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A SPECIAL CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. There are many special classes of univalent functions in the unit disc U. In this paper, we consider the special class $P^*(A,B,\alpha,\beta)$, $-1 \leq B < A \leq 1$, $-1 \leq B \leq 0$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, of univalent functions in the unit disc U. And it is the purpose of this paper to show some properties of this class.

1. Introduction

Let T denote the class of functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n (a_1 > 0; a_n \geq 0) \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z : |z| < 1\}$. Then a function $f(z)$ of T is said to be starlike of order α and type β if and only if

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha} \right| < \beta \qquad (z \in U) \tag{1.2}$$

for $\alpha(0 \leq \alpha < 1)$ and $\beta(0 < \beta \leq 1)$. We denote by $T(\alpha,\beta)$ the class of all starlike functions of order α and type β. Further a function $f(z)$ of T is said to be convex of order α and type β if and only if $zf'(z) \in T(\alpha,\beta)$. We denote by $C(\alpha,\beta)$ the class of all convex functions of order α and type β.

The classes $T(\alpha,\beta)$ and $C(\alpha,\beta)$ were studied by Gupta and Ahmad [2], Owa [7,8] and by Sekine, Owa and Nishimoto [10]. In particular, for $a_1 = 1$, these classes were studied by Gupta and Jain [3,4] and Owa [6].

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In this paper we introduce the class $P^*(A,B,\alpha,\beta)$, $-1 \le B < A \le 1$, $-1 \le B \le 0$, $0 \le \alpha < 1$ and $0 < \beta \le 1$, being defined as follows:

Definition 1. A function $f(z) \in T$ is in the class $P^*(A,B,\alpha,\beta)$, $-1 \le B < A \le 1$, $-1 \le B \le 0$, $0 \le \alpha < 1$, and $0 < \beta \le 1$, if and only if

$$\left|\frac{f'(z)-a_1}{[B+(A-B)(1-\alpha)]a_1 - Bf'(z)}\right| < \beta \qquad (z \in U). \tag{1.3}$$

Remark 1. (i) For $a_1 = A = 1$ and $B = -1$, the class $P^*(1,-1,\alpha,\beta) = P^*(\alpha,\beta)$ was studied by Gupta and Jain [4].

(ii) For $a_1 = A = 1$ and $B = -\mu(0 \le \mu \le 1)$, the class $P^*(1,-\mu,\alpha,\beta) = P^*(\alpha,\beta,\mu)$ was studied by Owa and Aouf [9].

(iii) Two subclasses $T(A,B,\alpha,\beta)$ and $C(A,B,\alpha,\beta)$ of T, obtained by taking $a_1 = 1$ and replacing $f'(z)$ with $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''(z)}{f'(z)}$, respectively in (1.3), have been studied by the author in [1].

2. Coefficient Estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.1). The $f(z)$ is in the class $P^*(A,B,\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(1-\beta B)a_n \le (A-B)\beta(1-\alpha)a_1. \tag{2.1}$$

The result is sharp.

Proof. Let $|z| = 1$. Then

$$\begin{aligned} &|f'(z)-a_1| - \beta|[B+(A-B)(1-\alpha)]a_1 - Bf'(z)| \\ &= |-\sum_{n=2}^{\infty} na_n z^{n-1}| - \beta|(A-B)(1-\alpha)a_1 + B\sum_{n=2}^{\infty} na_n z^{n-1}| \\ &\le \sum_{n=2}^{\infty} n(1-\beta B)a_n - (A-B)\beta(1-\alpha)a_1, \text{since} B \le 0 \\ &\le 0, \text{by hypothesis.} \end{aligned}$$

Hence, by maximum modulus principle, $f \in P^*(A,B,\alpha,\beta)$.

For the converse, assume that

$$\left|\frac{f'(z)-a_1}{[B+(A-B)(1-\alpha)]a_1 - Bf'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} na_n z^{n-1}}{(A-B)(1-\alpha)a_1 + B\sum_{n=2}^{\infty} na_n z^{n-1}}\right| < \beta$$

for $z \in U$. Since $|Re(z)| \leq |z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=2}^{\infty} na_n z^{n-1}}{(A-B)(1-\alpha)a_1 + B\sum_{n=2}^{\infty} na_n z^{n-1}}\right\} < \beta. \tag{2.2}$$

Choose values of z on the real axis so that $f'(z)$ is real. Upon clearing the denominator in (2.2) and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=2}^{\infty} na_n \leq (A-B)\beta(1-\alpha)a_1 + \beta B\sum_{n=2}^{\infty} na_n.$$

This completes the proof of Theorem 1.

