TAMKANG JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1997

A CLASS OF FUNCTIONS AND THEIR DEGREE OF APPROXIMATION BY ALMOST (N, p, α) METHOD

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Abstract. Qureshi [6] proved a theorem for the degree of approximation of a periodic function $\overline{f}$, conjugate to a 2π-periodic function f and belonging to the class Lip θ, by almost matrix mean of its conjugate series. The above theorem was further generalized by Qureshi and Nema [8] for a function belonging to the class $W(L^p, \Psi_1(t))$ by almost matrix mean. In the present paper we have discussed degree of approximation of above class of functions by almost (N, p, α) method.

1. Let $f(x)$ be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series associated with $f(x)$, is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

then

$$\overline{f}(x) \sim \sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx), \tag{1.2}$$

is called the conjugate series of $f(x)$.

A function $f \in \text{Lip}\ \theta$ if

$$f(x+h) - f(x) = O(|h|^\theta) \quad \text{for} \quad 0 < \theta \leq 1, \tag{1.3}$$

We define the norm $||\cdot||_p$ by

$$||f||_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, \quad p \geq 1$$

and the degree of approximation $E_n(f)$ by

$$E_n(f) = \min_{T_n} ||f - T_n||_p \qquad \text{(see Zygmund [7])}$$

where T_n is a trigonometrical polynomial of degree n. We say

$$f(x) \in \text{Lip}(\theta, q) \quad \text{for} \quad a \leq x \leq b \tag{1.4}$$

Received November 15, 1995.
Key words and phrases. Fourier series, almost matrix means, almost (N, p, α) mean, almost Nörlund means.

if

$$\Big(\int_a^b |f(x+h)-f(x)|^q dx\Big)^{1/q} \leq A|h|^\theta, \quad 0 \leq \theta < 1, q \geq 1,$$

where A is some constant. (see def. 5.38 of McFadden [2])

Given a positive increasing function $\Psi_1(t)$ and an integer $p > 1$, we notice (see Qureshi [4]) $f(x) \in \text{Lip}(\Psi_1(t), p)$ if

$$\Big(\int_0^{2\pi} |f(x+t)-f(x)|^p dx\Big)^{1/p} = O(\Psi_1(t)) \tag{1.5}$$

and that $f(x) \in W(L^p, \Psi_1(t))$ if

$$\Big(\int_0^{2\pi} |f(x+t)-f(x)|^p \sin^{\beta p} x\, dx\Big)^{\frac{1}{p}} = O(\Psi_1(t)), \quad \beta \geq 0. \tag{1.6}$$

In case $\beta = 0$, we find that our newly defined class $W(L^p, \Psi_1(t))$ coincides with the class $\text{Lip}(\Psi_1(t), p)$.

Lorentz [1] has defined:

Definition 1. A sequence $\{S_n\}$ is said to be almost convergent to a limit S, if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=p}^{n+p} S_k = S \tag{1.7}$$

uniformly with respect to p.

An almost convergence is a generalization of ordinary convergence. Qureshi [5] defined almost Nörlund means. Qureshi [3] have also defined almost triangular matrix means as:

Definition 2. If $(a_{n,k})$ $(n = 0, 1, \ldots, K = 0, 1, \ldots n)$; $a_{n,0} = 1$ be a triangular matrix with real or complex elements, then a series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{S_n\}$ is said to be almost triangular matrix summable to S, provided

$$\sigma_{n,p} = \sum_{k=0}^{n} a_{n,k}\, S_{k,p} \to S \quad \text{as} \quad n \to \infty \tag{1.8}$$

uniformly with respect to p, where

$$S_{k,p} = \frac{1}{K+1} \sum_{\mu=p}^{k+p} S\mu.$$

Definition 3. In above definition almost matrix summability reduces to almost (N, p, α) summability if

$$a_{n,k} = \frac{a_k p_{n-k}}{(p \times \alpha)_n} = \frac{a_k p_{n-k}}{r_n}$$

where

$$r_n = (p \times \alpha)_n = p_0\alpha_n + p_1\alpha_{n-1} + \cdots + p_n\alpha_0$$
$$= \sum_{k=0}^{n} \alpha_k\, p_{n-k}.$$

