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ON A GENERALIZATION OF CLOSE-TO-CONVEXITY OF COMPLEX ORDER

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Abstract. The class V_k of bounded boundary rotation is used to generalize the concept of close-to-convexity of complex order. A function $f: f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, analytic in the unit disc E, belongs to $T_k(b)$, $b \neq 0$ (complex) if and only if there exists a function $g \in V_k$ such that

$$Re\left\{1 + \frac{1}{b}\left(\frac{f'(z)}{g'(z)} - 1\right)\right\} > 0, \quad z \in E.$$

Some basic properties, rate growth of Hankel determinant and radii problems for the functions in $T_k(b)$ are studied.

1. Introduction

Let A denote the class of functions f given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$. By S, K, S^* and C, we denote the subclasses of A which are respectively univalent, close-to-convex, starlike and convex in E. Let P be the class of analytic functions h given by

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.2}$$

with $Re\ h(z) > 0$ for $z \in E$.

Let V_k, $k \geq 2$ be the class of functions of bounded boundary rotation and let P_k be the class of functions p analytic in E and have the representation

$$p(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t),$$

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where $\mu(t)$ is a function with bounded variation on $[-\pi, \pi]$ which satisfies the conditions

$$\int_{-\pi}^{\pi} d\mu(t) = 2, \int_{-\pi}^{\pi} |d\mu(t)| \leq k.$$

We note that $P_2 = P$.

It is known [13] that P_k is a convex set. Also f, given by (1.1), belongs to V_k if and only if $\frac{(zf'(z))'}{f'(z)} \in P_k$. It is clear that $p \in P_k$ if and only if

$$p(z) = (\frac{k}{4} + \frac{1}{2})p_1(z) - (\frac{k}{4} - \frac{1}{2})p_2(z), \tag{1.3}$$

where $p_1, p_2 \in P$.

We define the class $P(b)$ as follows.

Difinition 1.1. Let $b \neq 0$ be a complex number. Then an analytic function h, given by (1.2), is said to belong to $P(b)$, if and only if, there exists a function $p \in P$ such that

$$h(z) = bp(z) + (1 - b). \tag{1.4}$$

We note that, for $0 < b < 1$,

$$p(b) \subset P$$

and $P(1) = P$.

In the following we define a generalized concept of close-to-convexity of complex order.

Definition 1.2. Let f be analytic in E and be given by (1.1). Then $f \in T_k(b)$, $k \geq 2$, $b \neq 0$ (complex), if and only if, there exists a function $g \in V_k$ such that $\frac{f'(z)}{g'(z)} \in P(b)$ for $z \in E$.

We note that $T_k(1) = T_k$, a class of analytic functions introduced and studited in [9] and $T_2(1)$ is the class K of close-to-convex functions. Also $T_2(b) = K(b)$ consists entirely of close-to-convex functions of complex order introduced in [1] by Al-Amiri and Fernando.

2. Some Basic Properties of $T_k(b)$

Theorem 2.1. *$f \in T_k(b)$ if and only if*

$$f'(z) = \frac{(K_1'(z))^{\frac{k}{4}+\frac{1}{2}}}{(K_2'(z))^{\frac{k}{4}-\frac{1}{2}}},$$

where K_1 and K_2 are close-to-convex functions of complex order b.

Proof. From definition 1.2, we have

$$\begin{aligned} f'(z) &= g'(z)h(z),\ g \in V_k, h \in P(b) \\ &= \frac{(s_1(z)/z)^{\frac{k}{4}+\frac{1}{2}}}{(s_2(z)/z)^{\frac{k}{4}-\frac{1}{2}}} h(z),\ s_1, s_2 \in S^*, see[3]. \\ &= \frac{(K_1(z))^{\frac{k}{4}+\frac{1}{2}}}{(K_2(z))^{\frac{k}{4}-\frac{1}{2}}},\quad K_1, K_2 \in K(b). \end{aligned}$$

Theorem 2.2 *Let* $0 < b_1 < b_2$. *Then* $T_k(b_1) \subset T_k(b_2)$.

Proof. Let $f \in T_k(b_1)$. Then there exists a function $g \in V_k$ such that

$$\frac{f'(z)}{g'(z)} = b_1 h(z) + (1 - b_1),\ h \in P.$$

Now

$$1 + \frac{1}{b_2}\Big[\frac{f'(z)}{g'(z)} - 1\Big] = \frac{b_1}{b_2} h(z) + \Big(1 - \frac{b_1}{b_2}\Big).$$

Since $0 < b_1 < b_2$, we have $0 < \frac{b_1}{b_2} < 1$ and this means $0 < (1 - \frac{b_1}{b_2}) = \alpha_1 < 1$. Hence $Re[1 + \frac{1}{b_2}\{\frac{f'(z)}{g'(z)} - 1\}] > \alpha_1 > 0$. This implies that $f \in T_k(b_2)$, and this completes the proof.

