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ON ABSOLUTE NÖRLUND SUMMABILITY OF ORTHOGONAL SERIES

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Abstract. The purpose of this paper is to give a general theorem on the $|N,p_n;\delta|_k$ summability of orthogonal series, which generalizes a theorem due to Okuyama [1] related to summability of orthogonal series.

1. Introduction

Let $\sum a_n$ be a given infinite series with (s_n) as its n-th partial sum. If (p_n) is a sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n}p_v\to\infty \text{ as } n\to\infty,\quad (P_{-i}=p_{-i}=0, i\geq 1).$$

The sequence-to-sequence transformation

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}p_{n-v}s_v=\frac{1}{P_n}\sum_{v=0}^{n}P_v a_{n-v},\quad (P_n\neq 0)\tag{1.1}$$

defines the sequence (T_n) of the (N,p_n) means of the sequence (s_n) generated by the sequence of coefficients (p_n).

The series $\sum a_n$ is said to be summable $|N,p_n|_k$, $k\geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}(\frac{P_n}{P_n})^{k-1}|T_n-T_{n-1}|^k<\infty.\tag{1.2}$$

The case $k=1$ is reduced to the Nörlund summability $|N,p_n|$ and further, in the special case in which $p_n=A_n^{\delta-1}=\binom{n+\delta-1}{n}$ and $p_n=\frac{1}{n+1}$, the summability $|N,p_n|$ is the same as the summability $|C,\delta|$ and the absolute harmonic summability, respectively
The series $\sum a_n$ is said to be summable $|N,p_n;\delta|_k$, $k\geq 1$, $\delta\geq 0$, if (see [2])

$$\sum_{n=1}^{\infty}(\frac{P_n}{p_n})^{\delta k+k-1}|T_n-T_{n-1}|^k<\infty.$$

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Let $\{\Phi_n(x)\}$ be an orthonormal system defined in the interval (a, b). For a function $f(x) \in L^2(a, b)$ such that

$$f(x) \approx \sum_{n=0}^{\infty} a_n \Phi_n(x), \tag{1.3}$$

We denote by $E_n^{(2)}(f)$ the best approximation to $f(x)$ in the metric of L^2 by means of polynomials of $\Phi_0(x), \ldots, \Phi_{n-1}(x)$. It is well known that

$$E_n^{(2)}(f) = \Big(\sum_{j=n}^{\infty} |a_j|^2\Big)^{\frac{1}{2}}.$$

We put $\Delta\lambda_n = \lambda_n - \lambda_{n-1}$ for any sequence $\{\lambda_n\}$. A is a positive constant necessarily the same at each occurrence.

2. Preliminary Result

Dealing with the absolute Nörlund summability of orthogonal series, Okuyama [1] proved the following theorem.

Theorem A. *Let $1 \leq k \leq 2$ and $\{\lambda_n\}$ be a positive sequence. If $\{p_n\}$ is a positive sequence and the series*

$$\sum_{n=1}^{\infty} \frac{p_n}{p_n p_{n-1}^k} \left\{ \sum_{j=1}^{n} p_{n-j}^2 \Big\{ \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big\}^2 \lambda_j^2 |a_j|^2 \right\}^{\frac{k}{2}}.$$

converges, then the orthogonal series

$$\sum \lambda_n a_n \Phi_n(x) \tag{1.4}$$

is summable $|N, p_n|_k$ almost everywhere.

In this paper we shall prove the following theorem.

Theorem. *Let $1 \leq k \leq 2$ and $0 \leq \delta k < 1$. If $\{p_n\}$ and $\{\lambda_n\}$ are positive sequences and the series*

$$\sum_{n=1}^{\infty} \Big(\frac{p_m}{p_n}\Big)^{\delta k - 1} \frac{1}{p_{n-1}^k} \left\{ \sum_{j=1}^{n} p_{n-j}^2 \Big\{ \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \Big\}^2 \lambda_j^2 |a_j|^2 \right\}^{\frac{k}{2}}.$$

converges, then the orthogonal series $\sum \lambda_n a_n \Phi_n(x)$ is summable $|N, p_n; \delta|_k$ almost everywhere.

Proof of the Theorem. Let $T_n(x)$ be the n-th Nörlund mean of the series (1.4). Then we have by (1.1)

$$\Delta T_n(x) = T_n(x) - T_{n-1}(x) = \frac{p_n}{P_n P_{n-1}} \sum_{j=1}^{n} p_{n-j}\left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right)\lambda_j a_j \Phi_j(x)$$

Using the Hölder's inequality and the orthogonality

$$\int_a^b |\Delta T_n(x)|^k dx \le A\left\{\int_a^b |\Delta T_n(x)|^2 dx\right\}^{\frac{k}{2}}$$

$$= A\left(\frac{p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{j=1}^{n} p_{n-j}^2 \left\{\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right\}^2 \lambda_j^2 |a_j|^2\right\}^{\frac{k}{2}}$$

and then

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \int_a^b |\Delta T_n(x)|^k dx$$

$$\le A \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} \left(\frac{p_n}{P_n P_{n-1}}\right)^k \left\{\sum_{j=1}^{n} p_{n-j}^2 \left\{\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right\}^2 \lambda_j^2 |a_j|^2\right\}^{\frac{k}{2}}$$

$$= A \sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{\sum_{j=1}^{n} p_{n-j}^2 \left\{\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right\}^2 \lambda_j^2 |a_j|^2\right\}^{\frac{k}{2}}$$

which is convergent by the assumption and from the Beppo-Lèvi lemma we complete the proof.

In this theorem, if we take $\delta = 0$, then we get Theorem A.

References

[1] Y. Okuyama, *Absolute summability of Fourier series and orthogonal series*, Lecture Notes in Math., No. 1067, Springer-verlag, 1984.

[2] S. Umar and H. H. Khan, "On $|N_p, \gamma, \alpha|_k$ summability of infinite serie," *Indian J. Pure and Appl. Math.*, 8(1977), 752-757.

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