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ON THE NEIGHBOURHOODS OF STRONGLY CONVEX FUNCTIONS

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Abstract. In this paper neighbourhoods of strongly convex and strongly starlike function are determined.

1. Introduction

Let $H(E)$ denote the class of all functions f holomorphic in the open unit disc E in $\mathbb{C}$ and A be the class of all functions $f \in H(E)$ with the normalizations $f(0) = 0 = f'(0)-1$. Any $f \in A$ has the Taylor's expansion $f(z) = z + a_2z^2 + \dots$ in E. The convolution or Hadamard product of $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is defined as $(f*g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$. Clearly $f(z) = f(z) * f_1(z)$ and $zf'(z) = f(z) * f_2(z)$ where

$$f_1(z) = \frac{z}{1-z} \quad \text{and} \quad f_2(z) = \frac{z}{(1-z)^2}.$$

In this paper let us investigate the neighbourhoods of functions which are Strongly Starlike or Strongly Convex. These functions were introduced and discussed by D. A. Brannan and W. E. Kirwan [1] and also by J. Stankiewicz [5] and [6].

Definition 1. A function $f \in A$ is said to be Strongly Starlike of order α, $0 < \alpha \leq 1$ in E if for all $z \in E$, $\left|arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}$. The set of all such functions is denoted by $S^*(\alpha)$.

Clearly $S^*(1) = S^*$ (the class of all starlike functions).

$f \in S^*(\alpha)$ means that the image of E under $\frac{zf'(z)}{f(z)}$ lies in the region

$$\Omega = \{z \in \mathbb{C} : |arg\, z| < \frac{\alpha\pi}{2},\ 0 < \alpha \leq 1\}. \tag{1}$$

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Definition 2. Any function $f \in A$ is said to be Strongly Convex of order α in E if for all $z \in E$,

$$\left|arg(1 + \frac{zf''(z)}{f'(z)})\right| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \leq 1.$$

Let $K(\alpha)$ be the class of all strongly convex functions of order α.

Note. (i) $K(1) = K-$ the class of all convex functions.
(ii) $f \in K(\alpha) \Longleftrightarrow zf'(z) \in S^*(\alpha)$.

First let us state two lemmas (without proofs) which we need to establish our results in the sequel.

Lemma A.[2] *Let β', $\tau' \in \mathbb{C}$, $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $Re(\beta' h(z) + \tau') > 0$, $z \in E$ and let $p(z) = 1 + p_1 z + \cdots \in H(E)$. Then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \tau} \prec h(z) \Longrightarrow p(z) \prec h(z)$$

where the symbol $\prec$ denotes subordination.

Lemma B.[4] *If ϕ is a convex univalent function with $\phi(0) = 0 = \phi'(0) - 1$ in the unit disk E and g is starlike univalent in E, then for each analytic function F in E, the image of E under $\frac{(\phi * Fg)(z)}{(\phi * g)(z)}$ is a subset of the convex hull of $F(E)$.*

First let us establish an inclusion relation.

Theorem 1. *Let $f \in K(\alpha)$. Then $f \in S^*(\alpha)$.*

Proof. Let $p(z) = \frac{zf'(z)}{f(z)}$. Then since $f \in K(\alpha)$, $p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} \subset \Omega$, defined in (1). Since Ω is a convex domain, an application of Lemma A gives $\left[\frac{zf'(z)}{f(z)}\right]_{z\in E} = p(z)_{z\in E} \subset \Omega$, which shows that $f \in S^*(\alpha)$.

Definition 3. For $f \in S^*(\alpha)$ define $S^{*\prime}(\alpha)$ as a class of all functions

$$h(z) = \frac{f_2(z) - te^{\pm i\alpha\pi/2} f_1(z)}{1 - te^{\pm i\alpha\pi/2}};$$

$t \in \mathbb{R}^+$ where $f_1(z) = \frac{z}{1-z}$ and $f_2(z) = \frac{z}{(1-z)^2}$.

Now let us give a characterization for a function $f \in A$ to be in $S^*(\alpha)$ be means of convolution.