We now discuss a geometrical property for the class $T_k(b)$. Here we investigate the behaviour of the inclination of the tangent at a point $\omega(\theta) = f(re^{i\theta})$ to the image Γ_r of the circle $C_r = \{z : |z| = r\}$, $0 \le r < 1$, and θ is any number of interval $(0, 2\pi)$ under the mapping by means of a function f from the class $T_k(b)$.

We have

$$\phi(\theta) = \frac{\pi}{2} + \theta + \arg\ f'(re^{i\theta}) = \arg\ \frac{\partial}{\partial\theta} f(re^{i\theta}),$$

and for $\theta_2 > \theta_1, \theta_1, \theta_2 \in [0, 2\pi]$,

$$\phi(\theta_2) - \phi(\theta_1) = \theta_2 + \arg\ f'(re^{i\theta_2}) - \theta_1 - arg f'(re^{i\theta_1}).$$

Now, since

$$\theta + \arg\ f'(re^{i\theta}) = \theta + Re\{-i \ln f'(re^{i\theta})\},$$

then

$$\frac{\partial}{\partial\theta}\Big(\theta + \arg\ f'(re^{i\theta})\Big) = Re\Big\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\Big\}.$$

Therefore

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}\Big(\theta + \arg\ f'(re^{i\theta})\Big) = \int_{\theta_1}^{\theta_2} Re\Big\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\Big\} d\theta.$$

On the other hand

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta}\{\theta + \arg\ f'(re^{i\theta})\}d\theta = \theta_2 + \arg\ f'(re^{i\theta_2}) - \theta_1 - \arg f'(re^{i\theta_1})$$
$$= \phi(\theta_2) - \phi(\theta_1).$$

So, the integral on the left hand side of the last equality characterizes the increment of the angle of the incliṅation of the tangent to the curve Γ_r between the points $\omega(\theta_2)$ and $\omega(\theta_1)$ for $\theta_2 > \theta_1$.

We now have the following.

Theorem 2.3. *If $f \in T_k(b)$ and $0 \le r < 1$, $|2b-1| < 1$, then for $\theta_2 > \theta_1$, $\theta_1, \theta_2 \in [0, 2\pi]$,*

$$\int_{\theta_1}^{\theta_2} Re\Big\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\Big\}d\theta > -\frac{k}{2}\pi + 2\cos^{-1}\frac{2|b|r}{1-|2b-1|r^2}.$$

Proof. From definition 1.2, we can write

$$Re\frac{(zf'(z))'}{f'(z)} = Re\frac{(zg'(z))'}{g'(z)} + Re\frac{zh'(z)}{h(z)},\ g \in V_k, h \in P(b).$$

With $z = re^{i\theta}, 0 \le r < 1, \theta \in [0, 2\pi], \theta_1 < \theta_2$, we have

$$\int_{\theta_1}^{\theta_2} Re\Big\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\Big\}d\theta = \int_{\theta_1}^{\theta_2} Re\Big[1 + \frac{re^{i\theta} g''(re^{i\theta})}{g'(re^{i\theta})}\Big\}d\theta + \int_{\theta_1}^{\theta_2} Re\frac{re^{i\theta}h'(re^{re^{i\theta}})}{h(re^{i\theta})}d\theta. \tag{2.1}$$

It is known [3] that, for $g \in V_k$,

$$\int_{\theta_1}^{\theta_2} Re\Big\{1 + \frac{re^{i\theta} g''(re^{i\theta})}{g'(re^{i\theta})}\Big\}d\theta > -\Big(\frac{k}{2} - 1\Big)\pi. \tag{2.2}$$

Now in the second integral we observe that

$$\frac{\partial}{\partial\theta}\arg\ h(re^{i\theta}) = \frac{\partial}{\partial\theta}Re\{-i\ln h(re^{i\theta})\}$$
$$= Re\Big\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\Big\}.$$

Consequently

$$\int_{\theta_1}^{\theta_2} Re\Big\{\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}\Big\}d\theta = \arg h(re^{i\theta_2}) - \arg h(re^{i\theta_1}).$$

Hence

$$\max_{h\in P(b)}\Big|\int_{\theta_1}^{\theta_2} Re\frac{re^{i\theta}h'(re^{i\theta})}{h(re^{i\theta})}d\theta\Big| = \max_{h\in P(b)}|\arg h(re^{i\theta_2}) - \arg h(re^{i\theta_1})|.$$

Also, from (1.4), we have

$$1 + \frac{1}{b}(h(z) - 1) = p(z), \quad p \in P,$$

and, for $|z| = r < 1$, it is well-known that

$$\left| p(z) - \frac{1 + r^2}{1 - r^2} \right| \le \frac{2r}{1 - r^2}.$$

From this, we have

$$\left| h(z) - \frac{1 + (2b - 1)r^2}{1 - r^2} \right| \le \frac{2|b|r}{1 - r^2}.$$