Now the series $\sum_{n=0}^{\infty} u_n$ with the sequence of partial sums $\{S_n\}$ is said to be almost (N, p, α) summable to S, provided

$$\tau_{n,p} = \sum_{k=0}^{n} \frac{\alpha_k\, p_{n-k}\, S_{k,p}}{(p \times \alpha)_k} = \frac{1}{(p \times \alpha)_n} \sum_{k=0}^{n} \alpha_k\, p_{n-k}\, S_{k,p}$$

uniformly with respect to p, where

$$S_{k,p} = \frac{1}{K+1} \sum_{\mu=p}^{k+p} S_\mu.$$

2. Qureshi [6] proved the following theorems.

Theorem A. *The degree of approximation of a periodic function $\overline{f}$, conjugate to a 2π periodic function f and belonging to the class of Lip θ by almost Nörlund means of its conjugate series, is given by*

$$|\overline{f}(x) - \overline{t}_{n,p}(x)| = \begin{cases} O[(\frac{1}{n})^{\theta}]; & 0 < \theta < 1 \\ O[(\frac{1}{n}\log n)]; & \theta = 1 \end{cases} \tag{2.1}$$

where $\overline{t}_{n,p}(x)$ are the almost Nörlund means of the series (1.2) and the sequence $\{p_n\}$ is non-negative and non-increasing such that

$$\sum_{k=0}^{n} \frac{p_{n-k}}{K+1} = O(\frac{P_n}{n}).$$

Theorem B. *If a function $\overline{f(x)}$ is conjugate to a 2π periodic function $f(x)$ belonging to the class of Lip θ for $0 < \theta \leq 1$, then*

$$|\overline{\sigma}_{n,p} - \overline{f(x)}| = \begin{cases} O[(\frac{1}{n})^{\theta-1} \sum\limits_{k=0}^{n} \frac{a_{n,k}}{K+1}];\ 0 < \theta < 1 \\ O[(\log n) \sum\limits_{k=0}^{n} \frac{a_{n,k}}{K+1}];\ \theta = 1 \end{cases} \tag{2.2}$$

where $\overline{\sigma}_{n,p}$ are the almost triangular matrix means of the series (1.2).

The above theorems were further generalized by Qureshi and Nema [8] for a periodic function $f(x)$ and belonging to the class $W(L^p, \Psi_1(t))$.

However our theorem is as follows:

Theorem. *If $f(x)$ is periodic and belongs to the class $W(L^p, \Psi_1(t))$, then*

$$||\overline{\tau}_{n,p} - \overline{f}|| = O[\frac{\Psi_1(\frac{1}{n})\,(n)^{\beta+1+(\frac{1}{p})}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k\, p_{n-k}}{K+1}] \tag{2.3}$$

provided $\Psi_1(t)$ satisfies the following conditions:

$$\text{(a)}\ \ (\int_0^{\pi/n}(\frac{t|\Psi(t)|^p}{\Psi_{1(t)}})^p\sin^{\beta p}t\,dt)^{1/p} = O(\frac{1}{n})$$

$$\text{(b)}\ \ (\int_{\frac{\pi}{n}}^{\pi}(\frac{t^{-\delta}|\Psi(t)|}{\Psi_{1(t)}})^p\,dt)^{1/p} = O(n^{\delta})$$

where δ is an arbitrary number such that $q(1-\delta)-1>0$, condition (a) and (b) hold uniformly in x,

$$\Psi(t) = f(x+t) - f(x-t),$$

$\overline{f(x)}$ is a function conjugate to f, $\overline{\tau}_{n,p}$ are almost (N,p,α) means of the series (1.2) and $\{\frac{\alpha_k\,p_{n-k}}{r_n}\}_{k=0}^{n}$ is a non-negative sequence with respect to K.

3. Proof of the Theorem.

Let $\overline{S}_k$ be the k-th partial sum of the conjugate series (1.2). Then we have

$$\overline{S}_k(x) - \overline{f}(x) = -\frac{1}{\pi}\int_0^{\pi}\Psi(t)\frac{\cos(K+\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt$$

and

$$\begin{aligned}\overline{S}_{k,p} - f(x) &= \frac{1}{K+1}\sum_{\mu=p}^{k+p}(\overline{S}_\mu(x) - \overline{f}(x))\\ &= -\frac{1}{2\pi(K+1)}\int_0^{\pi}\Psi(t)\sum_{\mu=p}^{k+p}\frac{\cos(\mu+\frac{1}{2})^t}{\sin\frac{t}{2}}\,dt\\ &= \frac{1}{2\pi(K+1)}\int_0^{\pi}\Psi(t)\,\frac{\sin pt - \sin(K+p+1)t}{2\sin^2\frac{t}{2}}\,dt\end{aligned}$$

