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LOCAL AND UNIFORM NEAR SMOOTHNESS OF SOME BANACH SPACES

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Abstract. In this paper we give an estimate of the modulus of near smoothness of the space $c_0(E_i)$. In the case of the space $c_0(l_{p_i})$ the exact formula for this modulus is derived. Moreover, we show that the properties of near uniform smoothness and local near uniform smoothness are hereditary with respect to the product space $c_0(E_i)$.

In the last years the notions being fundamental in the classical geometry of Banach spaces, such as smoothness and uniform smoothness have been translated in terms of the measure of noncompactness. This way came into existence a branch of the geometry of Banach spaces involving compactness conditions (cf. [1, 3, 8, 9, 10] and references therein).

The aim of this paper is to study a few concepts of the theory mentioned above. At the beginning we start with some notation.

Let E be an infinite dimensional real Banach space with the dual E^*. Denote by B, B^*, S, S^* the unit balls and the unit spheres in E and E^*, respectively. For a bounded subset X of E let $\chi_E(X)$ denote the *Hausdorff measure of noncompactness* of X defined as the infimum of all numbers $r > 0$ such that X can be covered by a finite family of balls with radii r.

Recall [1] that the *modulus of near smoothness* of the space E is the function $\Sigma_E : [0,1] \to [0,1]$ defined by the formula

$$\Sigma_E(\varepsilon) = \sup\{\chi_{E^*}(F^*(x,\varepsilon)) : x \in S\},$$

where $F^*(x,\varepsilon) = \{f \in B^* : f(x) \geq 1-\varepsilon\}$.

Roughly speaking the modulus of near smoothness inform us about the noncompactness of the set of all hyperplanes supporting the unit sphere S at an arbitrary point.

A space E is called *nearly uniformly smooth* (NUS) [1] whenever $\lim_{\varepsilon\to 0} \Sigma_E(\varepsilon) = 0$. E is said to be *nearly smooth* (NS) if $\Sigma_E(0) = 0$. Moreover, a space E is referred to as *locally nearly uniformly smooth* ($LNUS$) [2] if $\lim_{\varepsilon\to 0} \chi_{E^*}(F^*(x,\varepsilon)) = 0$ for every $x \in S$.

Let us pay attention to some results obtained before.

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In the case of the classical sequence spaces c_0 and $l_p(1 < p < \infty)$ the following formulas can be derived [3] for $\varepsilon \in [0,1]$:

$$\Sigma_{c_0}(\varepsilon) = \varepsilon,$$

$$\Sigma_{l_p}(\varepsilon) = (1 - (1-\varepsilon)^q)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Thus both c_0 and l_p are NUS spaces.

The example of the space c_0 shows simultaneously that *a* NUS space has not to be reflexive. The other facts in this direction can be found in [1,2], for example.

The next concept which turns out to be connected with the properties defined above, is the property H^*. We say [2] that the norm $\|\cdot\|$ in a Banach space E has the property H^* whenever for any sequence $(f_n) \subset E^*$ converging weakly star to $f \in E^*$ and such that $\lim_{n\to\infty} \|f_n\|_{E^*} = \|f\|_{E^*}$ we have that $f_n \longrightarrow f$ in the norm of E^*.

Further, Let $(E_i, \|\cdot\|_i)$ be a sequence of Banach spaces. Then $c_0(E_i) = c_0(E_1, E_2, \ldots)$ is the Banach space of all sequences $x = (x_i)$, $x_i \in E_i$ for $i = 1, 2, \ldots$ such that $\lim_{n\to\infty} \|x_i\|_i = 0$, furnished with the norm

$$\|x_0\| = \max\{\|x_i\|_i : i \in \mathbb{N}\}.$$

In the similar way we define the space $l_p(E_i) = l_p(E_1, E_2, \ldots)(1 \leq p < \infty)$ under the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|_i^p\right)^{1/p}.$$

Let us mention that in [2, 3, 5, 6] it was shown that both the properties related to convexity such as NSC, NUC, $LNUC$, H and the properties connected with smoothness as NUS, $LNUS$, NS, H^* are hereditary with respect to the space $l_p(E_i)$. On the other hand it is easily seen that the space $c_0(E_i)$ is no longer NSC, NUC, $LNUC$ and it has no the property H. Nevertheless, if we assume that E_i is $NS(i = 1, 2, \ldots)$ then the space $c_0(E_i)$ has also this property (cf. [4]).

