2.2. (i) *The compatibility conditions for the system* (2.7) *for arbitrary initial conditions* (2.8) *are*

$$R_{\alpha\beta uv}\equiv 0,\quad\text{i.e.}\quad R_{\{r\}\beta uv}\equiv 0,\quad u,v=1,\dots,p.\tag{2.9}$$

(ii) *If the compatibility conditions* (2.9) *are satisfied, then there exist functions*

$$P^{<w_1,\dots w_p>}_{i_1\cdots i_m j_1\cdots j_m}(\mathbf{x}),\quad w_1,\dots,w_p\in\mathbf{N}_0,\quad i_1,\dots i_m,j_1,\dots,j_m\in\mathbf{Z},$$

in a neighbourhood of $(0,\dots,0,z_{p+1},\dots,z_n)$ *such that*

$$P^{<0,\dots,0>}_{i_1\dots i_m j_1\cdots j_m}=\delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_mj_m},\tag{2.10a}$$

$$P^{<w_1,\dots,w_u+1,\dots,w_p>}_{i_1\cdots i_m j_1\cdots j_m}=\frac{\partial}{\partial x_u}P^{<w_1,\dots,w_p>}_{i_1\cdots i_m j_1\cdots j_m}$$
$$+\sum_{t_1,\dots,t_m\in\mathbf{Z}}\Big(\sum_{s=1}^{m}i_s f_{s(t_1-i_1)\cdots(t_s-i_s+1)\cdot(t_m-i_m)u}\Big)p^{<w_1,\dots,w_p>}_{t_1\cdots t_m j_1\cdots j_m}.\tag{2.10b}$$

If $\psi(x_{p+1},\dots,x_n)\in W$, then the solution of (2.7) *and* (2.8) *in a neighbourhood of* $(0,\dots,0,z_{p+1},\dots,z_n)$ *is given by*

$$y_r = \sum_{w_1,\dots,w_p\in\mathbf{N}_0}\Big[\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_p)^{w_p}}{w_p!}\sum_{j_1,\dots,j_m\in\mathbf{Z}} P^{<w_1,\dots,w_p>}_{\{r\}j_1\cdots j_m}\psi_1^{j_1}\cdot\psi_2^{j_2}\cdots\psi_m^{j_m}\Big]. \quad (2.11)$$

This solution is unique with the given initial conditions in a neighborhood of $(0,\dots,0,$ $z_{p+1},\dots,z_n)$.

If we introduce the variables

$$y_\alpha = y_{i_1 i_2\cdots i_m} = y_1^{i_1}\cdot y_2^{i_2}\cdots y_m^{i_m},\quad (i_1,\dots,i_m\in\mathbf{Z}),$$

then the nonlinear system (2.7) is transformed into an infinitely dimensional linear system of PDEs with unknown functions y_α, $\alpha\in\mathbf{Z}^n$. The proof of Theorem 2.2 is analogous to the proof of Theorem 2.1 (see also [7, Theorem 3.2]).

3. Reduction of a System of Partial Differential Equations

In this section we return to [6], considering the algorithm for reduction of PDEs and the problem about the compatibility conditions for the derived system of PDEs with linear homogeneous algebraic constraints (see Theorem 1.1 and Proposition 1.2):

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \mathbf{F}(\mathbf{x})\cdot\mathbf{y}.$$

Let us consider the following homogeneous linear system of PDEs

$$\frac{\partial\mathbf{y}}{\partial x_i} + \mathbf{G}_i\cdot\mathbf{y} = \mathbf{0},\quad i=1,\dots,p\ (p\le n). \quad (3.1)$$

Let $\mathbf{f}_0(\mathbf{x},\mathbf{y}) = \mathbf{F}_0(\mathbf{x})\cdot\mathbf{y}$ for a matrix function $\mathbf{F}_0(\mathbf{x})$ be the vector function obtained by uniting the given algebraic constraints and algebraic constraints concerning the compatibility conditions:

$$\Big[\frac{\partial\mathbf{G}_i}{\partial x_j} - \mathbf{G}_i\cdot\mathbf{G}_j - \frac{\partial G_j}{\partial x_i} + \mathbf{G}_j\cdot\mathbf{G}_\mathrm{i}\Big]\cdot\mathbf{y} = \mathbf{0}\quad (1\le i,j\le p).$$

