TAMKANG JOURNAL OF MATHEMATICS
Volume 28, Number 3, Autumn 1997

OSCILLATIONS OF SOLUTIONS TO PARABOLIC EQUATIONS WITH DEVIATING ARGUMENTS

SATOSHI TANAKA AND NORIO YOSHIDA

Abstract. Nonlinear parabolic equations with deviating arguments are studied and sufficient conditions are derived for every solution of boundary value problems to be oscillatory in a cylindrical domain. Two kinds of boundary conditions are considered. Our approach is to reduce the multi-dimensional problem to a one-dimensional problem for differential inequalities of neutral type.

1. Introduction

We shall be concerned with the forced oscillations of the parabolic equation with deviating arguments

$$\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{\ell} h_i(t)u(x,\tau_i(t))\right)-a(t)\Delta u(x,t)-\sum_{i=1}^{k} b_i(t)\Delta u(x,\rho_i(t))$$
$$+c(x,t,(u(x,\sigma_i(t)))_{i=1}^m)=f(x,t),\quad (x,t)\in\Omega\equiv G\times(0,\infty), \tag{1}$$

where G is a bounded domain of $\mathbb{R}^n$ with piecewise smooth boundary ∂G and Δ is the Laplacian in $\mathbb{R}^n$. We assume throughout this paper that:

(H_1) $h_i(t)\in C^1([0,\infty);[0,\infty))$ $(i=1,2,\dots,\ell)$, $\tau_i(t)\in C^1([0,\infty);\mathbb{R}^1)$ $(i=1,2,\dots,\ell)$, $\sigma_i(t)\in C([0,\infty);\mathbb{R}^1)$ $(i=1,2,\dots,m)$, $\lim_{t\to\infty}\tau_i(t)=\infty$, $\lim_{t\to\infty}\sigma_i(t)=\infty$;

(H_2) $a(t),b_i(t)\in C([0,\infty);[0,\infty))$ $(i=1,2,\dots,k)$, $\rho_i(t)\in C([0,\infty);\mathbb{R}^1)$ $(i=1,2,\dots,k)$, $\lim_{t\to\infty}\rho_i(t)=\infty$, $f(x,t)\in C(\overline{\Omega};\mathbb{R}^1)$;

(H_3) $c(x,t,(\xi_i)_{i=1}^m)\in C(\overline{\Omega}\times\mathbb{R}^m;\mathbb{R}^1)$,

$$c(x,t,(\xi_i)_{i=1}^m)\ge\sum_{i=1}^m p_i(t)\varphi_i(\xi_i)\text{ in }\Omega\times(0,\infty)^m,$$

$$c(x,t,(\xi_i)_{i=1}^m)\le\sum_{i=1}^m p_i(t)\varphi_i(\xi_i)\text{ in }\Omega\times(-\infty,0)^m,$$

Received April 19, 1996
1991 *Mathematics Subject Classification.* 35B05, 35R10.
Key words and phrases. Oscillation, deviating arguments, parabolic equation.

where $(0,\infty)^j = (0,\infty) \times (0,\infty)^{j-1}$, $(-\infty,0)^j = (-\infty,0) \times (-\infty,0)^{j-1} (j = 1,2,\ldots,m)$, $p_i(t) \in C([0,\infty);[0,\infty))$ $(i = 1,2,\ldots,m)$, $\varphi_i(\xi) \in C(\mathbb{R}^1;\mathbb{R}^1)(i = 1,2,\ldots,m)$, $\varphi_i(-\xi) = -\varphi_i(\xi)$, $\varphi_i(\xi) > 0$ for $\xi > 0$, and $\varphi_i(\xi)$ is nondecreasing and convex in $(0,\infty)$.

We consider two kinds of boundary conditions:

$$u = \psi \quad \text{on} \quad \partial G \times (0,\infty), \tag{B$_1$}$$

$$\frac{\partial u}{\partial \nu} + \mu u = \tilde{\psi} \quad \text{on} \quad \partial G \times (0,\infty), \tag{B$_2$}$$

where ψ, $\tilde{\psi} \in C(\partial G \times (0,\infty), \mathbb{R}^1), \mu \in C(\partial G \times (0,\infty);[0,\infty))$ and ν denotes the unit exterior normal vector to ∂G.

There has been much current interest in studying the oscillations of parabolic equations with deviating arguments. We refer the reader to [1-5, 7, 10-13] for parabolic equations (or systems) without forcing term, and to [8, 9, 14-16] for the forced oscillations. In particular, the case where the coefficients $h_i(t)$ are positive in $[0,\infty)$ was considered in the papers [4, 9-11].

By a *solution* of the boundary value problems (1), (B$_i$) $(i = 1,2)$ we mean a function $u(x,t) \in C^2(\overline{G}\times[t_{-1},\infty);\mathbb{R}^1)\cap C^1(\overline{G}\times[\tilde{t}_{-1},\infty);\mathbb{R}^1)\cap C(\overline{G}\times[T_{-1},\infty);\mathbb{R}^1)$ which satisfies (1), (B$_i$) (i=1,2), where

$$t_{-1} = \min\left\{ \min_{i\in\{1,2,\ldots,k\}} \left\{ \inf_{t\geq 0} \rho_i(t) \right\}, 0 \right\},$$

$$\tilde{t}_{-1} = \min\left\{ \min_{i\in\{1,2,\ldots,\ell\}} \left\{ \inf_{t\geq 0} \tau_i(t) \right\}, 0 \right\},$$

$$T_{-1} = \min_{i\in\{1,2,\ldots,m\}} \left\{ \inf_{t\geq 0} \sigma_i(t) \right\}.$$

A solution u of the boundary value problems (1), (B$_i$) (i=1,2) is said to be *oscillatory* in Ω if u has a zero in $G \times (t,\infty)$ for any $t > 0$.