Finally, we can see that the equality in (2.1) is attained for the function

$$f(z) = a_1 z - \frac{(A-B)\beta(1-\alpha)a_1}{n(1-\beta B)} z^n \quad (n \geq 2). \tag{2.3}$$

Corollary 1. *let the function $f(z)$ difined by (1.1) be in the class $P^*(A,B,\alpha,\beta)$. Then we have*

$$a_n \leq \frac{(A-B)\beta(1-\alpha)a_1}{n(1-\beta B)} \qquad (n \geq 2). \tag{2.4}$$

The equality in (2.4) *is attained for the function $f(z)$ given by* (2.3).

3. Distortion Theorem

Theorem 2. *Let the function $f(z)$ defined by* (1.1) *be in the class $P^*(A,B,\alpha,\beta)$. Then we have*

$$a_1|z| - \frac{(A-B)\beta(1-\alpha)a_1}{2(1-\beta B)}|z|^2 \leq |f(z)| \leq a_1|z| + \frac{(A-B)\beta(1-\alpha)a_1}{2(1-\beta B)}|z|^2 \tag{3.1}$$

and

$$a_1 - \frac{(A-B)\beta(1-\alpha)a_1}{(1-\beta B)}|z| \leq |f'(z)| \leq a_1 + \frac{(A-B)\beta(1-\alpha)a_1}{(1-\beta B)}|z| \tag{3.2}$$

for $z \in U$. The equalities hold for the function

$$f(z) = a_1 z - \frac{(A-B)\beta(1-\alpha)a_1}{2(1-\beta B)} z^2. \tag{3.3}$$

Proof. In view of Theorem 1, we have

$$2(1-\beta B)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n(1-\beta B)a_n \leq (A-B)\beta(1-\alpha)a_1, \tag{3.4}$$

which gives

$$\sum_{n=2}^{\infty} a_n \leq \frac{(A-B)\beta(1-\alpha)a_1}{2(1-\beta B)}. \tag{3.5}$$

Consequently, we have

$$|f(z)| \geq a_1|z| - |z|^2\sum_{n=2}^{\infty} a_n \geq a_1|z| - \frac{(A-B)\beta(1-\alpha)a_1}{2(1-\beta B)}|z|^2. \tag{3.6}$$

and

$$|f(z)| \leq a_1|z| + |z|^2\sum_{n=2}^{\infty} a_n \leq a_1|z| + \frac{(A-B)\beta(1-\alpha)a_1}{2(1-\beta B)}|z|^2. \tag{3.7}$$

Furthermore, from Theorem 1, we have

$$\sum_{n=2}^{\infty} na_n \leq \frac{(A-B)\beta(1-\alpha)a_1}{(1-\beta B)}. \tag{3.8}$$

Hence we have

$$|f'(z)| \geq a_1 - |z|\sum_{n=2}^{\infty} na_n \geq a_1 - \frac{(A-B)\beta(1-\alpha)a_1}{(1-\beta B)}|z| \tag{3.9}$$

and

$$|f'(z)| \leq a_1 + |z|\sum_{n=2}^{\infty} na_n \leq a_1 + \frac{(A-B)\beta(1-\alpha)a_1}{(1-\beta B)}|z|. \tag{3.10}$$

This completes the proof of Theorem 2.

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $P^*(A,B,\alpha,\beta)$. Then the unit disc U is mapped by $f(z)$ onto a domain that contains the disc*

$$|w| < \frac{2(1-\beta B)-(A-B)\beta(1-\alpha)}{2(1-\beta B)}a_1. \tag{3.11}$$

The result is sharp with the extremal function $f(z)$ given by (3.3).