By applying the reduction algorithm (see Section 1) we obtain sucessively the functions

$$\mathbf{f}_0(\mathbf{x},\mathbf{y}) = \mathbf{F}_0(\mathbf{x})\cdot\mathbf{y},\quad \mathbf{f}_1(\mathbf{x},\mathbf{y}) = \mathbf{F}_1(\mathbf{x})\cdot\mathbf{y},\dots,\mathbf{f}_k(\mathbf{x},\mathbf{y}) = \mathbf{F}_\mathrm{k}(\mathbf{x})\cdot\mathbf{y}.$$

Let us partition the functions $\mathbf{F}_i$ as

$$\mathbf{F}_i = \begin{bmatrix}\bar{\mathbf{F}}_i\\ \tilde{\mathbf{F}}_i\end{bmatrix}$$

where the rows of $\bar{\mathbf{F}}_i$ are linearly independent, and the rows of $\tilde{\mathbf{F}}_i$ are linearly dependent on the rows of $\bar{\mathbf{F}}_i$. According to the definition of the final index k there exist matrix functions $\mathbf{V}_i$ such that

$$\frac{\partial \bar{\mathbf{F}}_k}{\partial x_i} - \bar{\mathbf{F}}_k \cdot \mathbf{G}_i = \mathbf{V}_i \cdot \bar{\mathbf{F}}_k. \tag{3.2}$$

Thus by differentiating the following equation

$$\bar{\mathbf{f}}_k(\mathbf{x}, \mathbf{y}) = \bar{\mathbf{F}}_k(\mathbf{x}) \cdot \mathbf{y} = \mathbf{0}$$

with respect to x_i we do not obtain new constraints. The last equality can be written in the following form:

$$[\bar{\mathbf{F}}'_k \bar{\mathbf{F}}''_k] \cdot \begin{bmatrix} \mathbf{y}' \\ \mathbf{y}'' \end{bmatrix} = \bar{\mathbf{F}}'_k \cdot \mathbf{y}' + \bar{\mathbf{F}}''_k \cdot \mathbf{y}'' = \mathbf{0}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{y}' \\ \mathbf{y}'' \end{bmatrix},$$

where $\bar{\mathbf{F}}'_k$ is a nonsingular matrix. Hence,

$$\mathbf{y}' = -\bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}'' \cdot \mathbf{y}''.$$

The system (3.1) decomposes into

$$\frac{\partial \mathbf{y}'}{\partial x_i} + \bar{\mathbf{G}}'_i \cdot \mathbf{y}' + \bar{\mathbf{G}}''_i \cdot \mathbf{y}'' = \mathbf{0}, \tag{3.3}$$

$$\frac{\partial \mathbf{y}''}{\partial x_i} + \tilde{\mathbf{G}}'_i \cdot \mathbf{y}' + \tilde{\mathbf{G}}''_i \cdot \mathbf{y}'' = \mathbf{0}, \tag{3.4}$$

Now we prove that the equation (3.3) is a consequence of (3.4). First note that

$$\begin{aligned}
&\frac{\partial}{\partial x_i}(\bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) = -\bar{\mathbf{F}}'^{-1}_k \cdot \frac{\partial \bar{\mathbf{F}}'_k}{\partial x_i} \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k + \bar{\mathbf{F}}'^{-1}_k \cdot \frac{\partial \bar{\mathbf{F}}''_k}{\partial x_i} \\
&= -\bar{\mathbf{F}}'^{-1}_k \cdot (\mathbf{V}_i \cdot \bar{\mathbf{F}}'_k + \bar{\mathbf{F}}_k \cdot \bar{\mathbf{G}}'_i) \cdot \bar{\mathbf{F}}^{-1}_k \cdot \bar{\mathbf{F}}''_k + \bar{\mathbf{F}}'^{-1}_k \cdot (\mathbf{V}_i \cdot \bar{\mathbf{F}}''_k + \bar{\mathbf{F}}_k \cdot \mathbf{G}''_i) \\
&= \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}_k \cdot (\mathbf{G}''_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k),
\end{aligned}$$

where we have used the partitioned identity (3.2), i.e.