Thus the values of $h(z)$ are contained in the circle of Apollonius whose diameter is the line segment from $\frac{1+(2b-1)r}{1+r}$ to $\frac{1-(2b-1)r}{1-r}$. The circle is centered at the point $\frac{1+(2b-1)r^2}{1+r^2}$ and has the radius $\frac{1|b|r}{1-r^2}$. So $|\arg h(z)|$ attains its maximum at points where a ray from the origin is tangent to the circle, that is, when

$$\arg h(z) = \pm \sin^{-1} \frac{2|b|r}{1 - |(2b - 1)|r^2}. \tag{2.3}$$

From the above observations, we see that

$$\begin{aligned} \max_{h \in P(b)} \left| \int_{\theta_1}^{\theta_2} Re \frac{re^{i\theta} h'(re^{i\theta})}{h(re^{i\theta})} d\theta \right| &\le 2 \sin^{-1} \frac{2|b|r}{1 - |2b - 1|r^2} \\ &= \pi - 2 \cos^{-1} \frac{2|b|r}{1 - |2b - 1|r^2}. \end{aligned} \tag{2.4}$$

Using (2.2) and (2.4) in (2.1), we obtain the required result.

Remark 2.1. If $f \in T_k(b)$ and b is real, then it can easily be shown that, for $\theta_2 > \theta_1, z = re^{i\theta}$

$$\int_{\theta_1}^{\theta_2} Re\left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta > -\left(\frac{k}{2} - 1 + |b|\right).$$

For $\beta_1 \ge 0$, the class $K(\beta_1)$ has been introduced in [5]. We notice that, if b is real,

(i) $T_k(b) \subset K(\frac{k}{2} + |b| - 1)$
(ii) $T_k(b)$ consists of univalent functions for $k + 2|b| \le 4$ whilst $f \in T_k(b)$ for $k + 2|b| > 4$ need not be finitely-valent.
(iii) It can easily be seen that $T_k(b)$ forms a subset of a linear-invariant family of order $(\frac{k}{2} + |b|)$.

From (2.3) and the well-known result

$$|\arg g'(z)| \le k \sin^{-1} r$$

for $g \in V_k$, we have the following.

Theorem 2.4. *Let $f \in T_k(b)$, $|2b-1| \leq 1$. Then*

$$|\arg f'(z)| \leq k \sin^{-1} r + \sin^{-1} \frac{2|b|r}{1-|2b-1|r^2}.$$

Next we prove a distortion theorem for $T_k(b)$.

Theorem 2.5. *Let $f \in T_k(b)$. Then*

$$\frac{(1-|2b-1|r)(1-r)^{\frac{k}{2}-1}}{(1+r)^{\frac{k}{2}+2}} \leq |f'(z)| \leq \frac{(1+r)^{\frac{k}{2}-1}(1+|2b-1|r)}{(1-r)^{\frac{k}{2}+2}}$$

The equality is attained for the function $f_0 \in T_k(b)$ defined by

$$f_0'(z) = \frac{(1+\delta_1 z)^{\frac{k}{2}-1}}{(1-\delta_2 z)^{\frac{k}{2}+2}}(1+(2b-1)\delta_1 z), \quad |\delta_1| = |\delta_2| = 1.$$

The proof is immediate when we use the distortion theorems for $g \in V_k$, see [13] and for $h \in P(b)$,

$$\frac{1-|2b-1|r}{1+r} \leq |h(z)| \leq \frac{1+|2b-1|r}{1-r}.$$

Speical Cases.

(i) For $k = 2$, f is close-to-convex of complex order b and we have

$$\frac{1-|2b-1|r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+|2b-1|r}{(1-r)^3}.$$

This reult is proved in [1].

(ii) For $b = 1$, we obtain the sharp bounds for $f \in T_k$ established in [9].

Theorem 2.6.(Covering theorem). *The image of E under functions in $T_k(b)$ contains the schlicht disc*

$$|z| < \frac{k+1-|2b-1|}{k(k+2)}.$$

Proof. Let d_r denote the radius of the largest schlicht disc centered at the origin contained in the image of $|z| < r$ under $f(z)$. Then there is a point z_0, $|z_0| = r$ such that $|f(z_0)| = d_r$. The ray from 0 to $f(z_0)$ lies entirely in the image of E and the inverse image of this way is a curve in $|z| < r$.

Thus

$$d_r = |f(z_0)| = \int_c |f'(z)||dz| \geq \int_0^r \frac{(1-b_1 t)}{1+t}\left(\frac{1-t}{1+t}\right)^{\frac{k}{2}-1} \cdot \frac{dt}{(1+t)^2},$$

where $b_1 = |2b - 1|$.

Let $\frac{1-t}{1+t} = \xi$. Then $\frac{-2}{(1+t)^2}dt = d\xi$.