In this paper we derive sufficient conditions for every solution of the boundary value problems (1), (B$_i$) (i=1,2) to be oscillatory. In Section 2 we reduce the multi-dimensional oscillation problem to a one-dimensional problems for neutral differential inequalities. Various sufficient conditions are given in Section 3 that a neutral differential inequality has no eventually positive solution. In Section 4 we establish the oscillation results for the boundary value problems for (1), (B$_i$) (i=1,2) by combining the results obtained in Sections 2 and 3.

2. Reduction to a One-dimensional Problem

The object of this section is to reduce the boundary value problems (1), (B$_i$) (i=1,2) to neutral differential inequalities with deviating arguments.

It is known that the first eigenvalue λ_1 of the eivenvalue problem

$$-\Delta w = \lambda w \quad \text{in} \quad G,$$
$$w = 0 \quad \text{on} \quad \partial G$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G.

The following notation will be used:

$$\begin{aligned}
F(t) &= \int_G f(x,t)\Phi(x)dx \cdot \left(\int_G \Phi(x)dx\right)^{-1}, \\
\Psi(t) &= \int_{\partial G} \psi(x,t)\frac{\partial \Phi}{\partial \nu}(x)dS \cdot \left(\int_G \Phi(x)dx\right)^{-1}, \\
\widetilde{F}(t) &= \frac{1}{|G|}\int_G f(x,t)dx, \\
\widetilde{\Psi}(t) &= \frac{1}{|G|}\int_{\partial G} \widetilde{\psi}(x,t)dS,
\end{aligned}$$

where $|G| = \int_G dx$.

Theorem 1. *Assume that* (H_1)-(H_3) *hold. If the differential inequalities*

$$\begin{aligned}
\frac{d}{dt}\left(y(t) + \sum_{i=1}^{\ell} h_i(t)y(\tau_i(t))\right) + \lambda_1 a(t)y(t) + \lambda_1 \sum_{i=1}^{k} b_i(t)y(\rho_i(t)) \\
+ \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \le \pm G(t) \qquad (\mathrm{I}_\pm)
\end{aligned}$$

have no eventually positive solutions, then every solution u of the problem (1), (B_1) *is oscillatory in Ω, where*

$$G(t) = F(t) - a(t)\Psi(t) - \sum_{i=1}^{k} b_i(t)\Psi(\rho_i(t)).$$

Proof. Assume to the contrary that there is a nonoscillatory solution $u(x,t)$ of the problem (1), (B_1), that is, there exists a number $t_0 > 0$ such that $u(x,t)$ has no zero in $G\times[t_0,\infty)$. We may assume that $u(x,t) > 0$ in $G\times[t_0,\infty)$ since the case where $u(x,t) < 0$ can be treated similarly. The hypotheses (H_1) and (H_2) imply that $u(x,\tau_i(t)) > 0$ $(i = 1,2,\dots,\ell)$, $u(x,\rho_i(t)) > 0$ $(i = 1,2,\dots,k)$, $u(x,\sigma_i(t)) > 0$ $(i = 1,2,\dots,m)$ in $G \times [t_1,\infty)$ for some $t_1 \ge t_0$. Multiplying (1) by $\Phi(x)(\int_G \Phi(x)dx)^{-1}$, integrating over G and using the hypothesis (H_3), we obtain

$$\begin{aligned}
\frac{d}{dt}\left(U(t) + \sum_{i=1}^{\ell} h_i(t)U(\tau_i(t))\right) - a(t)L_\Phi \int_G \Delta u(x,t)\Phi(x)dx \\
- \sum_{i=1}^{k} b_i(t)L_\Phi \int_G \Delta u(x,\rho_i(t))\Phi(x)dx \\
+ \sum_{i=1}^{m} p_i(t)L_\Phi \int_G \varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \le F(t),\ t \ge t_1, \qquad (2)
\end{aligned}$$

where $L_\Phi = (\int_G \Phi(x)dx)^{-1}$ and

$$U(t) = L_\Phi \int_G u(x,t)\Phi(x)dx.$$

It follows from Green's formula that

$$L_\Phi \int_G \Delta u(x,t)\Phi(x)dx = -\Psi(t) - \lambda_1 U(t), \ t \geq t_1. \tag{3}$$

Analogously we obtain

$$L_\Phi \int_G \Delta u(x,\rho_i(t))\Phi(x)dx = -\Psi(\rho_i(t)) - \lambda_1 U(\rho_i(t)), \ t \geq t_1. \tag{4}$$

Application of Jensen's inequality shows that

$$L_\Phi \int_G \varphi_i(u(x,\sigma_i(t)))\Phi(x)dx \geq \varphi_i(U(\sigma_i(t))), \ t \geq t_1. \tag{5}$$

Combining (2)-(5) yields

$$\begin{aligned}\frac{d}{dt}\left(U(t) + \sum_{i=1}^{\ell} h_i(t)U(\tau_i(t))\right) + \lambda_1 a(t)U(t) + \lambda_1 \sum_{i=1}^{k} b_i(t)U(\rho_i(t)) \\ + \sum_{i=1}^{m} p_i(t)\varphi_i(U(\sigma_i(t))) \leq G(t), \ t \geq t_1.\end{aligned}$$

Hence, $U(t)$ is an eventually positive solution of (I_+). This is a contradiction and the proof is complete.