$$\frac{\partial \bar{\mathbf{F}}'_k}{\partial x_i} - \bar{\mathbf{F}}_k \cdot \mathbf{G}'_i = \mathbf{V}_i \cdot \bar{\mathbf{F}}'_k,$$

$$\frac{\partial \bar{\mathbf{F}}''_k}{\partial x_i} - \bar{\mathbf{F}}_k \cdot \mathbf{G}''_i = \mathbf{V}_i \cdot \bar{\mathbf{F}}''_k,$$

Further, for (3.3) we obtain

$$\begin{aligned}
&\frac{\partial \mathbf{y}'}{\partial x_i} + \bar{\mathbf{G}}_i \cdot \mathbf{y}' + \bar{\mathbf{G}}''_i \cdot \mathbf{y}'' \\
&= -\frac{\partial}{\partial x_i}(\bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot \mathbf{y}'' - \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k \cdot \frac{\partial \mathbf{y}''}{\partial x_i} + (\bar{\mathbf{G}}''_i - \bar{\mathbf{G}}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot \mathbf{y}'' \\
&= -\bar{\mathbf{F}}'^{-1}_k \cdot \mathbf{F}_k \cdot (\mathbf{G}''_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot \mathbf{y}'' - \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k \cdot \frac{\partial \mathbf{y}''}{\partial x_i} + (\bar{\mathbf{G}}''_i - \bar{\mathbf{G}}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot \mathbf{y}'' \\
&= -\bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k \Big(\frac{\partial \mathbf{y}''}{\partial x_i} + (\tilde{\mathbf{G}}''_i - \tilde{\mathbf{G}}'_i \cdot \bar{\mathbf{G}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot \mathbf{y}'' \Big) = \mathbf{0}
\end{aligned}$$

under the condition (3.4), i.e.

$$\frac{\partial \mathbf{y}''}{\partial x_i} + (\tilde{\mathbf{G}}_i'' - \tilde{\mathbf{G}}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'') \cdot \mathbf{y}'' = \mathbf{0}. \tag{3.5}$$

Thus we obtain a new system of PDEs with the same form as the former one but with a smaller dimension. Now we are ready to prove the main theorem.

Theorem 3.1. *The compatibility conditions for the system of PDEs* (3.5) *are identically satisfied.*

Proof. By the definition of $\mathbf{F}_0$ there exist matrices $\mathbf{V}_{ij}$ such that

$$\frac{\partial \mathbf{G}_i}{\partial x_j} - \frac{\partial \mathbf{G}_j}{\partial x_i} - \mathbf{G}_i\mathbf{G}_j + \mathbf{G}_j\mathbf{G}_i = \mathbf{V}_{ij} \cdot \bar{\mathbf{F}}_k.$$

This equality decomposes into the following two equalities

$$\frac{\partial \mathbf{G}_i'}{\partial x_j} - \frac{\partial \mathbf{G}_j'}{\partial x_i} - \mathbf{G}_i\mathbf{G}_j' + \mathbf{G}_j\mathbf{G}_i' = \mathbf{V}_{ij} \cdot \bar{\mathbf{F}}_k',$$
$$\frac{\partial \mathbf{G}_i''}{\partial x_j} - \frac{\partial \mathbf{G}_j''}{\partial x_i} - \mathbf{G}_i\mathbf{G}_j'' + \mathbf{G}_j\mathbf{G}_i'' = \mathbf{V}_{ij} \cdot \bar{\mathbf{F}}_k''.$$

By elimination of $\mathbf{V}_{ij}$ we obtain

$$\begin{aligned}&\frac{\partial \mathbf{G}_i''}{\partial x_j} - \frac{\partial \mathbf{G}_i'}{\partial x_i} \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'' - \mathbf{G}_i(\mathbf{G}_j'' - \mathbf{G}_j' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'')\\ &= \frac{\partial \mathbf{G}_j''}{\partial x_j} - \frac{\partial \mathbf{G}_j'}{\partial x_i} \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'' - \mathbf{G}_j(\mathbf{G}_i'' - \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'').\end{aligned} \tag{3.6}$$