So

$$|f(z_0)| \geq -\frac{1}{4}\int_0^{\frac{1-r}{1+r}} (1-b_1)\xi^{\frac{k}{2}-1}d\xi - \frac{1}{4}\int_0^{\frac{1-r}{1+r}} (1+b_1)\xi^{\frac{k}{2}}d\xi$$
$$= -\frac{1}{2}\Big(\frac{1-r}{1+r}\Big)^{\frac{k}{2}}\Big[\frac{(1+b_1)}{k+2}\Big(\frac{1-r}{1+r}\Big) + \frac{(1-b_1)}{k}\Big] + \Big(\frac{k+(1-b_1)}{k(k+2)}\Big).$$

Now, by letting $r \to 1$, we obtain the required result.

3. Hankel Determinant Problem for $T_k(b)$

Hankel determinant of $f \in A$ and given by (1.1) is defined, for $q \geq 1$, $n \geq 1$, by

$$h_q(n) = \begin{vmatrix} a_n & a_{n+1}\cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots \\ \vdots & & \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For $f \in S^*$, Pommerenke [16] solved this problem completely. He showed that, if $f \in S^*$, $H_q(n) = 0(1)n^{2-q}$ and the exponent $(2-q)$ is best possible.

We shall investigate the rate of growth of $H_q(n)$ for $f \in T_k(b)$. We first prove the following.

Theorem 3.1. *Let $f \in T_k(b), k > 3$ and be given by (1.1). Then, for $m = 0, 1, 2, \ldots,$ there are numbers γ_m and $c_{m\mu}(\mu = 0, \ldots, m)$ that satisfy $|c_{mo}| = |c_{mm}| = 1$, and*

$$\sum_{k=0}^{\infty} \gamma_k \leq 3, \quad 0 \leq \gamma_m \leq \frac{2}{m+1} \tag{3.1}$$

such that

$$\sum_{mu=0}^{\infty} c_{m\mu}a_{n+\mu} = O(1).n^{\gamma_m+\frac{k}{2}-2}, \quad (n \to \infty).$$

The bounds (3.1) are best possible.

Proof. Since $f \in T_k(b)$, there exists $g \in V_k$ such that, for $z \in E$ and $h \in P(b)$, we have

$$f'(z) = g'(z)h(z). \tag{3.2}$$

Let $\tilde{K}(\beta)$ be the class of strongly close-to-convex functions of order β in the sense of Pommerenke [14]. It is known [4] that, for all $k > 2$, V_k is properly contained in $\tilde{K}(\beta)$ and $\beta = (\frac{k}{2} - 1)$. From this it follows that, if $g \in V_k$, $k > 2$, then

$$zg'(z) = s(z)p^{\frac{k}{2}-1}(z),$$

for some $s \in S^*$, $p \in P$.

Thus we can write (3.2) as

$$zf'(z) = s(z)p^{\frac{k}{2}-1}(z)h(z) \tag{3.3}$$

Now s can be represented as

$$s(z) = z\exp[\int_0^{2\pi} \log\frac{1}{1-ze^{-it}}d\mu(t)],$$

where $\mu(t)$ is an increasing function and $\mu(2\pi) - \mu(0) = 2$.

Let $\alpha_1 \geq \alpha_2 \geq \cdots$ be the jumps of $\mu(t)$ and $t = \theta_1, \theta_2, \ldots$ be the values at which these jumps occur. We may assume that $\theta_1 = 0$. Then $\alpha_1+\alpha_2+\cdots \leq 2$ and $\alpha_1+\alpha_2+\cdots+\alpha_q = 2$ for some q if and only if s is of the form

$$s(z) = z\prod_{j=1}^{q}(1-e^{-i\theta_j}z)^{-2/q}. \tag{3.4}$$

We define ϕ_m by

$$\phi_m(z) = \prod_{\mu=1}^{m}(1-e^{i\theta_\mu}z).$$

We consider three cases as in [16] and define η_m, for each case respectively, as follows.

(i) $0 \leq \alpha_1 \leq 1$ and $\eta_m = \alpha_{m+1}$ $(m = 0, 1, 2, \ldots)$

(ii) $1 < \alpha_1 < \frac{3}{2}$ and $\eta_0 = \alpha_1, \eta_1 = \max(\alpha_1 - 1, \alpha_2), \eta_2 = \max(\alpha_1 - 1, \alpha_3), \eta_m = \alpha_m$ for $m \geq 3$.

(iii) $\frac{3}{2} \leq \alpha_1 \leq 2$ and $\eta_0 = \alpha_1, \eta_1 = \max(\alpha_1 - 1, \alpha_2), \eta_m = \alpha_m (m \geq 2)$.

Then the first part, that is the the bounds (3.1), follows similarly as in [16]. For the rest, we need the following.

