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BEST COAPPROXIMATION IN LOCALLY CONVEX SPACES

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Let G be a non-empty subset of a normed liner space X. An element $g_0 \in G$ is called a best coapproximation to an element $x \in X$ if $\|g_0 - g\| \leq \|x - g\|$ for all $g \in G$. This concept was introduced by C. Franchetti and M. Furi [2] in 1972 and was subsequently studied by H. Berens, L. Hetzelt, P. L. Papini, Ivan Singer, the authors and few others (see [2], [3], [4], [5] and [8]). There are plenty of spaces which are not normable e.g. (i) the space $C(\mathbb{R})$ of all real (complex)-valued continuous functions on the real line $\mathbb{R}$ whose topology is defined by a family $\{p_n : n \in \mathbb{N}\}$ of semi-norms, $p_n(f) = \max_{|t| \leq n} |f(t)|$; (ii) the space $L^p_{l\alpha}(\mathbb{R})(1 < p < \infty)$ of all locally summable functions with the topology determined by the sequence of semi norms $p_n(f)[\int_{-\pi}^{\pi} |f(t)|^p dt]^{\frac{1}{p}}$, $n \in \mathbb{N}$. In order to discuss best coapproximation in such spaces, the concept was extended from normed linear spaces to locally convex spaces by Geetha S. Rao and S. Elmulai [6]. In this paper, we also discuss best coapproximation in locally convex spaces thereby extending some of the results proved in [5].

Let X be a Hausdorff locally convex linear topological space equipped with a family P of semi-norms, G a non-empty subset of X, $x \in X$ and $g_0 \in G$. Then g_0 is said to be a best coapproximation to x in G if $p(g_0 - g) \leq P(x - g)$ for every $g \in G$ and every $p \in P$. We denote by $R_G(x)$ the set of all best coapproximations to x in G. We consider R_G as a set-valued mapping from $D(R_G) = \{x \in X : R_G(x) \neq \phi\}$ into G. The ampping R_G satisfies the following properties:

Theorem 1. *If G is a subset of the locally convex space X, $x \in X$ and $g_0 \in G$ then*

(a) *G is contained in $D(R_G)$, the domain of R_G. Moreover, $R_G(x) = \{x\}$ for every $x \in G$,*
(b) *$R_G(x)$ is closed if G is closed,*
(c) *$R_G(x)$ is convex if G is convex, and*
(d) *$g_0 \in R_G(x)$ if and only if $g_0 \in R_G[tx + (1-t)g_0]$ for $t \geq 1$.*

Proof.

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(a) Suppose $g_0 \in G$. Then $g_0 \in R_G(g_0)$ since

$$p(g_0 - g) \le p(g_0 - g), \quad p \in P, \quad g \in G$$

Therefore $g_0 \in D(R_G)$ and so $G \subset D(R_G)$. Also, $g_0 \in R_G(g_0) \Rightarrow \{g_0\} \subset R_G(g_0)$. Now, supppose $y \in R_G(g_0)$. Then

$$p(y - g) \le p(g_0 - g) \quad \text{for all} \quad g \in G, \quad p \in P$$

and so

$$p(y - g_0) \le p(g_0 - g_0) = 0, \quad p \in P.$$

Therefore $y = g_0$ and consequently, $R_G(g_0) \subset \{g_0\}$. Hence $R_G(x) = \{x\}$ for all $x \in G$

(b) Suppose y is a limit point of $R_G(x)$. Then there exists a sequence $\{y_n\}$ in $R_G(x)$ such that $y_n \to y$. Now $y_n \in R_G(x) \Rightarrow p(y_n - g) \le p(x - g)$ for all $n \in N$, $g \in G$ and $p \in P \Rightarrow p(y - g) \le p(x - g)$ for all $g \in G$ and $p \in P$, i.e. $y \in R_G(x)$ since $y \in G$ by the closedness of G.

(c) Let $g_1, g_2 \in R_G(x)$. Then for all $g \in G$ and $p \in P$, $p(g_1 - g) \le p(x - g)$, $p(g_2 - g) \le p(x - g)$. Consider

$$\begin{aligned} p[\alpha g_1 + (1-\alpha)g_2 - g] &= p[\alpha(g_1 - g) + (1-\alpha)(g_2 - g)], \qquad 0 \le \alpha \le 1 \\ &\le \alpha p(g_1 - g) + (1-\alpha)p(g_2 - g) \\ &\le \alpha p(x - g) + (1-\alpha)p(x - g) \\ &= p(x - g). \end{aligned}$$

So, $\alpha g_1 + (1-\alpha)g_2 \in R_G(x)$. Hence $R_G(x)$ is convex.