Using the identity

$$\frac{\partial}{\partial x_j}(\mathbf{G}_i'' - \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'') = \frac{\partial \mathbf{G}_i''}{\partial x_j} - \frac{\partial \mathbf{G}_i'}{\partial x_j} \cdot \bar{\mathbf{F}}'^{-1} \cdot \bar{\mathbf{F}}_k'' - \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k \cdot (\mathbf{G}_j'' - \mathbf{G}_j' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k''),$$

from (3.6) we obtain

$$\begin{aligned}&\frac{\partial}{\partial x_j}(\mathbf{G}_i'' - \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'') + \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k \cdot (\mathbf{G}_j'' - \mathbf{G}_j' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'')\\ &\quad -\mathbf{G}_i(\mathbf{G}_j'' - \mathbf{G}_j' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'')\\ &= \frac{\partial}{\partial x_i}(\mathbf{G}_j'' - \mathbf{G}_j' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'') + \mathbf{G}_j' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k \cdot (\mathbf{G}_i'' - \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'')\\ &\quad -\mathbf{G}_j(\mathbf{G}_i'' - \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k''),\\ &\frac{\partial}{\partial x_j}(\mathbf{G}_i'' - \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'') - (\mathbf{G}_i - \mathbf{G}_i' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'') \cdot (\mathbf{G}_j'' - \mathbf{G}_j' \cdot \bar{\mathbf{F}}_k'^{-1} \cdot \bar{\mathbf{F}}_k'')\end{aligned}$$

$$= \frac{\partial}{\partial x_i}(\mathbf{G}''_j - \mathbf{G}'_j \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) - (\mathbf{G}_j - \mathbf{G}'_j \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot (\mathbf{G}''_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k),$$

$$\frac{\partial}{\partial x_j}(\mathbf{G}''_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) - (\mathbf{G}''_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot (\tilde{\mathbf{G}}''_j - \tilde{\mathbf{G}}'_j \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k)$$
$$= \frac{\partial}{\partial x_i}(\mathbf{G}''_j - \mathbf{G}'_j \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) - (\mathbf{G}''_j - \mathbf{G}'_j \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot (\tilde{\mathbf{G}}''_i - \tilde{\mathbf{G}}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k), \quad (3.7)$$

because

$$\mathbf{G}_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}_k = [\mathbf{G}'_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \bar{\mathbf{F}}'_k, \bar{\mathbf{G}}''_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k] = [0, \bar{\mathbf{G}}''_i - \mathbf{G}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k].$$

Decomposing the equality (3.7) into two equalities like (3.3) and (3.4), we obtain

$$\frac{\partial}{\partial x_j}(\tilde{\mathbf{G}}''_i - \tilde{\mathbf{G}}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) - (\tilde{\mathbf{G}}''_i - \tilde{\mathbf{G}}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot (\tilde{\mathbf{G}}''_j - \tilde{\mathbf{G}}'_j \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k)$$
$$= \frac{\partial}{\partial x_i}(\tilde{\mathbf{G}}''_j - \tilde{\mathbf{G}}'_j \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) - (\tilde{\mathbf{G}}''_j - \tilde{\mathbf{G}}'_j \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k) \cdot (\tilde{\mathbf{G}}''_i - \tilde{\mathbf{G}}'_i \cdot \bar{\mathbf{F}}'^{-1}_k \cdot \bar{\mathbf{F}}''_k)$$

i.e. the compatibility conditions for the system (3.5) are identically satisfied.

Next we discuss the usefulness of the previous result. Indeed, if a linear system (3.1) with some linear homogeneous algebraic constraints is given, then using the algorithm from [6] we reduce it to a system of dimension less than the former one without any constraints and the compatibility conditions are identically satisfied. Then using Theorem 2.1, we can expand the solution as a convergent functional series. Specially, let us consider the system (3.1) and let the compatibility conditions be not identically satisfied for all initial conditions. In [8] the necessary and sufficient conditions are given for the existence of a solution for given initial conditions. These conditions are algebraic constraints. Now applying the algorithm in [6], we reduce the given system to a system of dimension less than the former one and the compatibility conditions are identically satisfied for all initial conditions. Then Theorem 2.1 can be applied to obtain the required solution.

Remark. Finally note that if we consider a system of nonlinear equations instead of (3.1), then a theorem like Theorem 3.1 holds for that system because as it is shown in [7], the nonlinear systems can be considered as linear systems of infinitely many equations and infinitely many unknown functions y_α. The above discussion considering the initial conditions is still valid for nonlinear systems of PDEs.