Now we have :

$$\overline{\tau}_{n,p} - \overline{f}(x) = \sum_{k=0}^{n}\frac{\alpha_k\,p_{n-k}}{(p\times\alpha)_k}(\overline{S}_{k,p}(x) - \overline{f}(x))$$

$$= \frac{1}{(p\times\alpha)_n}\sum_{k=0}^{n}\alpha_k\, p_{n-k}\,\big(\overline{S}_{k,p}(x)-\overline{f}(x)\big)$$
$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\Psi(t)}{(p\times\alpha)_n}\sum_{K=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\frac{\cos(K+2p+1)t/2\sin(K+1)t/2}{\sin^2 t/2}dt$$
$$= \frac{1}{2\pi}\Big[\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\Big]\frac{\Psi(t)}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\frac{\cos(K+2p+1)t/2\sin(K+1)t/2}{\sin^2 t/2}dt$$
$$= I_1 + I_2, \quad \text{say.}$$

Now

$$I_1 = \frac{1}{2\pi}\int_0^{\pi/n}\frac{\Psi(t)}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(k+1)}\frac{\cos(K+2p+1)t/2\sin(K+1)t/2}{\sin^2 t/2}dt$$

Applying Holder's inequality and the fact $\Psi(t)\in W(L^p,\Psi_1(t))$, we have

$$I_1 \le \frac{1}{2\pi}\Big(\int_0^{\pi/n}\big((\frac{t|\Psi(t)}{\Psi_1(t)})\sin^{\beta} t\big)^p dt\Big)^{1/p}$$
$$\times\Big(\int_0^{\pi/n}\Big(\frac{\Psi_1(t)}{(p\times\alpha)_n t}\Big|\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\frac{\cos(K+2p+1)t/2\sin(K+1)t/2}{\sin^2 t/2\sin^{\beta} t}\Big|\Big)^q dt\Big)^{1/q}$$
$$= O[(\frac{1}{n})]O\Big[\Big(\Big(\int_0^{\pi/n}\frac{\Psi_1(t)}{(p\times\alpha)_n t}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\frac{(K+1)|\sin t/2|}{\sin^2 t/2\sin^{\beta} t}\Big)^q dt\Big)^{1/q}\Big] \text{ by condition (a).}$$
$$= O[(\frac{1}{n})]O\Big[\frac{1}{(p\times\alpha)_n}\sum_{k=0}^{n}\alpha_k p_{n-k}\Big(\int_0^{\pi/n}(\frac{\Psi_1(t)}{t^{\beta+2}})^q dt\Big)^{1/q}\Big],$$

applying mean-value theorem, we have

$$I_1 = O[(\frac{1}{n})]O\Big[\frac{1}{(p\times\alpha)_n}\sum_{k=0}^{n}\alpha_k p_{n-k}\Big(\Psi_1^2(\frac{\pi}{n})\int_1^{\pi/n}\frac{dt}{t^{(2+\beta)q}}\Big)^{1/q}\Big]$$
$$= O[(\frac{1}{n})]O\Big[\frac{1}{(p\times\alpha)_n}\sum_{k=0}^{n}\alpha_k p_{n-k}\Big((\Psi_1(\frac{\pi}{n})^q\frac{t^{-(2+\beta)q+1}}{-(2+\beta)q+1})_1^{\pi/n}\Big)^{1/q}\Big]$$
$$= O[(\frac{1}{n})]O\Big[\frac{1}{(p\times\alpha)_n}\sum_{k=0}^{n}\alpha_k p_{n-k}\Psi_1(\frac{\pi}{n})(n)^{\beta+1+(1/p)}\Big]$$
$$= O\Big[\frac{\Psi_1(\frac{1}{n})(n)^{\beta+1+(1/p)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big].$$

Since we have:

$$\frac{\Psi_1(\frac{1}{n})(n)^{\beta+1+(1/p)}}{n(p\times\alpha)_n}\sum_{k=0}^{n}\alpha_k p_{n-k} < \frac{\Psi_1(\frac{1}{n})(n)^{\beta+1+(1/p)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}$$

where $\frac{1}{p}+\frac{1}{q}=1$ such that $1 \le p \le \infty$.