4. Modified Hadamard Products

Let the functions $f_i(z)$ $(i = 1,\ldots,m)$ be defined by

$$f_i(z) = a_{1,i}z - \sum_{n=2}^{\infty} a_{n,i}z^n (a_{1,i} > 0; a_{n,i} \geq 0). \tag{4.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = a_{1,1}a_{1,2}z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n. \tag{4.2}$$

Theorem 3. *Let the functions $f_i(z)$ $(i = 1, 2)$ defined by (4.1) be in the class $P^*(A, B, \alpha, \beta)$. Then $f_1 * f_2(z)$ belongs to the class $P^*(A, B, \delta(A, B, \alpha, \beta), \beta)$, where*

$$\delta(A, B, \alpha, \beta) = 1 - \frac{(A - B)\beta(1 - \alpha)^2}{2(1 - \beta B)}. \tag{4.3}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [11], we need to find the largest $\delta = \delta(A, B, \alpha, \beta)$ such that

$$\sum_{n=2}^{\infty} \frac{n(1 - \beta B)}{(A - B)\beta(1 - \delta)a_{1,1}a_{1,2}} a_{n,1}a_{n,2} \leq 1. \tag{4.4}$$

Since from Theorem 1

$$\sum_{n=2}^{\infty} \frac{n(1 - \beta B)}{(A - B)\beta(1 - \alpha)a_{1,1}} a_{n,1} \leq 1 \tag{4.5}$$

and

$$\sum_{n=2}^{\infty} \frac{n(1 - \beta B)}{(A - B)\beta(1 - \alpha)a_{1,2}} a_{n,2} \leq 1, \tag{4.6}$$

by the Cauchy-Schwarz inequality we have

$$\sum_{n=2}^{\infty} \frac{n(1 - \beta B)}{(A - B)\beta(1 - \alpha)\sqrt{a_{1,1}a_{1,2}}} \sqrt{a_{n,1}a_{n,2}} \leq 1. \tag{4.7}$$

Thus it is sufficient to show that

$$\begin{aligned} &\frac{n(1 - \beta B)}{(A - B)\beta(1 - \delta)a_{1,1}a_{1,2}} a_{n,1}a_{n,2} \\ &\leq \frac{n(1 - \beta B)}{(A - B)\beta(1 - \alpha)\sqrt{a_{1,1}a_{1,2}}} \sqrt{a_{1,1}a_{1,2}} \quad (n \geq 2), \end{aligned} \tag{4.8}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1 - \delta)}{(1 - \alpha)} \sqrt{a_{1,1}a_{1,2}}. \tag{4.9}$$

Note that from Corollary 1

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(A-B)\beta(1-\alpha)\sqrt{a_{1,1}a_{1,2}}}{n(1-\beta B)}(n \geq 2). \tag{4.10}$$

Consequently, we need only to prove that

$$\frac{(A-B)\beta(1-\alpha)}{n(1-\beta B)} \leq \frac{(1-\delta)}{(1-\alpha)} \quad (n \geq 2), \tag{4.11}$$

or, equivalently, that

$$\delta \leq 1 - \frac{(A-B)\beta(1-\alpha)^2}{n(1-\beta B)} \quad (n \geq 2). \tag{4.12}$$

Since

$$D(n) = 1 - \frac{(A-B)\beta(1-\alpha)^2}{n(1-\beta B)} \quad (n \geq 2). \tag{4.13}$$

is an increasing function of $n(n \geq 2)$, letting $n = 2$ in (4.13), we obtain

$$\delta \leq D(2) = 1 - \frac{(A-B)\beta(1-\alpha)^2}{2(1-\beta B)}, \tag{4.14}$$

which completes the proof of Theorem 3.

Finally, by taking the functions $f_i(z)$ given by

$$f_i(z) = a_{1,i}z - \frac{(A-B)\beta(1-\alpha)a_{1,i}}{2(1-\beta B)}z^2 \quad (i = 1, 2) \tag{4.15}$$

we can see that the result is sharp.

Corollary 3. *For $f_1(z)$ and $f_2(z)$ as in Theorem 3, we have*

$$h(z) = \sqrt{a_{1,1}a_{1,2}}\; z - \sum_{n=2}^{\infty} \sqrt{a_{n,1}a_{n,2}}\; z^n \tag{4.16}$$

belonges to the class $P^*(A, B, \alpha, \beta)$.

Proof. This result follows form the Cauchy-Schwarz inequality (4.7). It is sharp for the same functions as in Theorem 3.