In the sequel we prove similar results for the "smothness" properties such as NUS, $LNUS$ and H^*.

We start with the following theorem.

Theorem 1. *Let* $r(\varepsilon) = \sup\{\Sigma_{E_i}(\varepsilon) : i \in \mathbb{N}\}$ *for* $\varepsilon \in [0,1]$*. Then*

$$\Sigma_{c_0(E_i)}(\varepsilon) \leq r(\sqrt{\varepsilon}) + 2\sqrt{\varepsilon} + 2\varepsilon. \tag{1}$$

Proof. For convenience denote by B, B^*, S, S^* the unit balls and the unit spheres in the spaces $c_0(E_i)$ and $(c_0(E_i))^* = l_1(E_i^*)$, respectively. The norms in E_i and E_i^* will be denoted by $\|\cdot\|_i$ while $\|\cdot\|_1$ stands for the norm in $l_1(E_i^*)$.

Now, fix a number $\varepsilon \in (0,1]$ and take a number $\eta > 0$ small enough. Choose $x = (x_i) \in S$ and a number γ in such a way that

$$\chi_{l_1(E_i^*)}(F^*(x,\varepsilon)) > \gamma > \Sigma_{c_0(E_i)}(\varepsilon) - \eta. \tag{2}$$

Then there exists a sequence $(f_n) \subset B^*$ satisfying the inequalities $1-\varepsilon \leq f(x)$ and $\gamma < \|f^n - f^m\|_1$ for $n, m \in \mathbb{N}$, $n \neq m$. Writting $f^n = (f_i^n)$, where $f_i^n \in E_i^*$ for every $i \in \mathbb{N}$ $(n = 1, 2 \ldots)$ we can write the last inequalities in the form

$$1-\varepsilon \leq \sum_{i=1}^{\infty} f_i^n(x_i) \quad \text{and} \quad \gamma < \sum_{i=1}^{\infty} \|f_i^n - f_i^m\|_i. \tag{3}$$

Applying the same argumentation as in [8] we may assume without loss of generality that

$$\|f_i^n\|_i \longrightarrow a_i, \quad f_i^n(x_i) \longrightarrow b_i, \quad \|f_i^n - f_i^m\|_i \longrightarrow c_i \tag{4}$$

when $n \longrightarrow \infty$, $m, n \longrightarrow \infty$ $(i = 1, 2, \ldots)$.
Further, for $\delta > 0$ consider the sets S_δ, T_δ and W_δ defined below:

$$\begin{aligned} S_\delta &= \{i \in \mathbb{N} : \|x_i\|_i > \delta \quad \text{and} \quad a_i > 0\}, \\ T_\delta &= \{i \in \mathbb{N} : \|x_i\|_i > \delta \quad \text{and} \quad a_i = 0\}, \\ W_\delta &= \{i \in \mathbb{N} : \|x_i\|_i \leq \delta\}. \end{aligned}$$

Obiously $\mathbb{N} = S_\delta \cup T_\delta \cup W_\delta$ in which T_δ is finite set. For $m \in \mathbb{N}$ large enough we have $\sum_{i \in T_\delta} f_i^m(x_i) \leq \delta$ and $\sum_{i \in W_\delta} f_i^m(x_i) \leq \delta$. This inequalities together with (3) yields $1 - \varepsilon - 2\delta \leq \sum_{i \in S_\delta} f_i^m(x_i)$ for $m \in \mathbb{N}$ sufficiently large.
Hence, putting $S = \{i \in \mathbb{N} : x_i \neq \theta \text{ and } a_i > 0\}$ and keeping in mind that $\cup_{\delta > 0} S_\delta = S$, for $m \to \infty$ and $\delta \to 0$ we get

$$1 - \varepsilon \leq \sum_{i \in S} b_i. \tag{5}$$

Further, observe that

$$1 - \varepsilon \leq \sum_{i \in W_\delta} f_i^m(x_i) + \sum_{i \in \mathbb{N} \setminus W_\delta} f_i^m(x_i) \leq \delta \sum_{i \in W_\delta} \|f_i^m\|_i + \left(1 - \sum_{i \in W_\delta} \|f_i^m\|_i\right).$$