Also, similarly as above

$$I_2 \le \Big(\int_{\pi/n}^{\pi}\Big|\frac{t^{-\delta}\sin^{\beta} t\Psi(t)}{\Psi_1(t)}\Big|^p dt\Big)^{1/p}$$
$$\times\Big(\int_{\pi/n}^{\pi}\Big|\frac{1}{(p\times n)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\cdot\frac{\cos(K+2p+1)t/2\sin(K+1)t/2\Psi_1(t)}{t^{-\delta+2}\sin^{\beta} t}\Big|^q dt\Big)^{1/q}$$
$$=O\Big[\Big(\int_{\pi/n}^{\pi}\Big(t^{-\delta}\frac{|\Psi(t)|}{\Psi_1(t)}\Big)dt\Big)^{1/p}\Big]O\Big[\frac{1}{\sin^{\beta}(\frac{1}{n})}\Big]$$
$$\times O\Big[\frac{1}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big(\int_{\pi/n}^{\pi}(\frac{\Psi_1(t)}{t^{-\delta+2}})^q dt\Big)^{1/q}\Big]$$
$$=O[(n)^{\delta}]O\Big[\frac{1}{(\frac{1}{n})^{\beta}}\Big]O\Big[\frac{1}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big(\int_1^n\Big(\frac{\Psi_1(1/y)}{y^{\delta-2}}\Big)\frac{dy}{y^2}\Big)^{1/q}\Big] \text{ by condition (b)}$$
$$=O[(n)^{\delta}]O[(n)^{\beta}]O\Big[\frac{\Psi_1(\frac{1}{n})}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big(\int_1^n\frac{dy}{y^{\delta q-2q+2}}\Big)^{1/q}\Big]$$
$$=O[(n)^{\delta}]O[(n)^{\beta}]O\Big[\frac{\Psi_1(\frac{1}{n})}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}(n)^{-\delta+2-(1/q)}}{(K+1)}\Big]$$
$$=O\Big[\frac{\Psi_1(\frac{1}{n})(n)^{\beta+2-(1/q)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big].$$

Hence

$$|\overline{\tau}_{n,p}(x)-\overline{f(x)}|=O\Big[\frac{\Psi_1(\frac{1}{n})(n)^{\beta+1+(1/p)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big]$$

uniformly for x, therefore,

$$\|\overline{\tau}_{n,p}(x)-\overline{f(x)}\|=\sup_{0\le x\le 2\pi}|\overline{\tau}_{n,p}(x)-\overline{f(x)}|=O\Big[\frac{\Psi_1(\frac{1}{n})(n)^{\beta+1+(1/p)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big]$$

This complete the proof of the theorem.

The following Corollaries can be derived from the theorem:

Corollary 1. *If $\beta=0$ and $\Psi_1(t)=t^{\gamma}$, then the degree of approximation of a function $\overline{f(x)}$, conjugate to a 2π-periodic function f belong to the class $Lip(\gamma,p)$, $0<\gamma\le 1$, is given by:*

$$\|\overline{\tau}_{n,p}(x)-\overline{f(x)}\|=O\Big[\frac{(\frac{1}{n})^{\gamma-1-(1/p)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big]$$

where $\overline{\tau}_{n,p}(x)$ are almost (N,p,α) means of the series (1.2) and $\{\frac{\alpha_k p_{n-k}}{(p\times\alpha)_n}\}$ is non-negative sequence with respect to K.

Proof. Since

$$\begin{aligned}|\overline{\tau}_{n,p}(x)-\overline{f(x)}| &= O\Big[\frac{\Psi_1(\frac{1}{n})(n)^{\beta+1+(1/p)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big]\\ &= O\Big[\frac{(\frac{1}{n})^{\gamma}(n)^{1+(1/p)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big]\\ &= O\Big[\frac{(\frac{1}{n})^{\gamma-1-(1/p)}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big]\end{aligned}$$

which completes the proof.

Corollary 2. *If $p\to\infty$ in Corollary 1, then*

$$\|\overline{\tau}_{n,p}(x)-\overline{f(x)}\| = O\Big[\frac{(\frac{1}{n})^{\gamma-1}}{(p\times\alpha)_n}\sum_{k=0}^{n}\frac{\alpha_k p_{n-k}}{(K+1)}\Big]$$

and we have theorem B, and if further $\alpha=1$, we have theorem A for $0<\alpha<1$. The proof of Corollary 2 is obvious.

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