Theorem 2. *Assume that* (H_1)-(H_3) *hold. If the differential inequalities*

$$\frac{d}{dt}\left(y(t) + \sum_{i=1}^{\ell} h_i(t)y(\tau_i(t))\right) + \sum_{i=1}^{m} p_i(t)\varphi_i(y(\sigma_i(t))) \leq \pm\widetilde{G}(t) \tag{$\widetilde{\mathrm{I}}_\pm$}$$

have no eventually positive solutions, then every solution u of the problem (1), (B_2) *is oscillatory in* Ω, *where*

$$\widetilde{G}(t) = \widetilde{F}(t) + a(t)\widetilde{\Psi}(t) + \sum_{i=1}^{k} b_i(t)\widetilde{\Psi}(\rho_i(t)).$$

Proof. The proof is quite similar to that of Theorem 1 and hence will be omitted (cf. [3, 16]).

3. Neutral Differential Inequalities

In this section we consider the neutral differential inequalities of the form

$$\frac{d}{dt}\left(y(t)+\sum_{i=1}^{\ell} h_i(t)y(\tau_i(t))\right)+\sum_{i=1}^{m} p_i(t)\widehat{\varphi}_i(y(\sigma_i(t))) \le q(t), \ t \ge t_0, \tag{6}$$

where t_0 is some positive number. We assume throughout this section that:

(H$_4$) $p_i(t) \in C([t_0,\infty);[0,\infty))$ $(i=1,2,\dots,m)$, $\widehat{\varphi}_i(\xi) \in C(\mathbb{R}^1;\mathbb{R}^1)$ $(i=1,2,\dots,m)$, $\widehat{\varphi}_i(-\xi)=-\widehat{\varphi}_i(\xi), \widehat{\varphi}_i(\xi)>0$ for $\xi>0, \widehat{\varphi}_i(\xi)$ is nondecreasing in $(0,\infty)$ and $q(t) \in C([t_0,\infty);\mathbb{R}^1)$.

We derive sufficient conditions for no solution of (6) to be eventually positive.

Theorem 3. *Assume that* (H$_1$) *and* (H$_4$) *hold, and that the following hypotheses hold:*

(H$_5$) $\sum_{i=1}^{\ell} h_i(t) \le 1, \tau_i(t) \ge t$ $(i=1,2,\dots,\ell)$;

(H$_6$) *there exists a function* $Q(t) \in C^1([t_0,\infty);\mathbb{R}^1)$ *such that* $Q(t)$ *is not eventually positive and* $Q'(t) \ge q(t)$.

Assume, moreover, that

$$\int_{t_0}^{\infty} p_j(s)\widehat{\varphi}_j\left(\left[\left(1-\sum_{i=1}^{\ell} h_i(\sigma_j(s))\right)[Q(\sigma_j(s))]_- + H(\sigma_j(s))\right]_+\right)ds=\infty$$

for some $j \in \{1,2,\dots,m\}$, *where*

$$[Q(t)]_\pm = \max\{\pm Q(t),0\},$$
$$H(t)=Q(t)-\sum_{i=1}^{\ell} h_i(t)Q(\tau_i(t)).$$

Then (6) *has no eventually positive solution.*

Proof. Suppose that $y(t)$ is an eventually positive solution of (6). The hypothesis (H$_1$) implies that there exists a number $t_1 \ge t_0$ such that $y(t)>0$, $y(\tau_i(t))>0$ $(i=1,2,\dots,\ell)$ and $y(\sigma_i(t))>0$ $(i=1,2,\dots,m)$ for $t \ge t_1$. We let

$$z(t)=y(t)+\sum_{i=1}^{\ell} h_i(t)y(\tau_i(t))-Q(t). \tag{7}$$

It follows from (6) and (H_6) that

$$z'(t) \leq -p_j(t)\widehat{\varphi}_j(y(\sigma_j(t))) \leq 0, \ t \geq t_1. \tag{8}$$

Therefore, $z(t)$ is eventually positive or eventually nonpositive. If $z(t)$ is eventually nonpositive, then $Q(t)$ is eventually positive by (7). This contradicts the hypothesis (H_6). Hence, we must have $z(t) > 0$ in $[t_2, \infty)$ for some $t_2 \geq t_1$. Since $z(t) > -Q(t)$ for $t \geq t_1$, we find that $z(t) > [Q(t)]_-$ for $t \geq t_2$. In view of the inequality $y(t) \leq z(t) + Q(t)$ and the fact that $z(t)$ is nonincreasing, we observe that

$$\begin{aligned} y(t) &= z(t) - \sum_{i=1}^{\ell} h_i(t) y(\tau_i(t)) + Q(t) \\ &\geq z(t) - \sum_{i=1}^{\ell} h_i(t)[z(\tau_i(t)) + Q(\tau_i(t))] + Q(t) \\ &\geq \Big(1 - \sum_{i=1}^{\ell} h_i(t)\Big) z(t) + H(t) \\ &\geq \Big(1 - \sum_{i=1}^{\ell} h_i(t)\Big)[Q(t)]_- + H(t), \ t \geq t_2. \end{aligned}$$