This implies $\sum_{i \in W_\delta} \|f_i^m\|_i \leq \frac{\varepsilon}{1-\delta}$. Hence, in view of (3) we obtain

$$\gamma < \sum_{i \in W_\delta} \|f_i^n - f_i^m\|_i + \sum_{i \in S_\delta} \|f_i^n - f_i^m\|_i + \sum_{i \in T_\delta} \|f_i^n - f_i^m\|_i \leq$$

$$\leq \frac{2\varepsilon}{1-\delta} + \sum_{i \in S_\delta} \|f_i^n - f_i^m\|_i + \sum_{i \in T_\delta} \|f_i^n - f_i^m\|_i.$$

Using (4) and taking into account that $c_i = 0$ for $i \in T_\delta$ we derive that

$$\gamma - \frac{2\varepsilon}{1-\delta} \leq \sum_{i \in S_\delta} c_i, \tag{6}$$

for $n, m \longrightarrow \infty$. Hence, letting $\delta \to 0$ we get

$$\gamma - 2\varepsilon \leq \sum_{i\in S} c_i. \tag{7}$$

Now, let us put $P = \{i \in S : \frac{b_i}{a_i \cdot \|x_i\|_i} > 1 - \sqrt{\varepsilon}\}$. Then the inequality (5) gives

$$\begin{aligned}
1 - \varepsilon &\leq \sum_{i\in S} b_i = \sum_{i\in P} b_i + \sum_{i\in S\setminus P} \frac{b_i}{a_i\|x_i\|_i} a_i\|x_i\|_i \\
&\leq \sum_{i\in P} b_i + (1-\sqrt{\varepsilon}) \sum_{i\in S\setminus P} a_i\|x_i\|_i \\
&\leq \sum_{i\in P} b_i + (1-\sqrt{\varepsilon}) \left(\sum_{i\in S} a_i\|x_i\|_i - \sum_{i\in P} b_i\right) \\
&\leq \sum_{i\in P} b_i + (1-\sqrt{\varepsilon}) \left(1 - \sum_{i\in P} b_i\right).
\end{aligned}$$

Hence, after simple calculation we obtain $1 - \sqrt{\varepsilon} \leq \sum_{i\in P} b_i$. Since $b_i \leq a_i$ we have $1 - \sqrt{\varepsilon} \leq \sum_{i\in P} a_i$ and consequently

$$\sum_{i\in S\setminus P} a_i = \sum_{i\in S} a_i - \sum_{i\in P} a_i \leq 1 - (1 - \sqrt{\varepsilon}) = \sqrt{\varepsilon},$$

what together with $c_i \leq 2a_i$ implies $\sum_{i\in S\setminus P} c_i \leq 2\sqrt{\varepsilon}$.

Further observe that by the inequality (7) we obtain

$$\sum_{i\in P} c_i = \sum_{i\in S} c_i - \sum_{i\in S\setminus P} c_i \geq \gamma - 2\varepsilon - 2\sqrt{\varepsilon} \quad i.e.$$

$$\sum_{i\in P} c_i \geq \gamma - 2\varepsilon - 2\sqrt{\varepsilon}. \tag{8}$$

Notice, that the following two cases are possible:

(i) $\gamma - 2\varepsilon - 2\sqrt{\varepsilon} \leq 0$,

(ii) $\gamma - 2\varepsilon - 2\sqrt{\varepsilon} > 0$.

In the case of (i) by (2) we get $\Sigma_{c_0(E_i)}(\varepsilon) < \eta + \gamma \leq \eta + 2\varepsilon + 2\sqrt{\varepsilon}$ which as $\eta \longrightarrow 0$ gives (1).

Now suppose that the case (ii) is satisfied and take $\delta > 0$ small enough. Then, there exists $i \in P$ such that

$$\frac{c_i}{a_i} \geq \gamma - 2\varepsilon - 2\sqrt{\varepsilon} - \delta > 0 \tag{9}$$

Indeed, if not then $c_i < a_i \cdot (\gamma - 2\varepsilon - 2\sqrt{\varepsilon} - \delta)$ for every $i \in P$.