Lemma 3.1. [16] *Let $\theta_1 < \theta_2 < \cdots < \theta_q < \theta_1 + 2\pi$ and let $\lambda_1, \ldots, \lambda_q$ be real, $\lambda > 0$, $\lambda \geq \lambda_j (j = 1, \ldots, q)$. If*

$$\psi(z) = \prod_{j=1}^{q}(1-e^{-i\theta_j}z)^{-\lambda_j} = \sum_{n=1}^{\infty} b_n z^n, \tag{3.5}$$

then

$$b_n = O(1)\cdot n^{\lambda-1} \quad \text{as } n \to \infty.$$

We now complete the proof of theorem 3.1. We write

$$\phi_m(z) = \sum_{\mu=0}^{m} c_{m\mu} z^{m-\mu}$$

and

$$\phi_m(z) \cdot zf'(z) = \sum_{n=1}^{m} b_{mn} z^{n+m} + \sum_{n=1}^{\infty} (n+m) a_{mn} z^{z+m}, \tag{3.6}$$

where

$$b_{mn} = \sum_{\nu=0}^{n} (n+\nu) c_{m-\nu} a_{n-\nu},$$
$$a_{mn} = \sum_{m\mu} c_{m\mu} a_{n+\mu}, |c_{m_0}| = |c_{mm}| = 1.$$

First let s in (3.3) be not of the form (3.4). Then $\alpha_1 + \alpha_2 + \cdots + \alpha_k < 2$ for $k \geq 1$ and in particular $\alpha_1 < 2$. Hence the number η_m defined as before satisfy

$$\eta_m < \frac{2}{m+1}, \quad \eta_0 + \eta_1 + \cdots < 3.$$

For $m \geq 0$, let

$$\delta_m = \frac{1}{3} \min \Big\{ \frac{2}{m+1} - \eta_m, \frac{1}{2^{m+1}} \Big(3 - \sum_{k=0}^{m} \eta_k \Big) \Big\},$$

and

$$\gamma_m = \eta_m + 2\delta_m.$$

Then $\delta_m > 0$ and $\gamma_m < \frac{2}{m+1}, \gamma_0 + \gamma_1 + \cdots < 3$. Now, it can easily be shown [15] that in each case (i), (ii), (iii),

$$\max_{|z|=r} |\phi_m(z) g(z)| = O(1) \cdot (1-r)^{-\eta_m - \delta_m} \tag{3.7}$$

Thus, from (3.6), (3.3) and Cauchy integral fromula, we have

$$\begin{aligned} (n+m)|a_{mn}| &\leq \frac{1}{r^{n+m}} \Big(\frac{1}{2\pi} \int_0^{2\pi} |\phi_m(z) s(z) p^{\frac{k}{2}-1}(z) h(z)| d\theta \Big) \\ &\leq \frac{1}{2\pi r^{n+m}} \max |\phi_m(z) s(z)| \Big(\int_0^{2\pi} |p^{\frac{k}{2}-1} h(z)| d\theta \Big) \\ &\leq \frac{1}{r^{n+m}} \max |\phi_m(z) s(z)| \Big(\frac{1}{2\pi} \int_0^{2\pi} |p^{k-2}(z)| d\theta \Big)^{\frac{1}{2}} \Big(\frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta \Big)^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

We shall need the following two lemmas.

Lemma 3.2. [6] *Let $p \in P$ for $z \in E$. Then, for $\lambda > 1$,*

$$\int_0^{2\pi} |p(re^{i\theta})|^{\lambda} d\theta < c(\lambda) \frac{1}{(1-r)^{\lambda-1}}.$$

Lemma 3.3. [12] *Let $h \in P(b)$ in E and be given by (1.2). Then*

$$\frac{1}{2\pi}\int_0^{2\pi}|h(re^{i\theta}|^2 d\theta \leq \frac{1+(4|b|^2-1)r^2}{1-r^2}.$$

Using these lemmas in (3.8), we have for $k>3$

$$(n+m)|a_{mn}| = O(1)\cdot(1-r)^{-\eta_m-\delta_m-(\frac{k}{2}-1)} \quad (r\to 1),$$

which implies

$$a_{mn} = O(1)n^{\gamma_m+\frac{k}{2}-2} \quad (n\to\infty).$$

We now consider the case when s in (3.3) has the form (3.4), that is, $\alpha_1+\alpha_2+\cdots+\alpha_q=2$ with $\gamma_m=\eta_m$. It follows that

$$\gamma_m \leq \frac{2}{m+1}, \gamma_0+\gamma_1+\cdots\leq 3,$$

and

$$\gamma_m = \frac{2}{m+1} \text{ implies that } m=q-1, \alpha_1=\cdots=\alpha_q.$$

Thus, for $k>3$

$$\begin{aligned}(n+m)|a_{mn}| &\leq \frac{1}{2\pi r^{n+m}}\int_0^{2\pi}|\phi_m(z)s(z)p^{\frac{k}{2}-1}(z)h(z)|d\theta \\ &\leq \frac{1}{r^{n+1}}\left(\frac{1}{2\pi}\int_0^{2\pi}|\phi_m(z)s(z)p^{\frac{k}{2}-1}(z)|^2 d\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}|h(z)|^2 d\theta\right)^{\frac{1}{2}}. \quad (3.9)\end{aligned}$$