Theorem 4. *Let the function $f_1(z)$ defined by (4.1) be in the class $P^*(A, B, \alpha, \beta)$ and the function $f_2(z)$ defined by (4.1) be in the class $P^*(A, B, \gamma, \beta)$, then $f_1 * f_2(z) \in P^*(A, B, \zeta(A, B, \alpha, \gamma, \beta), \beta)$, where*

$$\zeta(A, B, \alpha, \gamma, \beta) = 1 - \frac{(A-B)\beta(1-\alpha)(1-\gamma)}{2(1-\beta B)}. \tag{4.17}$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 3, we get

$$\zeta \leq D(n) = 1 - \frac{(A-B)\beta(1-\alpha)(1-\gamma)}{n(1-\beta B)} (n \geq 2). \tag{4.18}$$

Since the function $D(n)$ is an increasing function of n $(n \geq 2)$, letting $n = 2$ in (4.18), we obtain

$$\zeta \leq D(2) = 1 - \frac{(A-B)\beta(1-\alpha)(1-\gamma)}{2(1-\beta B)}, \tag{4.19}$$

which evidently proves Theorem 4. Finally the result is best possible for the functions

$$f_1(z) = a_{1,1}z - \frac{(A-B)\beta(1-\alpha)a_{1,1}}{2(1-\beta B)} z^2 \tag{4.20}$$

and

$$f_2(z) = a_{1,2}z - \frac{(A-B)\beta(1-\gamma)a_{1,2}}{2(1-\beta B)} z_2. \tag{4.21}$$

Corollary 4. *Let the functions $f_i(z)$ $(i = 1,2,3)$ defined by (4.1) be in the class $P^*(A,B,\alpha,\beta)$, then $f_1 * f_2 * f_3(z) \in P^*(A,B,\eta(A,B,\alpha,\beta),\beta)$, where*

$$\eta(A,B,\alpha,\beta) = 1 - \frac{(A-B)^2\beta^2(1-\alpha)^3}{4(1-\beta B)^2}. \tag{4.22}$$

The result is best possible for the functions

$$f_i(z) = a_{1,i}z - \frac{(A-B)\beta(1-\alpha)a_{1,i}}{2(1-\beta B)} z^2 \qquad (i = 1,2,3). \tag{4.23}$$

Proof. From Theorem 3, we have $f_i * f_2(z) \in P^*(A,B,\delta(A,B,\alpha,\beta),\beta)$, where δ is given by (4.3). We now use Theorem 4, we get
$f_1 * f_2 * f_3(z) \in P^*(A,B,\eta(A,B,\alpha,\beta),\beta)$, where

$$\eta(A,B,\alpha,\beta) = 1 - \frac{(A-B)\beta(1-\alpha)(1-\delta)}{2(1-\beta B)} = 1 - \frac{(A-B)^2\beta^2(1-\alpha)^3}{4(1-\beta B)^2}.$$

This completes the proof of Corollary 4.

5. Fractional Calculus

We begin with the statements of the following definitions of fractional calculus (that is, fractional derivatives and fractional integrals) which were defined by Owa [5].

Definition 2. The fractional integral of order $k (k > 0)$ is defined, for a function $f(z)$, by

$$D_z^{-k} f(z) = \frac{1}{\Gamma(k)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-k}} d\zeta, \tag{5.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{k-1}$ is removed by requiring $log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 3. The fractional derivative of order $k (0 \leq k < 1)$ is defined, for a function $f(z)$, by

$$D_z^k f(z) = \frac{1}{\Gamma(1-k)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^k} d\zeta, \tag{5.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-k}$ is removed, as in Definition 2.

Definition 4. Under the hypotheses of Definition 3, the fractional derivative of order $n+k$ $(0 \leq k < 1; n \in \mathbb{N}_0 = \{0, 1, \ldots\})$ is defined by

$$D_z^{n+k} f(z) = \frac{d^n}{dz^z} D_z^d f(z). \tag{5.3}$$

Theorem 5. *Let the function $f(z)$ defined by* (1.1) *be in the class* $P^*(A, B, \alpha, \beta)$. *Then we have*

$$|D_z^{-k} f(z)| \geq \frac{a_1 |z|^{1+k}}{\Gamma(2+k)} \left[1 - \frac{(A-B)\beta(1-\alpha)}{(2+k)(1-\beta B)} |z|\right] \tag{5.4}$$

and

$$|D_z^{-k} f(z)| \leq \frac{a_1 |z|^{1+k}}{\Gamma(2+k)} \left[1 + \frac{(A-B)\beta(1-\alpha)}{(2+k)(1-\beta B)} |z|\right] \tag{5.5}$$

for $k > 0$ and $z \in U$. The result is sharp.