This yields
$$\sum_{i\in P} c_i < (\gamma - 2\varepsilon - 2\sqrt{\varepsilon} - \delta)\sum_{i\in P} a_i \le \gamma - 2\varepsilon - 2\sqrt{\varepsilon} - \delta$$
but this contradicts (8).

Thus let us take $i \in P$ satisfying (9) and put $\bar{x} = \frac{x_i}{\|x_i\|_i}$, $g^n = \frac{f_i^n}{\|f_i^n\|_i}$. By the definition of the set P and (4) we have that $\lim_{n\to\infty} g^n(\bar{x}) = \frac{b_i}{a_i\|x_i\|_i} > 1 - \sqrt{\varepsilon}$ what means that $F^*(\bar{x}, \sqrt{\varepsilon}) \supset \{g^n : n \ge k\}$ for some $k \in \mathbb{N}$.
Further using (4) and (9) we obtain
$$\|g^n - g^m\|_i \longrightarrow \frac{c_i}{a_i} \ge \gamma - 2\varepsilon - 2\sqrt{\varepsilon} - \delta \quad \text{when} \quad n, m \to \infty$$

These facts imply
$$r(\sqrt{\varepsilon}) \ge \chi_{E_i^*}(F^*(\bar{x}, \sqrt{\varepsilon})) \ge \chi_{E_i^*}(\{g^n : n \ge k\}) \ge \frac{c_i}{a_i} \ge \gamma - 2\varepsilon - 2\sqrt{\varepsilon} - \delta$$
i.e. $\gamma \le \delta + 2\varepsilon + 2\sqrt{\varepsilon} + r(\sqrt{\varepsilon})$. Combining the last inequality and (2) we infer
$$\Sigma_{c_0(E_i)}(\varepsilon) < \eta + \gamma \le \eta + \delta + 2\varepsilon + 2\sqrt{\varepsilon} + r(\sqrt{\varepsilon}),$$
and the arbitrariness of η and δ allows us to obtain (1).

Thus the proof is complete.

Corollary. *The space $c_0(E_i)$ is NUS if and only if $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$.*

Indeed, in view of the inequalities
$$\Sigma_{E_j}(\varepsilon) \le r(\varepsilon) \le \Sigma_{c_0(E_i)}(\varepsilon) \quad \textit{for} \quad j = 1, 2, \ldots$$
we infer that if $c_0(E_i)$ is UNS then $\lim_{\varepsilon\to 0} r(\varepsilon) = 0$. The converse implication is a consequence of Theorem 1.

Theorem 2. *The space $c_0(E_i)$ is LNUS if and only if E_i is LNUS for $i = 1, 2, \ldots$*

Proof. The implication $\Longrightarrow$ is obvious. For the proof of the converse implication suppose contrary, i.e., there exists a number $\gamma > 0$ and $x = (x_i) \in S$ such that $\lim_{\varepsilon\to 0} \chi(F^*(x, \varepsilon)) > \gamma > 0$, where $\chi = \chi_{l_1(E_i^*)}$.
Let us fix $\delta > 0$ such that
$$\frac{\gamma}{2} < \gamma - \frac{2\varepsilon}{1-\delta} \quad \text{for} \quad \varepsilon \in [0, \delta]. \tag{10}$$

Take $\varepsilon_n \in (0, \delta]$ with ε_n monotone decreasing to zero. Repeating the argumentation in the proof of Theorem 1 we may assume that there exists a sequence $(f^n) \subset B^*$, $f^n = (f_i^n)$ such that $1 - \varepsilon_n \le \sum_{i=1}^{\infty} f_i^n(x_i)$, $\gamma < \sum_{i=1}^{\infty} \|f_i^n - f_i^m\|_i$ and $\|f_i^n\|_i \longrightarrow a_i$, $f_i^n(x_i) \longrightarrow b_i$, $\|f_i^n - f_i^m\|_i \longrightarrow c_i$ when $m, n \longrightarrow \infty (i = 1, 2, \ldots)$.