4. Example

Let the following system of four PDEs be given

$$\frac{\partial y_1}{\partial x_1} - \alpha \cdot (y_1 + y_2) = 0, \quad (4.1)$$

$$\frac{\partial y_2}{\partial x_1} + (x_1 - x_3) \cdot y_1 - (x_1 + x_2) \cdot y_2 = 0, \tag{4.2}$$

$$\frac{\partial y_1}{\partial x_2} - \beta \cdot (y_1 - y_2) = 0, \tag{4.3}$$

$$x_2 \cdot \frac{\partial y_2}{\partial x_2} - \beta \cdot (x_3 \cdot y_1 + x_1 \cdot y_2) = 0, \tag{4.4}$$

where $\alpha = \frac{1}{x_1+2x_2-x_3}$, $\beta = \frac{1}{x_1+x_3}$. If we define

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

then we bring the system (4.1)-(4.4) into the form (3.1), where $p = 2$ and

$$\mathbf{G}_1(x) = \begin{bmatrix} -\alpha & -\alpha \\ x_2 - x_3 & -x_1 - x_2 \end{bmatrix}, \quad \mathbf{G}_2(x) = \beta \cdot \begin{bmatrix} -1 & 1 \\ -\frac{x_3}{x_2} & -\frac{x_1}{x_2} \end{bmatrix}.$$

Hence we obtain

$$\mathbf{G}_1 \cdot \mathbf{G}_2 - \mathbf{G}_2 \cdot \mathbf{G}_1 = \beta \cdot \begin{bmatrix} \alpha \cdot \frac{x_3}{x_2} - x_2 + x_3 & -2 \cdot \alpha + \alpha \cdot \frac{x_1}{x_2} + x_1 + x_2 \\ -\alpha \cdot \frac{x_3}{x_2} + x_1 - x_2 + 2 \cdot x_3 & -\alpha \cdot \frac{x_3}{x_2} + x_2 - x_3 \end{bmatrix},$$

$$\frac{\partial \mathbf{G}_1}{\partial x_2} - \frac{\partial \mathbf{G}_2}{\partial x_1} = \begin{bmatrix} 2 \cdot \alpha^2 - \beta^2 & 2 \cdot \alpha^2 + \beta^2 \\ 1 - \beta^2 \cdot \frac{x_3}{x_2} & -1 - \beta^2 \cdot \frac{x_1}{x_2} + \frac{\beta}{x_2} \end{bmatrix}.$$

Thus

$$\mathbf{F}_1 = \frac{\partial \mathbf{G}_1}{\partial x_2} - \frac{\partial \mathbf{G}_2}{\partial x_1} - (\mathbf{G}_1 \cdot \mathbf{G}_2 - \mathbf{G}_2 \cdot \mathbf{G}_1)$$
$$= \begin{bmatrix} 2 \cdot \alpha^2 - \beta^2 - \beta \cdot (\alpha \cdot \frac{x_3}{x_2} - x_2 + x_3) & 2 \cdot \alpha^2 + \beta^2 - \beta \cdot (-2 \cdot \alpha + \alpha \cdot \frac{x_1}{x_2} + x_1 + x_2) \\ 1 - \beta^2 \cdot \frac{x_3}{x_2} - \beta \cdot (-\alpha \cdot \frac{x_3}{x_2} + x_1 - x_2 + 2 \cdot x_3) & -1 - \beta^2 \cdot \frac{x_1}{x_2} + \frac{\beta}{x_2} - \beta \cdot (-\alpha \cdot \frac{x_3}{x_2} + x_2 - x_3) \end{bmatrix}.$$