Proof. Let

$$\begin{aligned} F(z) =& \Gamma(2+k) z^{-k} D_z^{-k} f(z) \\ =& a_1 z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+k)}{\Gamma(n+1+k)} a_n z^n \\ =& a_1 z - \sum_{n=2}^{\infty} \psi(n) a_n z^n, \end{aligned} \tag{5.6}$$

where

$$\psi(n) = \frac{\Gamma(n+1)\Gamma(2+k)}{\Gamma(n+1+k)} \quad (n \geq 2). \tag{5.7}$$

Since

$$0 < \psi(n) \leq \psi(2) = \frac{2}{2+k}, \tag{5.8}$$

for $k > 0$ and $n \geq 2$. Therefore, by using (3.5) and (5.8), we can see that

$$\begin{aligned} |F(z)| \geq & a_1|z| - \psi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ \geq & a_1|z| - \frac{(A-B)\beta(1-\alpha)a_1}{(2+k)(1-\beta B)}|z|^2 \end{aligned} \tag{5.9}$$

which implies (5.4), and

$$\begin{aligned} |F(z)| \leq & a_1|z| + \psi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ \leq & a_1|z| + \frac{(A-B)\beta(1-\alpha)a_1}{(2+k)(1-\alpha B)}|z|^2 \end{aligned} \tag{5.10}$$

which shows (5.5). Further, equalities are attained for the function

$$D_z^{-k} f(z) = \frac{a_1 z^{1+k}}{\Gamma(2+k)} \Big[1 - \frac{(A-B)\beta(1-\alpha)}{(2+k)(1-\beta B)} z\Big] \tag{5.11}$$

or for the function $f(z)$ given by (3.3). This completes the proof of Theorem 5.

Corollary 5. *Under the hypotheses of Theorem 5, $D_z^{-k} f(z)$ $(k > 0, z \in U)$ is included in a disc with its center at the origin and the radius r_1 given by*

$$r_1 = \frac{a_1}{\Gamma(2+k)} \Big[1 + \frac{(A-B)\beta(1-\alpha)}{(2+k)(1-\beta B)}\Big]. \tag{5.12}$$

Theorem 6. *Let the function $f(z)$ defined by (1.1) be in the class $P^*(A, B, \alpha, \beta)$. Then we have*

$$|D_z^k f(z)| \geq \frac{a_1|z|^{1-k}}{\Gamma(2-k)} \Big[1 - \frac{(A-B)\beta(1-\alpha)}{(2-k)(1-\beta B)}|z|\Big] \tag{5.13}$$

$$|D_z^k f(z)| \leq \frac{a_1|z|^{1-k}}{\Gamma(2-k)} \Big[1 + \frac{(A-B)\beta(1-\alpha)}{(2-k)(1-\beta B)}|z|\Big] \tag{5.14}$$

for $0 \leq k < 1$ and $z \in U$. The result is sharp.

Proof. Let

$$\begin{aligned} G(z) = & \Gamma(2-k) z^k D_z^k f(z) \\ = & a_1 z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-k)}{\Gamma(n+1-k)} a_n z^n \\ = & a_1 z - \sum_{n=2}^{\infty} \Phi(n) n \, a_n z^n, \end{aligned} \tag{5.15}$$

where

$$\Phi(n) = \frac{\Gamma(n)\Gamma(2-k)}{\Gamma(n+1-k)} \quad (n \geq 2). \tag{5.16}$$

