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INEQUALITIES OF THE COMPLETE ELLIPTIC INTEGRALS

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Abstract. In this article, using Tchebycheff's integral inequality, the authors establish some estimates and inequalities for three kinds of the complete elliptic integrals.

1. Introduction

It is well-known that the elliptic integrals can not be represented by elementary functions, it is also called as Abel's integral. The complete elliptic integrals are classed into three kinds, they are defined as and denoted by

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta, \tag{1}$$

$$F(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \tag{2}$$

$$II(k,h) = \int_0^{\pi/2} \frac{d\theta}{(1+h\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}, \tag{3}$$

where $0 < k < 1$.

The first kind and second kind of the complete elliptic integral can be expanded into power-series as follows,

$$E(k) = \frac{\pi}{2}\Big\{1 - \sum_{n=1}^{\infty} \frac{(2n-2)!(2n)!}{2^{4n-1}(n-1)!(n!)^3}\cdot k^{2n}\Big\}, \tag{4}$$

$$F(k) = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{[(2n)!]^2}{2^{4n}(n!)^4}\cdot k^{2n}. \tag{5}$$

The first kind of the complete elliptic integral was estimated in [2, 3] by

$$\frac{\pi(a+b)}{4} \le \int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta \le \frac{\pi\sqrt{2(a^2+b^2)}}{4}, \tag{6}$$

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$$\int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}d\theta \geq \frac{\pi(2a^2+b^2)}{3a}, \tag{7}$$

where $0 < k < 1$, $k = \sqrt{a^2-b^2}/a$.

Recently, the upper bound of (6) was improved in [1] for $a \geq 7b$ by a better constant $(a+2b)\pi/6$.

Another inequality for the complete elliptic integral was estimated in [2, 12] by the following inequality

$$\frac{\pi}{6} < \int_0^1 (4-x^2-x^3)^{-1/2}dx < \frac{\pi\sqrt{2}}{8} \tag{8}$$

The first author improved (8) in [9, 10] and got

$$\frac{3}{10} + \frac{27\sqrt{2}}{160} < \int_0^1 (4-x^2-x^3)^{-1/2}dx < \frac{79}{192} + \frac{\sqrt{2}}{10} \tag{9}$$

In this article, by using Tchebycheff's integral inequality, we get some estimates and inequalities of three kinds of the complete elliptic integrals. The main results are as follows

$$\frac{\pi \arcsin k}{2k} < F(k) < \frac{\pi \ln(\frac{1+k}{1-k})}{4k}; \tag{10}$$

$$E(k) < \frac{16-4k^2-3k^4}{4(4+k^2)}F(k); \tag{11}$$

$$F(k) < \Big(1+\frac{h}{2}\Big)II(k,h), \quad -1 < h < 0, \quad \text{or} \quad h > \frac{k^2}{2-3k^2} > 0; \tag{12}$$

$$II(k,h)\cdot E(k) > \frac{\pi^2}{4\sqrt{1+h}}, \ -2 < 2h < k^2; \tag{13}$$

$$E(k) \geq \frac{16-28k^2+9k^4}{4(4-5k^2)}F(k), \quad k^2 \leq \frac{2}{3}. \tag{14}$$

For $0 < 2h < k^2$, the inequality (12) is reversed; for $h > k^2/(2-3k^2) > 0$, inequality (13) is reversed.

For our own conveniences, we state the Tchebycheff's integral inequality in [2, 4, 5], which we will use throughout this article repeatedly, as follows

Lemma. *Let $f, g : [a,b] \to R$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a,b] \to R$ be a positive, integrable function. Then*

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx. \tag{15}$$

If one of the functions f or g is nonincreasing and the other nondecreasing, then the inequality in (15) is reversed.

2. Proofs of Inequalities for Elliptic Integrals

2.1. Let $p(x) = 1$, $f(x) = (1 - k^2\sin^2 x)^{-1/2}$, $g(x) = \cos x$ or $\sin x$, $[a, b] = [0, \pi/2]$ in (15), then we could get

$$\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \int_0^{\pi/2} \cos x dx > \int_0^{\pi/2} dx \int_0^{\pi/2} \frac{\cos x dx}{\sqrt{1-k^2\sin^2 x}}$$

or

$$\int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2\sin^2 x}} \int_0^{\pi/2} \sin x dx < \int_0^{\pi/2} dx \int_0^{\pi/2} \frac{\sin x dx}{\sqrt{1-k^2\sin^2 x}}$$

By direct calculation one gets the inequality (10), which is the estimates of the second kind of the complete elliptic integral $F(k)$.