Now

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi_m(z)s(z)|^2|p(z)|^{k-2}d\theta \leq \left(\frac{1}{2\pi}\int_0^{2\pi}|\phi_m(z)s(z)|^4 d\theta\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^{2k-4}d\theta\right)^{\frac{1}{2}}. \quad (3.10)$$

When we write $|\phi_m(z)|^4$ in the form (3.5), the exponents $(-\lambda_j)$ satisfy $\lambda_j \leq 4\gamma_m (j=1,\ldots,q : m>0)$. Hence, by using lemma 2.1, we have

$$\int_0^{2\pi}|\phi_m(z)s(z)|^4 d\theta \leq A_1 n^{4\gamma_m-1} \quad (n\to\infty). \quad (3.11)$$

Also, since $k>3$, we use Lemma 3.2 to have

$$\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^{2k-4}d\theta \leq A_2 n^{2k-5} \quad (n\to\infty). \quad (3.12)$$

Thus, from (3.10), (3.11) and (3.12), we have

$$\frac{1}{2\pi}\int_0^{2\pi}|\phi_m(z)s(z)|^2|p(z)|^{k-2}d\theta \leq An^{2\gamma_m+k-3}. \quad (3.13)$$

Hence, using (3.13) and Lemma 3.3 in (3.9), we obtain

$$(n+m)|a_{mn}| \leq B(k,b)n^{\gamma_m+\frac{k}{2}-1} \quad (n \to \infty),$$

and this gives us

$$a_{mn} = O(1) \cdot n^{\gamma_m+\frac{k}{2}-2}$$

The function $s_0 : s_0(z) = z(1-z^q)^{-\frac{2}{q}} = \sum_{\nu=0}^{\infty} \binom{\frac{2}{q}+\nu-1}{\nu} z^{\nu q+1}$ shows that the bounds in (3.1) are best possible. This completes the proof of theorem 3.1.

We can now easily prove the following result.

Theorem 3.2. *Let $f \in T_k(b)$, $k > 3$ and f be given by (1.1). Then, for $q \geq 1, n \geq 1$,*

$$H_q(n) = O(1) \cdot n^{2+(\frac{k}{2}-2)q}.$$

The exponent $[2 + (\frac{k}{2} - 2q]$ is best possible.

In particular, for $q = 1, k > 3$

$$H_1(n) = a_n = O(1) \cdot n^{\frac{k}{2}} \quad (n \to \infty).$$

Remark 3.1. From remark 2.1, it is clear that, for b real, $T_k(b) \subset K(\frac{k}{2} + |b| - 1)$. Thus for $f \in T_k(b)$, b real, we can write

$$zf'(z) = s(z)p(z)^{\frac{k}{2}+|b|-1}, s \in S^*, p \in P.$$

Following the same techniques of Theorems 3.1 and 3.2 together with the remark 3.1, we have the following.

Theorem 3.3. *Let $f \in T_k(b)$, b real , $k + 2|b| > 3$. Then, for $q \geq 1, n \geq 1$*

$$H_q(n) = O(1) \cdot n^{2+q(\frac{k}{2}+|b|-3)}.$$

4. Some Radii Problems

In the following we find the radius of convexity for $f \in T_k(b)$.

Theorem 4.1. *Let $f \in T_k(b)$. Then f maps $|z| < r_0$ onto a convex domain, where r_0 is the least positive root of the equation*

$$T(r) = (1 + Re\mu) - (1 + k)(1 + Re\mu)r - (1 + k)(1 - Re\mu)r^2 + (1 - Re\mu)r^3 = 0, \quad (4.1)$$

where $\mu = \frac{1-b}{b}$ and $Re\ \mu \geq 0$.

This results is best possible for $b = 1$ where the extremel function $F_k \in T_k(1)$ is given by

$$F_k(z) = \frac{1}{k+2}\Big[(\frac{1+z}{1-z})^{\frac{k}{2}+1} - 1\Big].$$

Also, for $b = 1$, $k = 2$, r_0 is the radius of convexity for $f \in K$ and in this case $r_0 = 2-\sqrt{3}$.