Lemma 1. *If $x_i \neq \theta$ and $a_i > 0$ the* $\chi_{E_i^*}\left(F^*\left(\frac{x_i}{\|x_i\|_i}, \frac{2\varepsilon_m}{a_i\|x_i\|_i}\right)\right) \geq \frac{c_i}{2}$, $m = 1, 2, \ldots$

Proof. Observe that if $n > m$ then

$$f_i^n(x_i) \geq \|f_i^n\|_i \cdot \|x_i\|_i - \varepsilon_m \quad \text{for} \quad i \in \mathbb{N}. \tag{11}$$

Indeed, suppose the contrary, i.e., there exists $j \in \mathbb{N}$ such that $f_j^n(x_j) < \|f_j^n\|_j \cdot \|x_j\|_j - \varepsilon_m$. Then

$$1 - \varepsilon_m \leq 1 - \varepsilon_n \leq f_j^n(x_j) + \sum_{i \neq j} f_i^n(x_i) < -\varepsilon_m + \sum_{i \in \mathbb{N}} \|f_i^n\|_i \cdot \|x_i\|_i \leq 1 - \varepsilon_m$$

which gives a contradiction.
Let $x_i \neq \theta$ and $a_i > 0$. Put $g^n = \frac{f_i^n}{\|f_i^n\|_i}$. From (11) we have $g^n\left(\frac{x_i}{\|x_i\|_i}\right) \geq 1 - \frac{\varepsilon_m}{\|f_i^n\|_i \cdot \|x_i\|_i}$ and for each large enough positive integer n we have $g^n\left(\frac{x_i}{\|x_i\|_i}\right) \geq 1 - \frac{2\varepsilon_m}{a_i\|x_i\|_i}$. On the other hand $\|g^n - g^m\| \longrightarrow \frac{c_i}{a_i}$ when $n, m \longrightarrow \infty$. These yield $\chi_{E_i^*}\left(F^*\left(\frac{x_i}{\|x_i\|_i}, \frac{2\varepsilon_m}{a_i \cdot \|x_i\|_i}\right)\right) \geq \chi_{E_i^*}(\{g^n : n \in \mathbb{N}\}) \geq \frac{c_i}{2a_i} \geq \frac{c_i}{2}$, and the proof of Lemma 1 is complete.

For what follows let us observe that (6) and (10) imply $\frac{\gamma}{2} < \sum_{i \in S_\delta} c_i$. From the last inequality we obtain that there exists $j \in S_\delta$ such that $\frac{\gamma}{2p} < c_j$, where p denotes the cardinality of the set S_δ. For this j we derive from Lemma 1 that

$$\chi_{E_j^*}\left(F^*\left(\frac{x_j}{\|x_j\|_j}, \frac{2\varepsilon_m}{a_j \cdot \|x_j\|_j}\right)\right) \geq \frac{c_j}{2} > \frac{\gamma}{4p} > 0,$$

and by taking $m \longrightarrow \infty$ we get

$$\lim_{\varepsilon \to 0} \chi_{E_j^*}\left(F^*\left(\frac{x_j}{\|x_j\|_j}, \varepsilon\right)\right) = \lim_{m \to \infty} \chi_{E_j^*}\left(F^*\left(\frac{x_j}{\|x_j\|_j}, \frac{2\varepsilon_m}{a_j \cdot \|x_j\|_j}\right)\right) \geq \frac{\gamma}{4p} > 0.$$

But this contradicts to the assumption that the spaces E_i have the property $LNUS$ and completes the proof.

Theorem 3. *The space $c_0(E_i)$ has property H^* if and only if the spaces E_i have this property ($i = 1, 2, \ldots$).*

Proof. Suppose that E_i has property H^* for $i = 1, 2, \ldots$ Take arbitrary sequence $(f^n) = (f_i^n) \subset l_1(E_i^*)$ which is weakly star convergent to some $f = (f_i) \in l_1(E_i^*)$ and such that $\lim_{n \to \infty} \|f^n\|_1 = \|f\|_1$. We have to prove that $\lim_{n \to \infty} \sum_{i=1}^{\infty} \|f_i^n - f_i\|_i = 0$. Let (g^n) be an arbitrary subsequence of (f^n). By the diagonal procedure we may select a subsequence $(h^n) = (h_i^n)$ of (g^n) such that

$$\|h_i^n\|_i \longrightarrow a_i \quad \text{when} \quad n \longrightarrow \infty. \tag{12}$$

Obviously $\lim_{n\to\infty} \|h^n\|_1 = \|f\|_1$ what means

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \|h_i^n\|_i = \sum_{i=1}^{\infty} \|f_i\|_i. \tag{13}$$

