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ON THE EXTENSOIN OF BERNOULLI, EULER AND EULERIAN POLYNOMIALS

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Abstract. Here an attempt has been made to extend the Bernoulli, Euler and Eulerian polynomials in multiplication theorem and finite difference formula have been established.

1. Introduction

The study of Bernoulli, Euler and Eulerian polynomials has contributed much to our knowledge of the theory of numbers. These polynomials are of basic importance in several parts of Analysis and Calculus of Finite Differences and have application in various fields such as Statistics, Numerical Analysis etc. In recent years, the Eulerian numbers and Certain generalizations have been encountered in a number of Combinatorial problems (vide (1), (3), (4), (5), (6) for example). Of late, Singh and Rai (7) studied the extended polynomial set $B(n, h, a, k, x)$. This polynomial set was subjected to further investigation and Singh and Rai (8) succeeded in presenting novel two-variable extension of the same. A study of above polynomial set motivated us for consideration of following multi-variate extension (3.1) of the Bernoulli, Euler and Eulerian Polynomials and numbers as well as in the unified form from the point of view just described.

2. Preliminary Results

In 1964, Carlitz [2] extended the Bernoulli, Euler and Eulerian numbers and corresponding polynomials as

$$\frac{\log \mathcal{G}(s)}{\mathcal{G}(S)-1}=\sum_{n=1}^{\infty}\frac{\beta(n)}{n^s} \tag{2.1}$$

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$$\frac{(\mathcal{G}(s))^z \log \mathcal{G}(s)}{\mathcal{G}(s)-1} = \sum_{n=1}^{\infty} \frac{\beta(n,z)}{n^s} \tag{2.2}$$

$$\frac{1-\lambda}{\mathcal{G}(s)-\lambda} = \sum_{n=1}^{\infty} \frac{H(n,\lambda)}{n^s}, \qquad \lambda \neq 1 \tag{2.3}$$

$$\frac{(1-\lambda)\mathcal{G}(s)^z}{\mathcal{G}(s)-\lambda} = \sum_{n=1}^{\infty} \frac{H(n,\lambda,z)^n}{n^s}, \qquad \lambda \neq 1 \tag{2.4}$$

When $\beta(n)$, $\beta(n,z)$, $H(n,\lambda)$ and $H(n,\lambda,z)$ are the extended Bernoulli and Eulerian numbers as well as corresponding polynomials. In the same paper, Carlitz considered a slightly more general situation, namely, the polynomials $\beta(n,h,z)$ and $H(n,h,\lambda,z)$ generated by

$$\frac{h(\mathcal{G}(s))^{hz} \log \mathcal{G}(s)}{(\mathcal{G}(S))^h - 1)} = \sum_{n=1}^{\infty} \frac{\beta(n,h,z)}{n^s} \tag{2.5}$$

and,

$$\frac{(\lambda-1)(\mathcal{G}(s))^{hz}}{(\mathcal{G}(s)^h - \lambda)} = \sum_{n=1}^{\infty} \frac{H(n,h,\lambda,z)}{n^s}, \qquad \lambda \neq 0, \lambda \neq 1 \tag{2.6}$$

It may be of interest to note that

$$\epsilon(n,h,z) = H(n,h,-1,z) \tag{2.7}$$

where $\epsilon(n,h,z)$ are extended Eulerian polynomials defined as

$$\frac{2(\mathcal{G}(s))^{hz}}{((\mathcal{G}(s))^h + 1)} = \sum_{n=1}^{\infty} \frac{\epsilon(n,h,z)}{n^s} \tag{2.8}$$

It is familar that the formula

$$g(n) = \sum_{d/n} f(d), \quad (n = 1, 2, 3, \ldots) \tag{2.9}$$

$$f(n) = \sum_{cd=n} \mu(c) g(d), \quad (n = 1, 2, 3 \ldots) \tag{2.10}$$

where $\mu(n)$ is Mobius function, are equivalent. If in (2.9) and (2.10),. we take $n = p_1 p_2 \ldots p_r$ where p_r are distinct primes, it is easily verified that (2.9) and (2.10) reduce to