(d) Consider

$$\begin{aligned} p[tx + (1-t)g_0 - g] &= p[t(x - g) + (1-t)(g_0 - g)] \\ &\ge |t|p(x - g) - |1-t|p(g_0 - g) \\ &= tp(x - g) + (1-t)p(g_0 - g) \\ &\ge tp(g_0 - g) + (1-t)p(g_0 - g) \\ &= p(g_0 - g) \end{aligned}$$

i.e. $p(g_0 - g) \le p[tx + (1-t)g_0 - g]$ for all $g \in G$ and $t \ge 1$. Therefore $g_0 \in R_G[tx + (1-t)g_0]$ for $t \ge 1$. The converse part follows by taking $t = 1$.

Remarks. For normed linear spaces, these properties of R_G were observed in [5]. If G is a convex subset of the space X and $\tau(x, y) = \lim_{t \to 0^+} \frac{p(x+ty) - p(x)}{t} = \inf_{t>0} \frac{p(x+ty) - p(x)}{t}$ $p \in P$ is the tangent functional (see [1]) defined from $X \times X$ into R, we have:

Theorem 2. *An element $go \in G$ belongs to $R_G(x)$ if $\tau_p(go - g, x - go) \ge 0$ for every $g \in G$.*

Proof. Suppose $g \in G$, $p \in P$ and $t > 0$. Then

$$0 \le \tau_p(g_0 - g, x - g_0) = \lim_{t \to 0^+} \frac{p[(g_0 - g) + t(x - g_0)] - p(g_0 - g)}{t}$$

implies $p[(g_0 - g) + t(x - g_0)] \ge p(g_0 - g)$ and so $p(x - g) \ge p(g_0 - g)$ by putting $t = 1$, i.e. $g_0 \in R_G(x)$.

Some properties of the 'orthogonal retraction map' R'_G which is defined as:

an element $g_0 \in G$ belongs to $R'_G(x)$ if $\tau_p(g_0 - g, x - g_0) \ge 0$ for every $g \in G$, $p \in P$ were proved for normed linear spaces in [5]. We now consider this map in our setting of locally convex spaces.

If G is a subset of the locally convex space X, we say that $g_0 \in G$ belongs to $R'_G(x)$ if $\tau_p(g_0 - g, x - g_0) \ge 0$ for every $g \in G$. i.e. $p(g_0 - g) \le p[(g_0 - g) + t(x - g_0)]$ for every $g \in G$, $p \in P$ and $t \ge 0$.

Theorem 2 shows that if $g_0 \in R'_G(x)$ then $g_0 \in R_G(x)$. We also have:

Theorem 3. *$g_0 \in R'_G(x)$, if and only if $g_0 \in R'_G[tx + (1-t)g_0]$ for every $t \ge 0$.*

Proof. Let $g_0 \in R'_G(x)$, i.e. $p(g_0 - g) \le p[(g_0 - g) + t(x - g_0)]$ for every $g \in G$, $p \in P$ and $t \ge 0$. Suppose $s > 0$. Consider

$$\begin{aligned} p[s\{tx + (1-t)g_0 - g_0\} + (g_0 - g)] &= p(st(x - g_0) + g_0 - g] \\ &= p[t'(x - g_0) + g_0 - g], \ (t' = st) \\ &\ge p(g_0 - g) \end{aligned}$$

and so $g_0 \in R'_G[tx + (1-t)g_0]$. For $t = 0$, the result is obvious.

The converse part follows by taking $t = 1$.

Remark. For normed linear spaces, Theorems 2 and 3 were proved by Papini [5].

We now consider two properties (Theorems 4 and 5) which are sufficient for $R_G(x) = R'_G(x)$. For normed linear spaces, these were proved in [5].

Theorem 4. *Suppose that $R_G(x)$ satisfies:*

$$\text{if } g_0 \in R_G(x) \text{ then } g_0 \in R_G[tx + (1-t)g_0] \text{ for } 0 \le t \le 1 \tag{1}$$

Then $R_G(x) = R'_G(x)$.

Proof. Suppose $g_0 \in R'_G(x)$ then by Theorem 2 we get $g_0 \in R_G(x)$. Now suppose $g_0 \in R_G(x)$ then by (1), $g_0 \in R_G[tx + (1-t)g_0]$ for $0 \le t \le 1$ and so

$$\begin{aligned} p(g_0 - g) &\le p[tx + (1-t)g_0 - g], \quad g \in G \text{ and } p \in P \\ &= p[(g_0 - g) + t(x - g_0)]. \end{aligned}$$

This gives

$$\tau_p(g_0 - g, x - g_0) = \lim_{t \to 0^+} \frac{p[(g_0 - g) + t(x - g_0)] - p(g_0 - g)}{t} \ge 0$$

i.e., $g_0 \in R'_G(x)$. Hence $R_G(x) = R'_G(x)$.