If we fix $j \in \mathbb{N}$ we have $\sum\limits_{i=1}^{j} \|h_i^n\|_i \le \sum\limits_{i=1}^{\infty} \|h_i^n\|_i$, and when $n \longrightarrow \infty$ we get from (12) and (13) that $\sum\limits_{i=1}^{j} a_i \le \sum\limits_{i=1}^{\infty} \|f_i\|_i$. This implies

$$\sum_{i=1}^{\infty} a_i \le \sum_{i=1}^{\infty} \|f_i\|_i. \tag{14}$$

On the other hand, because $h^n = (h_1^n, h_2^n, \ldots)$ converges weakly star to $f = (f_1, f_2, \ldots)$ we infer that (h_i^n) converges weakly star to f_i for $i = 1, 2, \ldots$ which gives $\|f_i\|_i \le \liminf\limits_{n\to\infty} \|h_i^n\|_i$ i.e. $\|f_i\|_i \le a_i$. The last inequality together with (14) yields $\|f_i\|_i = a_i = \lim\limits_{n\to\infty} \|h_i^n\|_i$ for $i = 1, 2, \ldots$ and from the property H^* for E_i we obtain

$$\lim_{n\to\infty} \|h_i^n - f_i\|_i = 0 \quad \text{for} \quad i = 1, 2, \ldots \tag{15}$$

Further, let us fix $\varepsilon > 0$ and take $k \in \mathbb{N}$ so large that

$$\sum_{i=k+1}^{\infty} \|f_i\|_i \le \frac{\varepsilon}{5}. \tag{16}$$

Keeping in mind (15) and (13) we may choose $m \in \mathbb{N}$ so large that

$$\sum_{i=1}^{k} \|h_i^n - f_i\|_i \le \frac{\varepsilon}{5}, \tag{17}$$

and $\sum\limits_{i=1}^{\infty} \|h_i^n\|_i - \sum\limits_{i=1}^{\infty} \|f_i\|_i \le \frac{\varepsilon}{5}$ for $n \ge m$.
The last inequality gives

$$\sum_{i=1}^{k}(\|h_i^n\|_i - \|f_i\|_i) + \sum_{i=k+1}^{\infty} (\|h_i^n\|_i - \|f_i\|_i) \le \frac{\varepsilon}{5}.$$

Hence, by the inequalities (17) and (16) we obtain

$$\sum_{i=k+1}^{\infty} \|h_i^n\|_i \le \sum_{i=k+1}^{\infty} \|f_i\|_i + \left|\sum_{i=1}^{k}(\|h_i^n\|_i - \|f_i\|_i)\right| + \frac{\varepsilon}{5} \le \frac{\varepsilon}{5}$$
$$+ \sum_{i=1}^{k} \|h_i^n - f_i\|_i + \frac{\varepsilon}{5} \le \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \frac{3\varepsilon}{5} \quad \text{for} \quad n \ge m,$$

Linking the last inequality, (17) and (16) we derive

$$\sum_{i=1}^{\infty}\|h_i^n - f_i\|_i \leq \sum_{i=1}^{k}\|h_i^n - f_i\|_i + \sum_{i=k+1}^{\infty}(\|h_i^n\|_i - \|f_i\|_i)$$
$$\leq \frac{\varepsilon}{5} + \frac{3\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon \quad \text{for} \quad n \geq m,$$

which means that (h^n) converges to f in the norm of the space $l_1(E_i^*)$. The arbitrariness of the subsequence (g^n) implies the same for (f^n).
This completes the proof as the converse implication is obvious.