2.2. Letting

$$\begin{aligned} p(x) &= (1 + k^2\sin^2 x/2)(1 - k^2\sin^2 x), \\ g(x) &= (1 - k^2\sin^2 x)^{-1}, \\ f(x) &= \left[(1 + k^2\sin^2 x/2)\sqrt{1-k^2\sin^2 x}\right]^{-1}, \end{aligned}$$

for $x \in [0, \pi/2]$ in (15), then

$$\begin{aligned} &\int_0^{\pi/2} \sqrt{1-k^2\sin^2 x}dx \int_0^{\pi/2} (1 + k^2\sin^2 x/2)dx \\ <& \int_0^{\pi/2} (1 + k^2\sin^2 x/2)(1 - k^2\sin^2 x)dx \int_0^{\pi/2} (1 - k^2\sin^2 x)^{-1/2}dx \end{aligned}$$

By direct computation one can obtain inequality (11).

2.3. Set $g(x) = (1 + h\sin^2 x)\sqrt{1-k^2\sin^2 x}$, $x \in [0, \pi/2]$, then

$$g'(x) = \frac{\sin x\cos x[(2h - k^2) - 3hk^2\sin^2 x]}{\sqrt{1-k^2\sin^2 x}}$$

while $0 < 2h < k^2$, we have $g'(x) < 0$, $g(x)$ decreases; while $h > k^2/(2 - 3k^2) > 0$, we have $g'(x) > 0$, $g(x)$ increases; while $-1 < h < 0$, it is clear that $g(x)$ decreases.

If $-1 < h < 0$, or $h > k^2/(2 - 3k^2) > 0$, let

$$p(x) = (1 - k^2\sin^2 x)^{-1/2}, \quad f(x) = (1 + h\sin^2 x)^{-1}$$

and $[a, b] = [0, \pi/2]$, from (15) one has

$$II(k, h) \int_0^{\pi/2} (1 + h\sin^2 x)dx > F(k) \int_0^{\pi/2} dx$$

As a result, inequality (12) holds. If $0 < 2h < k^2$, inequality (12) is reversed.

2.4. Assume

$$f(x) = \sqrt{1 - k^2 \sin^2 x}, \quad g(x) = \left[(1 + h \sin^2 x)\sqrt{1 - k^2 \sin^2 x}\right]^{-1},$$

and $p(x) = 1, x \in [0, \pi/2]$ in (15). For $-2 < 2h < k^2$ we obtain

$$II(k,h) \cdot F(k) > \frac{\pi}{2} \int_0^{\pi/2} \frac{d\theta}{1 + h \sin^2 \theta} = \frac{\pi^2}{4\sqrt{1+h}}.$$

For $h > k^2/(2 - 3k^2) > 0$, since $g(x)$ is increasing, the reversed inequality of (13) is obtained.

2.5. For $h = k^2/(2 - 3k^2) > 0$, let

$$\begin{aligned} p(x) &= (1 + h \sin^2 x)(1 - k^2 \sin^2 x), \\ f(x) &= \left[(1 + h \sin^2 x)\sqrt{1 - k^2 \sin^2 x}\right]^{-1}, \\ g(x) &= (1 - k^2 \sin^2 x)^{-1} \end{aligned}$$

for $x \in [0, \pi/2]$, the inequality (15) implies (14).

Remark 1. As concrete examples we have the following estimates of the complete elliptic integrals

$$\frac{\pi^2}{4\sqrt{2}} < \int_0^{\pi/2} \left(1 - \frac{\sin^2 x}{2}\right)^{-1/2} dx < \frac{\pi \ln(1 + \sqrt{2})}{\sqrt{2}} \tag{16}$$

$$\int_0^{\pi/2} \left(1 + \frac{\cos x}{2}\right)^{-1} dx < \frac{\pi(\ln 3 - \ln 2)}{2} \tag{17}$$

$$\int_0^{\pi/2} \left(1 - \frac{\sin x}{2}\right)^{-1} dx = \int_{\pi/2}^{\pi} \left(1 + \frac{\cos x}{2}\right)^{-1} dx > \frac{\pi \ln 2}{2}. \tag{18}$$

These results are better than those in [2, p. 607].

Remark 2. Using Tchebycheff's integral inequality, we can refine Conte's inequality and some other inequalities relating to the incomplete gamma and the probability functions [8], obtain more particular inequalities [8], and verify the monotonicities of the generalized weighted mean values [6, 7, 11].

Remark 3. It is clear that, by the similar arguments, we can get inequalities of the incomplete elliptic integrals.

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