Noting that

$$0 < \Phi(n) \leq \Phi(2) = \frac{1}{2-k} \tag{5.17}$$

for $0 \leq k < 1$ and $n \geq 2$. Therefore, by using (3.8) and (5.17), we can see that

$$\begin{aligned} |G(z)| \geq & a_1|z| - \Phi(2)|z|^2 \sum_{n=2}^{\infty} n a_n \\ \geq & a_1|z| - \frac{(A-B)\beta(1-\alpha)a_1}{(2-k)(1-\beta B)}|z|^2 \end{aligned} \tag{5.18}$$

which implies (5.13), and

$$\begin{aligned} |G(z)| \leq & a_1|z| + \Phi(2)|z|^2 \sum_{n=2}^{\infty} n a_n \\ \leq & a_1|z| + \frac{(A-B)\beta(1-\alpha)a_1}{(2-k)(1-\beta B)}|z|^2 \end{aligned} \tag{5.19}$$

which implies (5.14). Further, equalities are attained by the function

$$D_z^k f(z) = \frac{a_1 z^{1-k}}{\Gamma(2-k)}\left[1 - \frac{(A-B)\beta(1-\alpha)}{(2-k)(1-\beta B)} z\right] \tag{5.20}$$

or by the function $f(z)$ given by (3.3). This completes the proof of Theorem 6.

Corollary 6. *Under the hypotheses of Theorem 6, $d_z^k f(z)$ $(0 \leq k < 1, z \in U)$ is included in a disc with its center at the origin and the radius r_2 given by*

$$r_2 = \frac{a_1}{\Gamma(2-k)}\left[1 + \frac{(A-B)\beta(1-\alpha)}{(2-k)(1-\beta B)}\right]. \tag{5.21}$$

6. Fractional Integral Operator

We need the following definition of fractional integral operator given by Srivastava, Saigo and Owa [12].

Definition 5. For real numbers $\zeta > 0$, γ and τ, the fractional operator $I_{0,z}^{\zeta,\gamma,\tau}$ is defined by

$$I_{0,z}^{\zeta,\gamma,\tau} f(z) = \frac{z^{-\zeta-\gamma}}{\Gamma(\zeta)} \int_0^z (z-t)^{\zeta-1} F(\zeta+\gamma, -\tau; \zeta; 1-\frac{t}{z}) f(t) dt, \tag{6.1}$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\epsilon}), \quad z \longrightarrow 0,$$

where

$$\epsilon > Max(0, \gamma - \tau) - 1,$$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{6.2}$$

with $(\nu)_n$ being the Pochhammer symbol

$$(\nu)_n = \frac{\Gamma(\nu + n)}{\Gamma(n)} = \begin{cases} 1 & (n = 0), \\ \nu(\nu + 1) \ldots (\nu + n - 1) & (n \in N = \{1, 2, \ldots\}) \end{cases}. \tag{6.3}$$

and the multiplicity of $(z - t)^{\zeta - 1}$ is removed by requiring $log(z - t)$ to be real when $z - t > 0$.

Remark 2. For $\gamma = -\zeta$, we note that

$$I_{0,z}^{\zeta, -\zeta, \tau} f(z) = D_z^{-\zeta} f(z).$$

In order to prove our result for the fractional integral operator, we have to recall the following lemma due to Srivastava, Saigo and Owa [12].

Lemma 1. If $\zeta > 0$ and $n > \gamma - \tau - 1$, then

$$I_{0,z}^{\zeta, \gamma, \tau} z^n = \frac{\Gamma(n + 1)\Gamma(n - \gamma + \tau + 1)}{\Gamma(n - \gamma + 1)\Gamma(n + \zeta + \tau + 1)} z^{n - \gamma}. \tag{6.4}$$

With the aid of Lemma 1, we have

Theorem 7. *Let $\zeta > 0$, $\gamma < 2$, $\gamma + \tau > -2$, $\gamma - \tau < 2$, $\gamma(\zeta + \tau) \leq 3\zeta$. If the function $f(z)$ defined by* (1.1) *is in the class $P^*(A, B, \alpha, \beta)$, then*

$$\begin{aligned} \left| I_{0,z}^{\zeta, \gamma, \tau} f(z) \right| \geq & \frac{a_1 \Gamma(2 - \gamma + \tau) |z|^{1 - \gamma}}{\Gamma(2 - \gamma)\Gamma(2 + \zeta + \tau)} \\ & \cdot \left\{ 1 - \frac{(A - B)\beta(1 - \alpha)(2 - \gamma + \tau)}{(1 - \beta B)(2 - \gamma)(2 + \zeta + \tau)} |z| \right\} \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \left| I_{0,z}^{\zeta, \gamma, \tau} f(z) \right| \leq & \frac{a_1 \Gamma(2 - \gamma + \tau) |z|^{1 - \gamma}}{\Gamma(2 - \gamma)\Gamma(2 + \zeta + \tau)} \\ & \cdot \left\{ 1 + \frac{(A - B)\beta(1 - \alpha)(2 - \gamma + \tau)}{(1 - \beta B)(2 - \gamma)(2 + \zeta + \tau)} |z| \right\} \end{aligned} \tag{6.6}$$

for $z \in U_O$, *where*

$$U_O = \begin{cases} U & (\gamma \le 1) \\ U - \{O\} & (\gamma > 1). \end{cases}$$