Theorem 2. *$f \in S^*(\alpha)$ if and only if $\frac{(f*H)(z)}{z} \neq 0$, $z \in E$ and for all $H(z) \in S^{*\prime}(\alpha)$.*

Proof. First let us assume that $\frac{(f*H)(z)}{z} \neq 0$ for all $H \in S^{*\prime}(\alpha)$ and $z \in E$. Hence

$$\frac{(f*H)(z)}{z} = \frac{f(z) * \frac{z}{(1-z)^2} - te^{\pm i\alpha\pi/2} f(z) * \frac{z}{1-z}}{z(1 - te^{\pm i\alpha\pi/2})}, \quad t \in \mathbb{R}^+ \neq 0.$$

Equivalently $\frac{zf'(z)}{f(z)} + te^{\pm i\alpha\pi/2}$, $t \in \mathbb{R}^+$. As $t \in \mathbb{R}^+$, $te^{\pm i\alpha\pi/2}$ covers the straight line $\arg w = \pm\alpha\pi/2$ and $\frac{zf'(z)}{f(z)} = 1$ at $z = 0$, Hence

$$\frac{zf'(z)}{f(z)} \in \Omega = \left[z \in \mathbb{C} : \left|arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}\right] \text{ or } f \in S^*(\alpha).$$

Conversely let $f \in S^*(\alpha)$. Then

$$\frac{zf'(z)}{f(z)} \neq te^{\pm i\alpha\pi/2}, \; t \in \mathbb{R}^+ \tag{2}$$

Now

$$\begin{aligned}\frac{(f*H)(z)}{z} &= \frac{f(z) * f_2(z) - te^{\pm i\alpha\pi/2} f_1(z)}{z(1 - te^{\pm i\alpha\pi/2})}; \\ &= \frac{\frac{zf'(z)}{f(z)} - te^{\pm i\alpha\pi/2}}{(1 - te^{\pm i\alpha\pi/2})}\left[\frac{f(z)}{z}\right].\end{aligned}$$

(2) gives $\frac{(f*H)(z)}{z} \neq 0$ in E which completes the proof of the theorem.

The notion of δ-neighbourhood was first introduced by St. Ruscheweyh [3].

Definition 4. For $\delta \geq 0$ the δ-neighbourhood of $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ is defined by

$$N_\delta(f) = \left[g(z) = z + \sum_{k=2}^{\infty} b_k z^k : \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta\right].$$

Lemma 1. *Let $H(z) = z + \sum_{k=2}^{\infty} h_n z^n \in S^{*\prime}(\alpha)$. Then*

$$|h_n| \leq \frac{n}{\sin\frac{\alpha\pi}{2}}.$$

Proof. Since $H(z) \in S^{*\prime}(\alpha)$, we have

$$\begin{aligned}H(z) &= \frac{1}{1 - te^{\pm i\alpha\pi/2}}\left[\frac{z}{(1-z)^2} - te^{\pm i\alpha\pi/2}\frac{z}{1-z}\right] \\ &= z + \sum_{k=2}^{\infty} h_n z^n.\end{aligned}$$

Then comparing the coefficients on either sides we get,

$$\begin{aligned}
|h_n| &= \left|\frac{n - te^{\pm i\alpha\pi/2}}{1 - te^{\pm i\alpha\pi/2}}\right|. \text{ Hence} \\
|h_n|^2 &= \frac{(n - t\cos(\alpha\pi/2))^2 + t^2\sin^2(\alpha\pi/2)}{(1 - t\cos(\alpha\pi/2))^2 + t^2\sin^2(\alpha\pi/2)}; \\
&= \frac{n^2 - 2nt\cos(\alpha\pi/2) + t^2}{1 - 2t\cos(\alpha\pi/2) + t^2} \\
|h_n|^2 &= 1 + \frac{n^2 - 1 - 2t(n-1)\cos(\alpha\pi/2)}{1 - 2t\cos(\alpha\pi/2) + t^2} \\
&\le 1 + \frac{n^2 - 1}{1 - 2t\cos(\alpha\pi/2) + t^2} \quad \text{since } t \ge 0; \\
&\le \max_t \left[1 + \frac{n^2 - 1}{1 - 2t\cos(\alpha\pi/2) + t^2}\right] \\
&\le 1 + \frac{n^2 - 1}{\sin^2(\alpha\pi/2)} = \frac{n^2 - \cos^2(\alpha\pi/2)}{\sin^2(\alpha\pi/2)}.
\end{aligned}$$