Since $y(t) > 0$ for $t \geq t_1$, we have

$$y(t) \geq \left[\Big(1 - \sum_{i=1}^{\ell} h_i(t)\Big)[Q(t)]_- + H(t)\right]_+, \ t \geq t_2.$$

Since $\sigma_j(t) \geq t_2$ in $[t_3, \infty)$ for some $t_3 \geq t_2$ and $\widehat{\varphi}_j(\xi)$ is nondecreasing ,we see from (8) that

$$z'(t) + p_j(t)\widehat{\varphi}_j\left(\left[\Big(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(t))\Big)[Q(\sigma_j(t))]_- + H(\sigma_j(t))\right]_+\right) \leq 0, \ t \geq t_3.$$

Integrating the above inequality over $[t_3, t]$ yields

$$\begin{aligned} \int_{t_3}^{t} p_j(s)\widehat{\varphi}_j\left(\left[\Big(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\Big)[Q(\sigma_j(s))]_- + H(\sigma_j(t))\right]_+\right) ds& \\ \leq -z(t) + z(t_3) \leq z(t_3), \ t \geq t_3.& \end{aligned}$$

This contradicts the hypothesis and the proof is complete.

Theorem 4. *Assume that* (H_1), (H_4) *and* (H_5) *hold. Assume, moreover, that*

$$\liminf_{t \to \infty} \int_{t_0}^{t} q(s) ds < \infty, \tag{9}$$

$$\int_{t_0}^{\infty} p_j(s)\widehat{\varphi}_j([I[q](\sigma_j(s))]_-) ds = \infty$$

for some $j \in \{1, 2, \ldots, m\}$, *where*

$$I[q](t) = \max_{i \in \{1,2,\ldots,\ell\}} \int_t^{\tau_i(t)} q(s)ds.$$

Then (6) *has no eventually positive solution.*

Proof. Let $y(t)$ be an eventually positive solution of the differential inequality (6). Then $y(t) > 0$, $y(\tau_i(t)) > 0$ $(i = 1, 2, \ldots, \ell)$ and $y(\sigma_i(t)) > 0$ $(i = 1, 2, \ldots, m)$ in $[t_1, \infty)$ for some $t_1 \geq t_0$. There is a function $\tau^*(t)$ for which

$$y(\tau^*(t)) = \max_{i \in \{1,2,\ldots,\ell\}} y(\tau_i(t)).$$

Integration of (6) over $[t, \tau^*(t)]$ yields

$$\begin{aligned} I[q](t) \geq & \int_t^{\tau^*(t)} q(s)ds \\ \geq & y(\tau^*(t)) + \sum_{i=1}^{\ell} h_i(\tau^*(t)) y(\tau_i(\tau^*(t))) - y(t) - \sum_{i=1}^{\ell} h_i(t) y(\tau_i(t)) \\ \geq & y(\tau^*(t)) - y(t) - \Big(\sum_{i=1}^{\ell} h_i(t) \Big) y(\tau^*(t)) \\ \geq & -y(t), \ t \geq t_1. \end{aligned}$$

Since $y(t) > 0$ in $[t_1, \infty)$, we have $y(t) \geq [I[q](t)]_-$ in $[t_1, \infty)$. We see from (6) that

$$\frac{d}{dt}\Big(y(t) + \sum_{i=1}^{\ell} h_i(t) y(\tau_i(t)) \Big) + p_j(t) \widehat{\varphi}_j([I[q](\sigma_j(t))]_-) \leq q(t), \ t \geq t_2$$

for some $t_2 \geq t_1$. An integration of the above inequality over $[t_2, t]$ implies that

$$\int_{t_2}^t p_j(s) \widehat{\varphi}_j([I[q](\sigma_j(s))]_-) ds \leq \int_{t_2}^t q(s)ds + c, \ t \geq t_2,$$

where $c = y(t_2) + \sum\limits_{i=1}^{\ell} h_i(t_2) y(\tau_i(t_2))$. Hence,

$$\int_{t_2}^{\infty} p_j(s) \widehat{\varphi}_j([I[q](\sigma_j(s))]_-) ds \leq \liminf_{t \to \infty} \int_{t_2}^t q(s)ds + c < \infty.$$

This contradicts the hypothesis and completes the proof.

Remark 1. Suppose that the hypothesis (H_6) holds. Since $\liminf_{t \to \infty} Q(t) \leq 0$, we obtain

$$\liminf_{t \to \infty} \int_{t_0}^t q(s)ds \leq \liminf_{t \to \infty} \int_{t_0}^t Q'(s)ds = \liminf_{t \to \infty} Q(t) - Q(t_0) \leq -Q(t_0) < \infty.$$

Hence, (9) holds. Conversely, let (9) hold. Setting

$$Q(t) = \begin{cases} \int_{t_0}^{t} q(s)ds, & \alpha < 0, \\ \int_{t_0}^{t} q(s)ds - \alpha - 1, & 0 \le \alpha < \infty, \end{cases}$$

where $\alpha = \liminf_{t\to\infty} \int_{t_0}^{t} q(s)ds$, we see that $Q'(t) = q(t)$ and $Q(t)$ is not eventually positive, and therefore (H_6) holds. Hence, the hypothesis (H_6) holds if and only if (9) holds.