The equalities in (6.5) *and* (6.6) *are attained by the function* $f(z)$ *given by* (3.3).

Proof. By using Lemma 1, we have

$$I_{0,z}^{\zeta,\gamma,\tau} f(z) = \frac{a_1 \Gamma(2-\gamma+\tau)}{\Gamma(2-\gamma)\Gamma(2+\zeta+\tau)} z^{1-\gamma} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\gamma+\tau+1)}{\Gamma(n-\gamma+1)\Gamma(n+\zeta+\tau+1)} a_n z^{n-\gamma}. \tag{6.7}$$

Letting

$$\begin{aligned} H(z) =& \frac{\Gamma(2-\gamma)\Gamma(2+\zeta+\tau)}{\Gamma(2-\gamma+\tau)} z^{\gamma} I_{0,z}^{\zeta,\gamma,\tau} f(z) \\ =& a_1 z - \sum_{n=2}^{\infty} h(n) a_n z^n, \end{aligned} \tag{6.8}$$

where

$$h(n) = \frac{(2-\gamma+\tau)_{n-1}(1)_n}{(2-\gamma)_{n-1}(2+\zeta+\tau)_{n-1}} \qquad (n \ge 2), \tag{6.9}$$

we can see that $h(n)$ is non-increasing for integers $n \ge 2$, and we have

$$0 < h(n) \le h(2) = \frac{2(2-\gamma+\tau)}{(2-\gamma)(2+\zeta+\tau)}. \tag{6.10}$$

Therefore, by using (3.5) and (6.10), we have

$$\begin{aligned} |H(z)| \ge& a_1|z| - h(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ \ge& a_1|z| - \frac{(A-B)\beta(1-\alpha)(2-\gamma+\tau)a_1}{(1-\beta B)(2-\gamma)(2+\zeta+\tau)} |z|^2 \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} |H(z)| \le& a_1|z| - h(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ \le& a_1|z| - \frac{(A-B)\beta(1-\alpha)(2-\gamma+\tau)a_1}{(1-\beta B)(2-\gamma)(2+\zeta+\tau)} |z|^2. \end{aligned} \tag{6.12}$$

This complets the proof of Theorem 7.

Remark 3. Taking $\gamma = -\zeta = -k$ in Theorem 7, we get the result of Theorem 5.

Remark 4. Owa [7] considered the class $P_0^*(\alpha,\beta)$ of functions $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n (a_n \ge 0; a_1 > 0)$ analytic and univelent in U and satisfying

$$\left| \frac{f'(z) - 1}{f'(z) + (1-2\alpha)} \right| < \beta, z \in U, \tag{i}$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

One can easily verify that the condition (i) is equivelent to

$$f'(z) = \frac{1 + \beta(1 - 2\alpha)\omega(z)}{1 - \beta\omega(z)}, z \in U, \tag{ii}$$

where $\omega(z)$ is a function analtyic in U and satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in U$. Since $f'(z) = a_1 - \sum_{n=2}^{\infty} n a_n z^{n-1}$, it follows that the constant term in the Taylor expansion of both sides of (ii) is not the same except when $a_1 = 1$. It seems, therefore, that the class $P_0^*(\alpha, \beta)$ has not been defined by Owa [7] in proper way. in fact, the correct form of (i) must be

$$\left| \frac{f'(z) - a_1}{f'(z) + (1 - 2\alpha)a_1} \right| < \beta, z \in U, \tag{iii}$$

where we put $A = -1$ $B = 1$ in (1.3). Consequently, the correct form of (ii) is

$$f'(z) = a_1 \frac{1 + \beta(1 - 2\alpha)\omega(z)}{1 - \beta\omega(z)}, z \in U. \tag{iv}$$

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