$$g_r = \sum_{j=0}^{r} \binom{r}{j} f_j; \quad (r = 0, 1, 2 \ldots) \tag{2.11}$$

$$f_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} g_j; \quad (r = 0, 1, 2, \ldots) \tag{2.12}$$

respectively, where, for brevity, we put

$$f_r = f(p_1p_2p_3\cdots p_r), \quad g_r = (p_1p_2p_3\cdots p_r)$$

The equivalence of (2.11) and (2.12) is, of course, well known. The fact that the second equivalence is implied by first is perhaps not quite so familiar. It should be emphasized that $f(n)$ and $g(n)$ are arbitrary arithmetic functions subject only to (2.9) or equivalently (2.10), similar remark applies to f_r and g_r.
Given a sequence

$$f_r \quad (r = 0, 1, 2, \ldots) \tag{2.13}$$

we define an extended sequence

$$f(n), \quad (n = 0, 1, 2, \ldots) \tag{2.14}$$

such that

$$f(p_1p_2p_3, \cdots p_r) = f_r, \quad \text{where } p_j \text{ are distinct prime;} \tag{2.15}$$

Clearly, the extended sequence (2.14) is not uniquely determined by means of (2.15). If the sequence g_r is related to f_r by means of (2.11), the sequence $g(n)$ defined by means of (2.9) furnishes an extension of the sequence g_r.

If we associate with the sequence f_r the (formal) power series

$$F_t = \sum_{r=0}^{\infty} f_r \frac{t^r}{r!} \tag{2.16}$$

then (2.11) is equivalent to

$$G_t = e^t F_t, \quad \text{where} \quad G(t) = \sum_{r=0}^{\infty} g_r \frac{t^r}{r!}$$

We associate with the sequence $f(n)$ the (formal) Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f_n}{n^s} \tag{2.18}$$

Then (2.9) is equivalent to

$$G(S) = \mathcal{G}(s)F(s) \tag{2.19}$$

where

$$G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \qquad \mathcal{G}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

It is well known that generating functions play an important role in the study of various useful properties of the polynomial sets which they generate. More systematic attacks have been made in this direction by Srivastava and Manocha [9]. The generating function concept led to the study of the following general class of polynomials.

3. Extended Polynomial Set

We define the polynomial set $P(n, h_1, h_2, \ldots, h_k, a_1, a_2, \ldots, a_k, m, i_1, i_2, \ldots, i_k, x_1, x_2, \ldots, x_k)$ by the following generating relation.

$$\frac{m^2(\frac{h_1}{m}\log \mathcal{G}(s))^{i_1}(\frac{h_2}{m}\log \mathcal{G}(s))^{i_2}\cdots(\frac{h_k}{m}\log \mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1}-a_1)((\mathcal{G}(s))^{h_2}-a_2)\cdots((\mathcal{G}(s))^{h_k}-a_k)} \times (\mathcal{G}(s))^{h_1x_1+h_2x_2+\cdots+h_kx_k}$$
$$=\sum_{n=1}^{\infty}\frac{P(n, h_1, h_2, \ldots, h_k, a_1, a_2, \ldots a_k, m, i_1, i_2, \ldots, i_k, x_1x_2, \ldots x_k)}{n^s} \tag{3.1}$$

Where $a_1, a_2, \ldots, a_k$, m are non-zero real numbers, $i_1, i_2, \ldots, i_k$ are non-negative integers and $(h_1, h_2, \ldots, h_k) \neq 0$ and $\mathcal{G}(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$

We, now, notice following conditions

1. $i_3 = i_4 = \ldots i_k = 0$
2. $h_3 = h_4 = \ldots h_k = 1$ (Since, $h_1, h_2, \ldots, h_k \neq 0$)
3. $a_3 = a_4 = \ldots a_k = 0$ and
4. under the stated conditions, $\mathcal{G}(s) = 1$, only when $\mathcal{G}(1) = 1$, $n = 1$.

If we apply above substitutions to the polynomial set $P(n, h_1, h_2, \ldots, h_k, a_1, a_2, \ldots a_k, m, i_1, i_2, \ldots, i_k, x_1, x_2, \ldots, x_k)$ this polynomial set reduces to polynomial set $P(n, h, k, a, b, m, i, j, x, y)$ defined by Singh and Rai [8]. Here, it is interesting to note that all the properties established for the polynomial set given by Singh and Rai [8] will spontaneously hold good for (3.1). In addition to these properties, some striking new properties will be set up. Categorically speaking, there will appear a plenty of properties in comparison to earlier.