Remark 2. In the case $h_i(t) \equiv 0$ $(i = 1, 2, \ldots, \ell)$, we see that

$$\left[\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\right)[Q(\sigma_j(s))]_- + H(\sigma_j(s))\right]_+ = [Q(\sigma_j(s))]_+.$$

Hence, Theorem 3 is a generalization of the result of Yoshida [17, Theorem 2].

We assume that the following hypothesis (H_7) holds:

(H_7) $h_\kappa(t) > 0$ and $\tau_\kappa(t)$ is strictly increasing for some $\kappa \in \{1, 2, \ldots, \ell\}$.

We introduce the following notation:

$$\overline{\tau}_i(t) = \begin{cases} \tau_\kappa^{-1}(t), & i = \kappa, \\ \tau_\kappa^{-1}(\tau_i(t)), & i \ne \kappa, \end{cases}$$

$$\overline{\sigma}_i(t) = \tau_\kappa^{-1}(\sigma_i(t)) \quad (i = 1, 2, \ldots, m),$$

$$\overline{h}_i(t) = \begin{cases} \dfrac{1}{h_\kappa(\tau_\kappa^{-1}(t))}, & i = \kappa, \\ \dfrac{h_i(t)}{h_\kappa(\tau_\kappa^{-1}(\tau_i(t)))}, & i \ne \kappa, \end{cases}$$

where $\tau_\kappa^{-1}(t)$ is the inverse function of $\tau_\kappa(t)$. Let $y(t)$ be an eventually positive solution of the differential inequality (6). We set $w(t) = h_\kappa(t)y(\tau_\kappa(t))$. Then we find that $w(t)$ is an eventually positive solution of the differential inequality

$$\frac{d}{dt}\left(w(t) + \sum_{i=1}^{\ell} \overline{h}_i(t)w(\overline{\tau}_i(t))\right) + \sum_{i=1}^{m} p_i(t)\widehat{\varphi}_i(\overline{h}_\kappa(\sigma_i(t))w(\overline{\sigma}_i(t))) \le q(t). \tag{10}$$

Therefore, if the differential inequality (10) has no eventually positive solution, then the differential inequality (6) has also no eventually positive solution. By the same arguments as were used in the proofs of Theorems 3 and 4, we can obtain the following two theorems.

Theorem 5. *Assume that* (H_1), (H_4), (H_6) and (H_7) *hold, and that the following hypothesis* (H_8) *holds:*

(H_8) $\sum_{i=1}^{\ell} \overline{h}_i(t) \le 1$, $\overline{\tau}_i(t) \ge t$ $(i = 1, 2, \ldots, \ell)$.

Assume, moreover, that

$$\int_{t_0}^{\infty} p_j(s)\widehat{\varphi}_j\left(\overline{h}_\kappa(\sigma_j(s))\left[\left(1 - \sum_{i=1}^{\ell} \overline{h}_i(\overline{\sigma}_j(s))\right)[Q(\overline{\sigma}_j(s))]_- + \overline{H}(\overline{\sigma}_j(s))\right]_+\right) ds = \infty$$

for some $j \in \{1, 2, \ldots, m\}$, *where*

$$\overline{H}(t) = Q(t) - \sum_{i=1}^{\ell} \overline{h}_i(t) Q(\overline{\tau}_i(t)).$$

Then (6) *has no eventually positive solution.*

Theorem 6. *Assume that* (H_1), (H_4), (H_7), (H_8) *and* (9) *hold, Assume, moreover, that*

$$\int_{t_0}^{\infty} p_j(s) \widehat{\varphi}_j(\overline{h}_\kappa(\sigma_j(s))[\overline{I}[q](\overline{\sigma}_j(s))]_-) ds = \infty$$

for some $j \in \{1, 2, \ldots, m\}$, *where*

$$\overline{I}[q](t) = \max_{i \in \{1,2,\ldots,\ell\}} \int_t^{\overline{\tau}_i(t)} q(s) ds.$$

Then (6) *has no eventually positive solution.*

Remark 3. If (H_8) holds, then we see that $\sum_{i=1}^{\ell} h_i(t) \geq 1$, and $\tau_1(t) \leq t$ in the case of $\ell = 1$.

4. Oscillation of Parabolic Equations

In this section we obtain sufficient conditions for every solution u of the problems (1), (B_i) ($i = 1, 2$) to be oscillatory in Ω.

Theorem 7. *Assume that* (H_1)-(H_3) *and* (H_5) *hold, and that the following hypothesis* (H_9) *holds:*

(H_9) *there exists a function* $Q(t) \in C^1([t_0, \infty); \mathbb{R}^1)$ *such that* $Q(t)$ *is oscillatory and* $Q'(t) = G(t)$.