Proof. We can write

$$\begin{aligned} zf'(z) &= zg'(z)h(z), g \in V_k, h \in P(b) \\ &= zg'(z)[bp(z) + (1-b)], p \in P. \end{aligned}$$

Differentiating logarithmically, we have

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zp'(z)}{p(z) + \frac{1-b}{b}}.$$

Thus

$$Re\frac{(zf'(z))'}{f'(z)} \geq Re\frac{(zg'(z))'}{g'(z)} - \Big|\frac{zp'(z)}{p(z)+\mu}\Big|, \quad \mu = \Big(\frac{1-b}{b}\Big).$$

It is known [13] that for $g \in V_k$

$$Re\frac{(zg'(z))'}{g'(z)} \geq \frac{r^2 - kr + 1}{1-r^2}, z = re^{i\theta}, 0 \leq r < 1,$$

and for $p \in P$, $Re\mu \geq 0$,

$$\Big|\frac{zp'(z)}{p(z)+\mu}\Big| \leq \frac{2r}{(1-r)[1+r+Re\mu(1-r)]}, \quad see[7].$$

Hence we have

$$\begin{aligned} Re\frac{(zf'(z))'}{f'(z)} &\geq \frac{r^2-kr+1}{1-r^2} - \frac{2r}{(1-r)[1+r+Re\mu(1-r)]} \\ &= \frac{(1+Re\mu) - (k+1)(1+Re\mu)r - (1+k)(1-Re\mu)r^2 + (1-Re\mu)r^3}{(1-r^2)[1+\gamma+Re\mu(1-r)]}. \end{aligned}$$

This implies that $Re\frac{(zf'(z))'}{f'(z)} > 0$ for $|z| < r_0$ where r_0 is the least positive root of $T(r) = 0$ give by (4.1).

We note here that $T(0) = 1 + Re\mu > 0$ and $T(1) = -2k < 0$ which means that $T(r) = 0$ has at least one zero in (0,1).

Theorem 4.2. *Let $f \in T_k(b)$ and F be defined, for $0 < \alpha < 1$, by*

$$F(z) = \frac{1}{\alpha}z^{1-\frac{1}{\alpha}}\int_0^z \xi^{\frac{1}{\alpha}-2}f(\xi)d\xi.$$

Then F is close-to-convex of complex order b for $|z| < r_1$, where

$$r_1 = \frac{1}{2}[k - \sqrt{k^2 - 4}]. \tag{4.2}$$

This result is sharp.

Proof. Since $f \in T_k(b)$, there exists a function $g \in V_k$ such that $\frac{f'(z)}{g'(z)} \in P(b)$. Let

$$G(z) = \frac{1}{\alpha} z^{1-\frac{1}{\alpha}} \int_0^z \xi^{\frac{1}{\alpha}-2} g(\xi) d\xi.$$

Then, from a special case of a result proved in [11], we see that G is convex for $|z| < r_1$ and this radius is best possible.

Now

$$\begin{aligned}\frac{F'(z)}{G'(z)} &= \frac{z^{\frac{1}{\alpha}-1} f(z) - (\frac{1}{\alpha} - 1) \int_0^z \xi^{\frac{1}{\alpha}-2} f(\xi) d\xi}{z^{\frac{1}{\alpha}-1} g(z) - (\frac{1}{\alpha} - 1) \int_0^z \xi^{\frac{1}{\alpha}-2} g(\xi) d\xi} \\ &= \frac{\int_0^z \xi^{\frac{1}{\alpha}-1} f'(\xi) d\xi}{\int_0^z \xi^{\frac{1}{\alpha}-1} g'(\xi) d\xi} = \frac{N(z)}{D(z)},\end{aligned}$$

and

$$\frac{N'(z)}{D'(z)} = \frac{f'(z)}{g'(z)} \in P(b).$$

Since $g \in V_k$, we know that g is convex for $|z| < r_1$ and so zg' is starlike for $|z| < r_1$. Now, using a similar technique of Libera [8], we can easily show that $\frac{N(z)}{D(z)} \in P(b)$ for $|z| < r_1$ where r_1 is given by (4.2). This completes the proof.

We have the following special cases.

(i) For $k = 2$, f is close-to-convex of complex order b. Then F is also close-to-convex of complex order b in E.
(ii) For $b = 1$, $f \in T_k$. Then F is close-to-convex (hence univalent) in $|z| < r_1$.
(iii) When $b = 1$, $k = 2$, $\frac{1}{\alpha}$ a positive integer, we obtain a result proved by Bernardi [2].
(iv) Libera [8] proved this result with $b = 1$, $k = 2$ and $\alpha = \frac{1}{2}$.

Theorem 4.3. *Let $f \in T_k(b)$ with respect to $h \in V_k$. Let $g \in V_k$ and for α, β positively real with $\alpha + \beta = 1$, let*

$$F(z) = \int_0^z (f'(\xi))^\alpha (g'(\xi))^\beta d\xi$$

and

$$H(z) = \int_0^z (h'(\xi))^\alpha (g'(\xi))^\beta d\xi.$$

Then $F \in T_k$ *with respect to* H *for* $|z| < r_2$ *where* r_2 *is given by*

$$r_2 = \Big[\frac{1}{|b| + \sqrt{|b|^2 - 2Reb + 1}}\Big]. \tag{4.3}$$

Proof. We first note that $H \in V_k$ since

$$\begin{aligned}\frac{(zH'(z))'}{H'(z)} &= \frac{\alpha(zh'(z))'}{h'(z)} + \frac{\beta(zg'(z))'}{g'(z)} \\ &= \alpha p_1(z) + \beta p_2(z), p_1, p_2 \in P_k \\ &= p_3(z), p_3 \in P_k \text{ as } P_k \text{ is a convex set.}\end{aligned}$$