It can be verified that $\det \mathbf{F}_1 = 0$, thus $\mathbf{F}_1 = \begin{bmatrix} \bar{\mathbf{F}}_1 \\ \tilde{\mathbf{F}}_1 \end{bmatrix}$, where $\bar{\mathbf{F}}_1$ and $\tilde{\mathbf{F}}_1$ are (linearly dependent) vector-rows. It can be verified also that, by further differentiation, new linearly independent rows cannot be obtained, thus $k = 1$. Following the theory in Section 3, we partition the vector function $\mathbf{y}$ into $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix}$. It can be verified that the equation

$$\bar{\mathbf{F}}_1 \cdot \mathbf{y} = 0 \tag{4.5}$$

is equivalent to

$$y' = \frac{x_1 + x_2}{x_2 - x_3} \cdot y''.$$

The system (4.1)-(4.4) becomes

$$\frac{\partial y'}{\partial x_1} - \alpha \cdot (y' + y'') = 0, \tag{4.6}$$

$$\frac{\partial y''}{\partial x_1} + (x_1 - x_3) \cdot y' - (x_1 + x_2) \cdot y'' = 0, \tag{4.7}$$

$$\frac{\partial y'}{\partial x_2} - \beta \cdot (y' - y'') = 0, \tag{4.8}$$

$$\frac{\partial y''}{\partial x_2} - \beta \cdot (\frac{x_3}{x_2} \cdot y' + \frac{x_1}{x_2} \cdot y'') = 0. \tag{4.9}$$

According to the theory in Section 3, the equations (4.6) and (4.8) are consequences of the equations (4.7) and (4.9), respectively. Thus the original system of PDEs (4.1)-(4.4) is reduced to (3.5), i.e.

$$\frac{\partial y''}{\partial x_1} = 0, \tag{4.10}$$

$$\frac{\partial y''}{\partial x_2} - \frac{1}{x_1 - x_3} \cdot y'' = 0. \tag{4.11}$$

According to Theorem 3.1, the compatibility conditions for this system are identically satisfied. Thus, we have reduced the original system of four PDEs with two unknown functions not satisfying the compatibility conditions identically to the system of two PDEs with one unknown function satisfying the compatibility conditions identically. For obtaining the solution, we can apply the theory of Section 2, i.e. the functional expansion method presented by Theorem 2.1. Suppose the following initial conditions

$$y_1(0,0,x_3) = 0, \quad y_2(0,0,x_3) = \varphi(x_3) \tag{4.12}$$

are given, where φ is a function. Note that the initial conditions have to satisfy the equation (4.5) for $x_1 = x_2 = 0$, i.e. $\bar{\mathbf{F}}_1(0,0,x_3) \cdot \mathbf{y}(0,0,x_3) = 0$, and they are chosen to satisfy it. The initial condition for the reduced system (4.10)-(4.11) is $y''(0,0,x_3) = \varphi(x_3)$.

We have to find (scalar) functions $P^{<w_1,w_2>}(x_1,x_2,x_3)$ satisfying

$$P^{<0,0>} = 0,$$

$$P^{<w_1+1,w_2>} = \frac{\partial}{\partial x_1} P^{<w_1,w_2>},$$

$$P^{<w_1,w_2+1>} = \frac{\partial}{\partial x_2} P^{<w_1,w_2>} - \frac{1}{x_2 - x_3} \cdot P^{<w_1,w_2>}.$$

The functions $P^{<w_1,w_2>} = 0$ for $w_1 \neq 0$ and

$$P^{<0,w_2>} = (-1)^{w_2} \cdot \frac{w_2!}{(x_2 - x_3)^{w_2}}.$$

According to (2.5) the functional expansion for y'' is given by

$$y'' = \Big(\sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty}\frac{(-x_1)^{w_1}}{w_1!}\cdot\frac{(-x_2)^{w_2}}{w_2!}\cdot P^{<w_1,w_2>}\Big)\varphi(x_3)$$

$$= \Big(\sum_{w_2=0}^{\infty}\frac{(-x_2)^{w_2}}{w_2!}\cdot P^{<0,w_2>}\Big)\cdot\varphi(x_3) = \varphi(x_3)\cdot\sum_{w_2=0}^{\infty}\Big(\frac{x_2}{x_2-x_3}\Big)^{w_2}.$$

For $|\frac{x_2}{x_2-x_3}| < 1$ this series converges to $y'' = -\frac{(x_2-x_3)}{x_3}\cdot\varphi(x_3)$ which means that the solution of the system (4.1)-(4.4) with Cauchy initial conditions (4.12) is given by

$$y_1 = -\frac{(x_1+x_2)}{x_3}\cdot\varphi(x_3),\quad y_2 = -\frac{(x_2-x_3)}{x_3}\cdot\varphi(x_3).$$

The present example was included following the referee's suggestion.

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