Example. Let us take the spaces l_{p_i}, $p_i > 1$, $i = 1, 2, \ldots$ and put $\overline{q} = \sup\{q_i : i \in \mathbb{N}\}$, where $\frac{1}{p_i} + \frac{1}{q_i} = 1$. We prove that if $\overline{q} < \infty$ then

$$\Sigma_{c_0(l_{p_i})}(\varepsilon) = (1 - (1-\varepsilon)^{\overline{q}})^{\frac{1}{\overline{q}}}. \tag{18}$$

In the case $\overline{q} = \infty$ we show that

$$\Sigma_{c_0(l_{p_i})}(\varepsilon) = \begin{cases} 0 \text{ for } \varepsilon = 0, \\ 1 \text{ for } \varepsilon \in (0,1]. \end{cases} \tag{19}$$

Proof. In what follows we will use the inequality

$$\left(\sum_{k=1}^{\infty} s_k^{\frac{1}{\omega_k}}\right)^{\omega} + \left(\sum_{k=1}^{\infty} t_k^{\frac{1}{\omega_k}}\right)^{\omega} \leq \left(\sum_{k=1}^{\infty}(s_k + t_k)^{\frac{1}{\omega_k}}\right)^{\omega} \tag{20}$$

where $1 < \omega_k \leq \omega$ and $s_k, t_k \geq 0$ for $k = 1, 2, \ldots$, which is a consequence of a reasoning similar to the proof of Minkowski inequality.

We will also need the following Lemma.

Lemma 2. [5]. *Let E be a space with Schauder basis (e_n) and let R_n be the n-remainder operator $R_n\left(\sum_{i=1}^{\infty}\alpha_i e_i\right) = \sum_{i=n+1}^{\infty}\alpha_i e_i$.*

Denote by $\|R_n\|$ the norm of the operator R_n. If $\|R_n\| = 1$ for $n = 1, 2, \ldots$ then

$$\chi(X) = \limsup_{n\to\infty}(\sup\{\|R_n x\| : x \in X\}) \quad \text{for} \quad X \subset E. \tag{21}$$

In what follows denote by $\|\cdot\|_1$ the norm of the space $(c_0(l_{p_i}))^* = l_1(l_{q_i})$.
Let $f \in l_1(l_{q_i})$. We describe $f = (f^i) = (f_j^i)$, where $f^i \in l_{q_i}$, $f_j^i \in \mathbb{R}$. Denote by $e_{n,k}$ the natural basis in $l_1(l_{q_i})$, i.e.,

$$(e_{n,k})_j^i = \begin{cases} 1 \text{ for } i = n \text{ and } j = k \\ 0 \text{ for } i \neq n \text{ or } \quad j \neq k. \end{cases}$$

Further, let h denotes one-to-one mapping between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

Put

$$e_n = e_{h(n)}. \tag{22}$$

It is easy to check that (e_n) is the Schauder basis in $l_1(l_{q_i})$ and $\|R_n\| = 1$.

We prove now the following Lemma.

Lemma 3. *If $\overline{q} < \infty$ and (e_n) is the basis in $l_1(l_{q_i})$ defined in (22) then*

$$\|R_n f\|_1^{\overline{q}} + \|(I - R_n)f\|_1^{\overline{q}} \leq \|f\|_1^{\overline{q}} \tag{23}$$

for $f \in l_1(l_{q_i})$ and $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$ and $f = (f^i) = (f_j^i) \in l_1(l_{q_i})$. Put $J_k = \{i \in \mathbb{N} : (k,i) \in h(\{1,2,\dots,n\})\}$. Applying the inequality (20) for $s_k = \left(\sum\limits_{i \in J_k} |f_i^k|^{q_k}\right)$, $t_k = \left(\sum\limits_{i \in \mathbb{N}\setminus J_k} |f_i^k|^{q_k}\right)$, $\omega_k = q_k$, $\omega = \overline{q}$, and keeping in the mind that $\|R_n f\|_1 = \sum\limits_{k=1}^{\infty} \left(\sum\limits_{i \in \mathbb{N}\setminus J_k} |f_i^k|^{q_k}\right)$, $\|(I - R_n)f\|_1 = \sum\limits_{k=1}^{\infty} \left(\sum\limits_{i \in J_k} |f_i^k|^{q_k}\right)^{\frac{1}{q_k}}$ and $\|f\|_1 = \sum\limits_{k=1}^{\infty} \left(\sum\limits_{i=1}^{\infty} |f_i^k|^{q_k}\right)^{\frac{1}{q_k}}$ we derive (23), which finishes the proof of our lemma.