Therefore

$$|h_n| \le \frac{\sqrt{n^2 - \cos^2(\alpha\pi/2)}}{\sin(\alpha\pi/2)} < \frac{n}{\sin(\alpha\pi/2)}.$$

Lemma 2. *For $f \in A$ and for every $\epsilon \in \mathbb{C}$ such that $|\epsilon| < \delta$, if $F_\epsilon(z) = \frac{f(z)+\epsilon z}{1+\epsilon} \in S^*(\alpha)$ then for every $H \in S^{*\prime}(\alpha)$, $\left|\frac{(f*H)(z)}{z}\right| \ge \delta$, $z \in E$.*

Proof. Let $F_\epsilon \in S^*(\alpha)$. Then by Theorem 2, $\frac{(F_\epsilon * H)(z)}{z} \neq 0$, for all $H \in S^{*\prime}(\alpha)$, $z \in E$. Equivalently

$$\frac{(f*H)(z) + \epsilon z}{(1+\epsilon)z} \neq 0 \quad \text{in } E \text{ or } \quad \frac{(f*H)(z)}{z} \neq -\epsilon$$

which shows that $\left|\frac{(f*H)(z)}{z}\right| \ge \delta$.

Theorem 3. *For $f \in A$ and $\epsilon \in \mathbb{C}$, $|\epsilon| < \delta < 1$ assume $F_\epsilon(z) \in S^*(\alpha)$. Then $N_{\delta\sin(\alpha\pi/2)}(f) \subset S^*(\alpha)$.*

Proof. Let $H \in S^{*\prime}(\alpha)$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is in $N_{\delta'}(f)$. Then

$$\begin{aligned} \left|\frac{(g*H)(z)}{z}\right| &= \left|\frac{(f*H)(z)}{z} + \frac{((g-f)*H)(z)}{z}\right| \\ &\geq \left|\frac{(f*H)(z)}{z}\right| - \left|\frac{(g-f)(z)*H(z)}{z}\right| \\ &\geq \delta - \left|\sum_{k=2}^{\infty}\frac{(b_k-a_k)h_k z^k}{z}\right| \qquad \text{by Lemma 2;} \end{aligned}$$

thus

$$\begin{aligned} \left|\frac{(g*H)(z)}{z}\right| &\geq \delta - |z|\sum_{k=2}^{\infty}|h_k|\,|b_k-a_k|; \\ &> \delta - \frac{1}{\sin(\alpha\pi/2)}\sum_{k=2}^{\infty}|b_k-a_k|k \text{ by Lemma 1} \\ &> \delta - \frac{\delta'}{\sin(\alpha\pi/2)} = 0 \text{ for } \delta' = \delta\sin(\alpha\pi/2). \end{aligned}$$

Thus $\frac{(g*H)(z)}{z} \neq 0$ in E for all $H \in S^{*\prime}(\alpha)$ which means by Theorem 2, $g \in S^*(\alpha)$; in otherwords $N_{\delta\sin(\alpha\pi/2)}(f) \subset S^*(\alpha)$.

Next let us show that the class $S^*(\alpha)$ is closed under convolution with functions f which are convex univalent in E, that is $(f*g)(z) \in S^*(\alpha)$ whenever $f \in K$ and $g \in S^*(\alpha)$.

Theorem 4. Let $f(z) \in K$, $g(z) \in S^*(\alpha)$. Then $(f*g)(z) \in S^*(\alpha)$.