Now

$$\frac{F'(z)}{H'(z)} = \frac{(f'(z))^\alpha (g'(z))^\beta}{(h'(z))^\alpha (g'(z))^\beta} = \Big(\frac{f'(z)}{h'(z)}\Big)^\alpha = (p(z))^\alpha.$$

Since $p \in P$ for $|z| < r_2$ where r_2 is given by (4.3), see [1], it follows that $p^\alpha \in P$ for $|z| < r_2$ which implies that $F \in T_k$ for $|z| < r_2$.

Theorem 4.4. *Let* $f \in V_k$ *and let*

$$F(z) = bz^{2-\frac{1}{b}}[z^{\frac{1}{b}-1}f(z)]'.$$

Then $F \in T_k(b)$ *for all* $|z| < r_1$ *where* r_1 *is given by* (4.2). *This result is sharp.*

Proof. Let $F'(z) = b[(\frac{1}{b} - 1)f'(z) + (zf'(z))']$
Then

$$\frac{F'(z)}{f'(z)} = b\Big[\frac{(zf'(z))'}{f'(z)} + \Big(\frac{1}{b} - 1\Big)\Big] = bH(z) + (1 - b)$$

Since $H \in P_k$, it follows that $H \in P$ for $|z| < r_1$ and the radius r_1 is best possible, see [13]. This implies that, for $|z| < r_1 \cdot F \in T_k(b)$.

Remark 4.1. Since r_1 is the radius of convexity for $g \in V_k$, we can conclude that $f \in T_k(b)$ is close-to-convex of complex order b for $|z| < r_1$ where r_1 is given by (4.2).

Following essentially the same technique used in [10], we can prove:

Theorem 4.5. *Let* $F \in T_2(b)$ *and let, for* $0 < \lambda < 1$,

$$f(z) = (1 - \lambda)F(z) + \lambda zF'(z).$$

Then $f \in T_2(b)$ *for* $|z| < r_\lambda$, *where*

$$r_\lambda = \Big[\frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}}\Big].$$

This result is best possible.

References

[1] H. S. Al-Amiri and T. S. Fernando, "On close-to-convex functions of complex order," *Int. J. Math. and Math. Sci.*, 13(1990), 321-330.

[2] S. D. Bernardi, "Convex and starlike univalent functions," *Trans. Amer. Math. Soc.*, 135(1969), 429-446.

[3] D. A. Brannan, "On functions of bounded boundary rotation," *Proc. Edin. Math. Soc.*, 2(1968/69), 339-347.

[4] D. A. Brannan, J. G. Clunie and W. E. Kirwan, "On the coefficient problem for functions of bounded boundary rotation," *Ann. Acad. Sci. Fenn. Series AI*, 523(1973), 1-18.

[5] A. W. Goodman, "On close-to-convex functions of higher order," *Ann. Univ. Sci. Budapest Eötvös Sect. Math.*, 15(1972), 17-30.

[6] W. K. Hayman, "On functions with positive real part," *J. Lond. Math. Soc.*, 36(1961), 34-48.

[7] R. J. Libera, "Some radius of convexity problems," *Duke Math. J.*, 31(1964), 143-158.

[8] R. J. Libera, "Some classes of regular univalent functions," *Proc. Amer. Math. Soc.*, 16(1965), 753-758.

[9] K. I. Noor, "On a generalization of close-to-convexity," Int. *J. Math. and Math. Sci.*, 6(1983), 327-334.

[10] K. I. Noor, F. Al-Oboudi and N. Al-Dihan, "On the radius of univalence of convex combinations of analytic functions," *Int. J. Math. and Math. Sci.*, 6(1983), 335-340.

[11] K. I. Noor, "On some analytic functions of class $P_k(\alpha)$," *C. R. Math. Rep. Acad. Sci.*, Canada, 12(1990), 69-74.

[12] K. I. Noor, "On close-to-convex functions of complex order and related subclasses, Analysis," *Geometry and Groups: A Riemann Legacy Volume (ed. H. M. Srivastava and Th. M. Rassias)* Hadronic Press, U. S. A., (1994), 313-335.

[13] B. Pinchuk, "Functions with bounded boundary rotation," *I. J. Math.*, 10(1971), 7-16.

[14] Ch. Pommerenke, "On close-to-convex analytic functions," *Trans. Amer. Math. Soc.*, 114(1965), 176-186.

[15] Ch. Pommerenke, "On starlike and close-to-convex functions," *Proc. Amer. Math. Soc.*, 13(1963), 290-304.

[16] Ch. Pommerenke, "On the coefficients and Hankel determinants of univalent functions," *J. Lond. Math. Soc.*, 41(1966), 111-122.

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