Assume, moreover, that any one of the following conditions holds:

$$\int_0^{\infty} a(s) \left[\left(1 - \sum_{i=1}^{\ell} h_i(s)\right) [Q(s)]_{\mp} \pm H(s) \right]_+ ds = \infty;$$

$$\int_0^{\infty} b_j(s) \left[\left(1 - \sum_{i=1}^{\ell} h_i(\rho_j(s))\right) [Q(\rho_j(s))]_{\mp} \pm H(\rho_j(s)) \right]_+ ds = \infty$$

for some $j \in \{1, 2, \ldots, k\}$;

$$\int_0^{\infty} p_j(s) \varphi_j \left(\left[\left(1 - \sum_{i=1}^{\ell} h_i(\sigma_j(s))\right) [Q(\sigma_j(s))]_{\mp} \pm H(\sigma_j(s)) \right]_+ \right) ds = \infty \qquad (11)$$

for some $j \in \{1, 2, \ldots, m\}$. Then every solution u of the problem (1), (B_1) is oscillatory in Ω.

Proof. Theorem 3 implies that the differential inequalities $(I_\pm)$ have no eventually positive solutions. Hence, the conclusion follows from Theorem 1.

By combining Theorem 1 with Theorems 4-6, we obtain the following results.

Theorem 8. *Assume that (H_1)-(H_3) and (H_5) hold, and that*

$$\limsup_{t\to\infty} \int_0^t G(s)ds > -\infty, \tag{12}$$

$$\liminf_{t\to\infty} \int_0^t G(s)ds < \infty. \tag{13}$$

Assume, moreover, that any one of the following conditions holds:

$$\int_0^\infty a(s)[I[\pm G](s)]_- ds = \infty;$$

$$\int_0^\infty b_j(s)[I[\pm G](\rho_j(s))]_- ds = \infty \text{ for some } j \in \{1, 2, \ldots, k\};$$

$$\int_0^\infty p_j(s)\varphi_j\Big([I[\pm G](\sigma_j(s))]_-\Big)ds = \infty \text{ for some } j \in \{1, 2, \ldots, k\}. \tag{14}$$

Then every solution u of the problem (1), (B_1) is oscillatory in Ω.

Theorem 9. *Assume that (H_1)-(H_3), (H_7)-(H_9) hold. Assume, moreover, that any one of the following conditions holds:*

$$\int_0^\infty a(s)\overline{h}_\kappa(s)\left[\Big(1 - \sum_{i=1}^{\ell} \overline{h}_i(\overline{\tau}_\kappa(s))\Big)[Q(\overline{\tau}_\kappa(s))]_\mp \pm \overline{H}(\overline{\tau}_\kappa(s))\right]_+ ds = \infty;$$

$$\int_0^\infty b_j(s)\overline{h}_\kappa(\rho_j(s))\left[\Big(1 - \sum_{i=1}^{\ell} \overline{h}_i(\overline{\rho}_j(s))\Big)[Q(\overline{\rho}_j(s))]_\mp \pm \overline{H}(\overline{\rho}_j(s))\right]_+ ds = \infty$$

for some $j \in \{1, 2, \ldots, k\}$, where $\overline{\rho}_j(t) = \tau_\kappa^{-1}(\rho_j(t))$;

$$\int_0^\infty p_j(s)\varphi_j\left(\overline{h}_\kappa(\sigma_j(s))\left[\Big(1 - \sum_{i=1}^{\ell} \overline{h}_i(\overline{\sigma}_j(s))\Big)[Q(\overline{\sigma}_j(s))]_\mp \pm \overline{H}(\overline{\sigma}_j(s))\right]_+\right)ds = \infty \tag{15}$$

for some $j \in \{1, 2, \ldots, m\}$. Then every solution u of the problem (1), (B_1) is oscillatory in Ω.

Theorem 10. *Assume that (H_1)-(H_3), (H_7), (H_8), (12) and (13) hold. Assume, moreover, that any one of the following conditions holds:*

$$\int_0^\infty a(s)\overline{h}_\kappa(s)[\overline{I}[\pm G](\overline{\tau}_\kappa(s))]_- ds = \infty;$$

$$\int_0^\infty b_j(s)\overline{h}_\kappa(\rho_j(s))[\overline{I}[\pm G](\overline{\rho}_j(s))]_- ds = \infty \quad \textit{for some } j \in \{1,2,\ldots,k\};$$

$$\int_0^\infty p_j(s)\varphi_j(\overline{h}_\kappa(\sigma_j(s))[\overline{I}[\pm G](\overline{\sigma}_j(s))]_-)ds = \infty \quad \textit{for some } j \in \{1,2,\ldots,k\}.$$

Then every solution u of the problem (1), (B_1) *is oscillatory in* Ω.

Theorem 2 combined with Theorems 3-6 yields the following theorems.

Theorem 11. *Assume that* (H_1), (H_3) *and* (H_5) *hold, and that the following hypothesis* (H_{10}) *holds:*

(H_{10}) *there exists a function* $Q(t) \in C^1([0,\infty);\mathbb{R}^1)$ *such that* $Q(t)$ *is oscillatory and* $Q'(t) = \widetilde{G}(t)$.

Assume, moreover, that (11) *holds for some* $j \in \{1,2,\ldots,m\}$. *Then every solution u of the problem* (1), (B_2) *is oscillatory in* Ω.