Now, let u suppose that $\overline{q} < \infty$. Take $\delta > 0$, $x = (x^i) = (x_j^i) \in S_{c_0(l_{p_i})}$ and $f = (f^i) = (f_j^i) \in F^*(x, \varepsilon)$. Choose $n_0 \in \mathbb{N}$ which satisfies $\max\{\|x_i\|_i : i \geq n_0 + 1\} \leq \delta$. Further observe that there exists $m_0 \in \mathbb{N}$ such that

$$\left(\sum_{j=m_0+1}^{\infty} |x_j^i|^{p_i}\right)^{\frac{1}{p_i}} \leq \frac{\delta}{n_0} \quad \text{for} \quad i = 1, 2, \dots, n_0. \tag{24}$$

This implies

$$1 - \varepsilon \leq f(x) \leq \sum_{i=1}^{n_0} f^i(x^i) + \sum_{i=n_0+1}^{\infty} \|f^i\|_i \cdot \|x^i\|_i \leq \sum_{i=1}^{n_0} \left(\left(\sum_{j=1}^{m_0} |f_j^i|^{q_i}\right)^{\frac{1}{q_i}} \cdot \right.$$

$$\left. \cdot \left(\sum_{j=1}^{m_0} |x_j^i|^{p_i}\right)^{\frac{1}{p_i}} \right) + \sum_{i=1}^{n_0} \left(\sum_{j=m_0+1}^{\infty} |f_j^i|^{q_i}\right)^{\frac{1}{q_i}} \cdot \frac{\delta}{n_0} + \delta$$

$$\leq \sum_{i=1}^{n_0} \left(\sum_{j=1}^{m_0} |f_j^i|^{q_i}\right)^{\frac{1}{q_i}} + 2\delta \quad \text{i.e.}$$

$$1 - \varepsilon - 2\delta \leq \sum_{i=1}^{n_0} \left(\sum_{j=1}^{m_0} |f_j^i|^{q_i}\right)^{\frac{1}{q_i}}.$$

Let us find $n_1 \in \mathbb{N}$ such that $h(\{1,2,\ldots,n_1\}) \supset \{1,\ldots,n_0\} \times \{1,\ldots,m_0\}$. This inclusion and the previous inequality yield

$$1-\varepsilon-2\delta \leq \sum_{i=1}^{n_0}\left(\sum_{j=1}^{m_0}|f_j^i|^{q_i}\right)^{\frac{1}{q_i}} \leq \|(I-R_n)f\|_1 \quad \text{for} \quad n \geq n_1.$$

Hence, in the light of (23) we obtain $\|R_n f\|_1 \leq (1-(1-\varepsilon-2\delta)^{\overline{q}})^{\frac{1}{\overline{q}}}$. for $f \in F^*(x,\varepsilon)$ and $n \geq n_1$ which by (21) implies

$$\chi_{l_1(l_{q_i})}(F^*(x,\varepsilon)) = \limsup_{n\to\infty}(\sup\{\|R_n f\|_1 : f \in F^*(x,\varepsilon)\}) \leq (1-(1-\varepsilon-2\delta)^{\overline{q}})^{\frac{1}{\overline{q}}}.$$

Consequently, in view of the arbitrariness of δ and $x \in S_{c_0(l_{p_i})}$ we get $\Sigma_{c_0(l_{p_i})}(\varepsilon) \leq (1-(1-\varepsilon)^{\overline{q}})^{\frac{1}{\overline{q}})}$. On the other hand

$$\Sigma_{c_0(l_{p_i})}(\varepsilon) \geq \sup\{\Sigma_{l_{p_i}}(\varepsilon) = (1-(1-\varepsilon)^{q_i})^{\frac{1}{q_i}} : i \in \mathbb{N}\} = (1-(1-\varepsilon)^{\overline{q}})^{\frac{1}{\overline{q}}}$$

and (18) is proved.

In the case $\overline{q} = \infty$ we derive

$$1 \geq \Sigma_{c_0(l_{p_i})}(\varepsilon) \geq \sup\{\Sigma_{l_{p_i}}(\varepsilon) = (1-(1-\varepsilon)^{q_i})^{\frac{1}{q_i}} : i \in \mathbb{N}\} = 1,$$

for $\varepsilon \in (0,1]$. Because the property NS of space E_i is transfered to $c_0(E_i)$ and l_{p_i} is NS then $c_0(l_{p_i})$ is NS i.e. $\Sigma_{c_0(l_{p_i})}(0) = 0$ what gives (19).

This ends the proof.

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