Proof. Since $g \in S^*(\alpha) \subset S^*$ the class of starlike functions and $f \in K$ and Ω defined by (1) is a convex domain, an application of Lemma B gives

$$\begin{aligned} \frac{z(f*g)'(z)}{(f*g)(z)} &= \frac{f*\left[\frac{zg'(z)}{g(z)}\right]g(z)}{(f*g)(z)} \\ &\subset \bar{c}_0\frac{zg'(z)}{g'(z)} = \Omega, \quad z \in E. \end{aligned}$$

This shows that $(f*g)(z) \in S^*(\alpha)$.

Theorem 5. *If* $f \in K(\alpha)$, *then* $\frac{f(z)+\epsilon z}{1+\epsilon} \in S^*(\alpha)$ *for* $|\epsilon| < \frac{1}{4}$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$\frac{f(z)+\epsilon z}{1+\epsilon} = \frac{z(1+\epsilon) + \sum_{n=2}^{\infty} a_n z^n}{1+\epsilon}$$
$$= \frac{f(z) * \left[z(1+\epsilon) + \sum_{n=2}^{\infty} z^n\right]}{1+\epsilon}$$
$$= f(z) * \frac{\left[z - \frac{\epsilon}{1+\epsilon} z^2\right]}{1-z} = f(z) * h(z)$$

where $h(z) = \dfrac{\left[z - \frac{\epsilon}{1+\epsilon} z^2\right]}{1-z}$. Now

$$\frac{zh'(z)}{h(z)} = \frac{\left[z - \frac{2\epsilon}{1+\epsilon} z^2\right]}{\left[z - \frac{\epsilon}{1+\epsilon} z^2\right]} + \frac{z}{1-z} = \frac{-\rho z}{1-\rho z} + \frac{1}{1-z},$$

where $\rho = \frac{\epsilon}{1+\epsilon}$. Hence $|\rho| < \frac{\epsilon}{1-|\epsilon|} < \frac{1}{3}$ gives $|\epsilon| < \frac{1}{4}$. Thus $Re\left[\frac{zh'(z)}{h(z)}\right] \geq \frac{1-2|\rho|\,|z|-|\rho|\,|z|^2}{(1-|\rho|\,|z|)(1+|z|)} > 0$ if $|\rho| < |z|^2 + 2|z|) - 1 < 0$. This inequality holds for all $\rho < 1/3$. and $|z| < 1$, which is true for $|\epsilon| < 1/4$. Therefore h is starlike in the unit disk and so $\int_0^z \frac{h(t)}{t} dt$ is convex.

But $h(z) * \log[\frac{1}{1-z}] = \int_0^z \frac{h(t)}{t} dt$ and so $h(z) * \log\left[\frac{1}{1-z}\right]$ is convex in E and

$$(f * h)(z) = (h * f)(z) = h(z) * \left[zf'(z) * \log\left[\frac{1}{1-z}\right]\right]$$
$$= zf'(z) * \left[(h(z) * \log\left[\frac{1}{1-z}\right]\right]$$

$f(z) \in K(\alpha) \implies zf'(z) \in S^*(\alpha)$ and $h(z) * \log\left[\frac{1}{1-z}\right]] \in K$. Now by Theorem 4 $h(z) * \left[zf'(z) * \log\left[\frac{1}{1-z}\right]\right]$ is in $S^*(\alpha)$. Thus $(f * h)(z) = \frac{f(z)+\epsilon z}{1+\epsilon} \in S^*(\alpha)$ for $|\epsilon| < 1/4$.

Theorem 6. *Let $f \in K(\alpha)$. Then $N_{1/4\ \sin(\alpha\pi/2)}(f) \subset S^*(\alpha)$.*

Proof. Let $f \in K(\alpha)$. Then from Theorem 5 we have $\frac{f(z)+\epsilon}{1+\epsilon} \in S^*(\alpha)$ for $|\epsilon| < 1/4$. Then an application of Theorem 3 gives $N_{1/4\ \sin(\alpha\pi/2)}(f) \subset S^*(\alpha)$.

When $\alpha = 1$ we get a result of St. Ruscheweyh [3] as a special case.

Refercence

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