Theorem 12. *Assume that* (H_1)-(H_3) *and* (H_5) *hold, and that*

$$\limsup_{t\to\infty} \int_0^t \widetilde{G}(s)ds > -\infty, \tag{16}$$

$$\liminf_{t\to\infty} \int_0^t \widetilde{G}(s)ds < \infty. \tag{17}$$

Assume, moreover, that

$$\int_0^\infty p_j(s)\varphi_j\Big([I[\pm\widetilde{G}](\sigma_j(s))]_-\Big)ds = \infty$$

for some $j \in \{1,2,\ldots,m\}$. *Then every solution u of the problem* (1), (B_2) *is oscillatory in* Ω.

Theorem 13. *Assume that* (H_1)-(H_3), (H_7), (H_8) *and* (H_{10}) *hold. Assume, moreover, that* (15) *holds for some* $j \in \{1,2,\ldots,m\}$. *Then every solution u of the problem* (1), (B_2) *is oscillatory in* Ω.

Theorem 14. *Assume that* (H_1)-(H_3), (H_7), (H_8), (16) *and* (17) *hold. Assume, moreover, that*

$$\int_0^\infty p_j(s)\varphi_j\Big(\overline{h}_\kappa(\sigma_j(s))[\overline{I}[\pm\widetilde{G}](\overline{\sigma}_j(s))]_-\Big)ds = \infty \tag{18}$$

for some $j \in \{1,2,\ldots,m\}$. *Then every solution u of the problem* (1), (B_2) *is oscillatory in* Ω.

Example 1. Let us consider the problem

$$\begin{aligned}\frac{\partial}{\partial t}\Big(u(x,t) + \frac{1}{2}u(x,t+2\pi)\Big) - u_{xx}(x,t) - u_{xx}(x,t-\frac{3}{2}\pi)&\\ +\frac{5}{2}u(x,t-\frac{1}{2}\pi) = \sin x \sin t,\ (x,t) \in (0,\pi)\times(0,\infty),& \qquad (19)\\ u(0,t) = u(\pi,t) = 0,\ t > 0.& \qquad (20)\end{aligned}$$

Here $n = 1$, $G = (0, \pi)$, $\ell = k = m = 1$, $h_1(t) = \frac{1}{2}$, $\tau_1(t) = t + 2\pi$, $a(t) = 1$, $b_1(t) = 1$, $\rho_1(t) = t - \frac{3}{2}\pi$, $\sigma_1(t) = t - \frac{1}{2}\pi$, $p_1(t) = \frac{5}{2}$, $\varphi_1(\xi) = \xi$, $f(x,t) = \sin x \sin t$. It is easily seen that $\lambda_1 = 1$, $\Phi(x) = \sin x$, $\Psi(t) = 0$ and $F(t) = G(t) = \frac{1}{4}\pi \sin t$. Theorems 8-10 are not applicable to the problem (19),(20). The hypotheses (H_1)-(H_3) and (H_5) are fulfilled, and $Q(t) \equiv -\frac{1}{4}\pi \cos t$ satisfies hypothesis (H_9). An easy calculation shows that (11) holds, and therefore the hypotheses of Theorem 7 are satisfied. Hence, every solution u of the problem (19), (20) is oscillatory in $(0,\pi) \times (0,\infty)$. Indeed, $u = \sin x \sin t$ is such a solution.

Example 2. Let us consider the problem

$$\frac{\partial}{\partial t}\Big(u(x,t) + u(x,t+\pi)\Big) - u_{xx}(x,t) - u_{xx}(x,t-\frac{3}{2}\pi)$$
$$+u(x,t-\frac{1}{2}\pi) = \sin x \sin t, \ (x,t) \in (0,\pi) \times (0,\infty) \tag{21}$$

with the boundary condition (20). Here $n = 1$, $G = (0,\pi)$, $\ell = k = m = 1$, $h_1(t) = 1$, $\tau_1(t) = t + \pi$, $a(t) = 1$, $b_1(t) = 1$, $\rho_1(t) = t - \frac{3}{2}\pi$, $\sigma_1(t) = t - \frac{1}{2}\pi$, $p_1(t) = 1$, $\varphi_1(\xi) = \xi$, $f(x,t) = \sin x \sin t$. It is easy to check that $G(t) = \frac{1}{4}\pi \sin t$. Theorems 7,9 and 10 do not apply to the problem (20), (21). The hypotheses (H_1)-(H_3) and (H_5) are fulfilled. We easily see that (12)-(14) hold. Therefore, Theorem 8 implies that every solution u of the problem (20), (21) is oscillatory in $(0,\pi) \times (0,\infty)$. One such solution is $u = \sin x \sin t$.

Example 3. We consider the problem

$$\frac{\partial}{\partial t}\Big(u(x,t) + 2u(x,t-2\pi)\Big) - u_{xx}(x,t) - u_{xx}(x,t-\frac{3}{2}\pi)$$
$$+4u(x,t-\frac{1}{2}\pi) = \cos x \sin t, \ (x,t) \in (0,\frac{\pi}{2}) \times (0,\infty), \tag{22}$$
$$-u_x(0,t) = 0, \ u_x(\frac{\pi}{2},t) = -\sin t, \ t > 0. \tag{23}$$