Form Theorems 2, 3 and 4 we obtain:

Corollary. *$g_0 \in R'_G(x)$ if and only if $g_0 \in R_G[tx + (1-t)g_0]$ for $0 \le t \le 1$ (and so for every $t \ge 0$).*

For a linear subspace G, this result was proved in [6] (Proposition 3.2).

Theorem 5. *Suppose that $R_G(x)$ satisfies:*

$$\text{if } g_0 \in R_G(x) \text{ and } g \in G \text{ then } (1-t)g_0 + tg \in G \text{ for } t \ge 1. \tag{2}$$

Then $R_G(x) = R'_G(x)$.

Proof. Suppose $g_0 \in R'_G(x)$ then by Theorem 2 we get $g_0 \in R_G(x)$. Now suppose $g_0 \in R_G(x)$ and $g \in G$. Then for $t \ge 1$

$$\begin{aligned} p[(x-g_0) + t(g_0 - g)] &= p[x - \{(1-t)g_0 + tg\}] \\ &\ge p[g_0 - \{(1-t)g_0 + tg\}] \quad (\text{by } \ (2)) \\ &= tp(g_0 - g.) \end{aligned}$$

This implies

$$p[s(x-g_0) + (g_0 - g)] \ge p(g_0 - g), \quad s = \frac{1}{t}, \quad 0 \le s \le 1$$

and so,

$$\tau(g_0 - g, x - g_0) = \lim_{s \to 0^+} \frac{p[(g_0 - g) + s(x - g_0)] - p(g_0 - g)}{s} \ge 0$$

i.e., $g_0 \in R'_G(x)$.

This theorem implies:

If G is a subset of the space X satisfying $g_1, g_2 \in G \Rightarrow tg_1 + (1-t)g_2 \in G$ for $t \ge 1$ then $R_G(x) = R'_G(x)$.

Remark 1. An element $g_o \in G$ is said to be a strong best coapproximation of x by elements of G if there exists an $r > 0$ ($r \le 1$) such that for every $g \in G$ and $p \in P$, $p(x-g) \ge p(g_0 - g) + rp(x - g_0)$. Clearly, strong best coapproximation implies best coapproximation. The notion of strong best coapproximation has been discussed by P. L. Papini [5] in normed linear spaces and by Geetha S. Rao and S. Elumalai [6] in locally convex spaces. For a subset G of the space X, we say that $g_0 \in R'_G(x)$ strongly if $x \in G$ and there exists an $r > 0 (r \le 1)$ such that $\tau_P(g_0 - g, x - g_0) \ge rp(x - g_0)$ for every $g \in G$, $x \ne g_0$ and $p \in P$. Clearly, $g_0 \in R'_G(x)$ strongly implies $g_0 \in R_G(x)$ strongly. Suppose $g_0 \in R_G(x)$ strongly and (2) is satisfied then

$$\tau_p(g_0 - g, x - g_0) = \lim_{t \to 0^+} \frac{p[(g_0 - g) + t(x - g_0)] - p(g_0 - g)}{t}$$

$$= \lim_{\lambda\to\infty} p[\lambda(g_0 - g) + (x - g_0)] - p[\lambda(g_0 - g)]$$
$$= \lim_{\lambda\to\infty} p[x - \{\lambda g + (1-\lambda)g_0\}] - p[g_0 - \{\lambda g + (1-\lambda)g_0\}]$$
$$\geq rp(x - g_0)$$

i.e., $g_0 \in R'_G(x)$ strongly. It will be interesting to study this strong best coapproximation map in the context of locally convex spaces.

Remark 2. As a counterpart to the notion of 'sun' available in literature (notion first introduced by N. V. Efimov and S. B. Stekin in the theory of best approximation in normed linear spaces-see e.g. [7]), we may intorduce the notion of 'cosun' in the locally convex space X as:

A subset G of the space X is called a 'cosun' if for each $x \in X$ there is at least one $g_0 \in R_G(x)$ such that $g_0 \in R_G[\lambda x + (1-\lambda)g_0]$ for each $\lambda \geq 0$.

In the theory of best coapproximation, cosuns have been discussed in normed linear spaces by L. Hetzelt, W. Westphal and few others (see [8]).

It will be interesting to study consuns in locally convex spaces.

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