Here $n = 1$, $G = (0, \frac{\pi}{2})$, $\ell = k = m = 1$, $h_1(t) = 2$, $\tau_1(t) = t - 2\pi$, $a(t) = 1$, $b_1(t) = 1$, $\rho_1(t) = t - \frac{3}{2}\pi$, $\sigma_1(t) = t - \frac{1}{2}\pi$, $p_1(t) = 4$, $\varphi_1(\xi) = \xi$, $f(x,t) = \cos x \sin t$. It is readily verified that $\widetilde{\Psi}(t) = -\frac{2}{\pi}\sin t$, $\widetilde{F}(t) = \frac{2}{\pi}\sin t$, $\widetilde{G}(t) = -\frac{2}{\pi}\cos t$. Theorems 11, 12 and 14 are not applicable to the problem (22), (23). The hypotheses (H_1)-(H_3), (H_7) and (H_8) are fulfilled, and $Q(t) \equiv -\frac{2}{\pi}\sin t$ satisfies (H_{10}). An easy computation shows that (15) holds. From Theorem 13 it follows that every solution u of the problem (22), (23) is oscillatory in $(0, \frac{\pi}{2}) \times (0,\infty)$. In fact, there exists an oscillatory solution $u = \cos x \sin t$.

Example 4. We consider the problem

$$\frac{\partial}{\partial t}\Big(u(x,t) + u(x,t-\pi)\Big) - u_{xx}(x,t) - u_{xx}(x,t-\frac{3}{2}\pi)$$
$$+u(x,t-\frac{1}{2}\pi) = \cos x \sin t, \ (x,t) \in (0,\frac{\pi}{2}) \times (0,\infty) \tag{24}$$

with the boundary condition (23). Here $n = 1$, $G = (0, \frac{\pi}{2})$, $\ell = k = m = 1$, $h_1(t) = 1$, $\tau_1(t) = t - \pi$, $a(t) = 1$, $b_1(t) = 1$, $\rho_1(t) = t - \frac{3}{2}\pi$, $\sigma_1(t) = t - \frac{1}{2}\pi$, $p_1(t) = 1$, $\varphi_1(\xi) = \xi$, $f(x,t) = \cos x \sin t$. It is easily checked that $\widetilde{G}(t) = -\frac{2}{\pi}\cos t$. The hypotheses ($H_1$)-($H_3$), ($H_7$) and ($H_8$) are fulfilled. We easily observe that (16)-(18) hold. Applying Theorem 14, we conclude that every solution u of the problem (23), (24) is oscillatory in $(0, \frac{\pi}{2}) \times (0, \infty)$. For example, $u = \cos x \sin t$ is such a solution.

References

[1] Ya. V. Bykov and T. Ch. Kultaev, "Oscillation of solutions of a class of parabolic equations," Izv, Akad. Nauk Kirgiz. SSR No. 6 (1983), 3-9 (Russian).

[2] L. Ya. Bykova and A. I. Marusich, "Oscillation properties of solutions of a class of parabolic equations with delay," Izv. Akad. Nauk Kirgiz. SSR No. 2 (1989), 3-10 (Russian).

[3] B. T. Cui, "Oscillation theorems of nonlinear parabolic equations of neutral type," *Math. J. Toyama Univ.* 14 (1991), 113-123.

[4] B. T. Cui, "Oscillation properties for parabolic equations of neutral type," *Comment. Math. Univ. Carolin.* 33 (1992), 581-588.

[5] K. Gopalsamy, "Oscillations in parabolic neutral systems," *Proc. Center Math. Appl. Austral. Nat. Univ.* 30 (1992), 128-141.

[6] M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, "Oscillations of second order neutral delay differential equations," *Rad. Mat.* 1 (1985), 267-274.

[7] K. Kreith and G. Ladas, "Allowable delays for positive diffusion processes," *Hiroshima Math. J.* 15 (1985), 437-443.

[8] T. Kusano and N. Yoshida, "Oscillation of parabolic equations with oscillating coefficients," *Hiroshima Math. J.* 24 (1994), 123-133.

[9] D. P. Mishev, "Oscillation of the solutions of parabolic equations of neutral type with "maxima"," *Godishnik Vissh. Uchebn. Zaved. Prilozhna Mat.* 25 (1989), 19-28.

[10] D. P. Mishev, "Necessary and sufficient conditions for oscillation of neutral type of parabolic differential equations, " *C. R. Acad. Bulgare Sci.* 44 (1991), 11-14.

[11] D. P. Mishev, "Oscillation of the solutions of non-linear parabolic equations of neutral type," *Houston J. Math.* 18 (1992), 259-269.

[12] D. P. Mishev and D. D. Bainov, "Oscillation of the solutions of parabolic differential equations of neutral type," *Appl. Math. Comput.* 28 (1988), 97-111.

[13] N. Yoshida, "Oscillation of nonlinear parabolic equations with functional arguments," *Hiroshima Math. J.* 16 (1986), 304-314.

[14] N. Yoshida, "Forced oscillations of solutions of parabolic equations," *Bull. Austral. Math. Soc.* 36 (1987), 289-294.

[15] N. Yoshida, "Forced oscillations of parabolic equations with deviating arguments," *Math. J. Toyama Univ.* 15 (1992), 131-142.

[16] N. Yoshida, "Forced oscillations of nonlinear parabolic equations with functional arguments," *Analysis* 15 (1995), 71-84.

[17] N. Yoshida, "Nonlinear oscillation of first order delay differential equations," *Rocky Mountain J. Math.* 25 (1996), 361-373.

Department of Mathematics, Faculty of Science, Toyama University, Toyama 930, Japan.