The two-fold advantages of this may be explained: firstly, it is a generalized polynomial set in the sense that it unifies Bernoulli, Euler and Eulerian polynomials which, in turn, are obtainable from it on specializing various parameters involved therein; secondly, some striking new properties of these polynomials follow as direct consequences of itself.

We state the following relationships between our polynomial set and other polynomials

1. Extended Bernoulli Polynomials

Taking

$$a_1 = a_2 = \cdots = a_{k-1} = a_k = i_1 = i_2 = \ldots = i_{k-1} = i_k = 1$$

we arrive at

$$P(n, h_1, h_2, h_2, \ldots, h_k, 1, 1, \ldots, 1, 1, \ldots, 1, x_1, x_2, \ldots, x_k)$$
$$=\beta(n, h_1, h_2, \ldots, h_k; x_1, x_2, \ldots, x_k) \tag{3.2}$$

(II) Extended Bernoulli numbers

$a_1 = a_2 = \ldots = a_k = m = i_1 = i_2 = \cdots = i_k = 1$ and $x_1 = x_2 = \cdots = x_k = 0$

Substituting, we get

$$\begin{aligned}&(n, h_1, h_2, h_2, \ldots, h_k, 1, \ldots, 1; 1, \ldots, 1, 0, 0, \ldots)\\ =&\beta(n, h_1, h_2, \ldots, h_k)\end{aligned} \tag{3.3}$$

(III) Extended Euler polynomials

when $a_1 = a_2 = \cdots = a_k = -1; i_1 = i_2 = \cdots = i_k = 0$ and $m = 2$

we are led to

$$\begin{aligned}&P(n, h_1, h_2, h_2, \ldots, h_k, -1, -1, \ldots, -1, 2, 0, \ldots, 0, x_1, x_2, \ldots x_k)\\ =&\epsilon(n, h_1, h_2, \ldots, h_k; x_1, x_2, \ldots x_k)\end{aligned} \tag{3.4}$$

(IV) Extended Euler numbers

Putting

$$a_1 = a_2 = \cdots = a_k = -1, i_1 = i_2 = \ldots i_k = 0$$
$$m = 2 \quad \text{and} \quad x_1 = x_2 = \cdots = x_k = 0$$

we get

$$\begin{aligned}&P(n, h_1, h_2, \ldots, h_k, -1, -1 \ldots - 1, 2, 0, \ldots, 0, 0, \ldots, 0)\\ =&\epsilon(n, h_1, h_2, \ldots, h_k)\end{aligned} \tag{3.5}$$

(V) Extended Eulerian polynomials

Yet another interesting special case of the polynomial set would occur when we let

$i_1 = i_2 = \ldots = i_k = 0$ and $m = 1$. Thus we obtain

$$\begin{aligned}&P(n, h_1, h_2, h_2 \ldots, h_k, a_1, a_2, \ldots, a_k, 1, 0, \ldots, 0, x_1, x_2, \ldots, x_k)\\ =&\frac{1}{(1-a_1) \times (1-a_2) \cdots (1-a_k)} H(n, h_1, h_2, \ldots, h_k, a_1, a_2, \ldots, a_k, x_1, x_2, \ldots, x_k)\end{aligned} \tag{3.6}$$

(VI) Extended Eulerian numbers

Letting

$i_1 = i_2 = \ldots = i_k = 0$ and $m = 1$ and $x_1 = x_2 = \ldots x_k = 0$, we have

$$\begin{aligned}&P(n, h_1, h_2, \ldots, h_k, a_1, a_2, \ldots, a_k, 1, 0, \ldots, 0, 0, 0, 0, \ldots, 0)\\ =&\frac{1}{(1-a_1)(1-a_2) \cdots (1-a_k)} \times H(n, h_1, h_2, \ldots, h_k, a_1, a_2, \ldots, a_k)\end{aligned} \tag{3.7}$$

the above extended Bernoulli, Euler and Eulerian polynomials and their corresponding numbers are due to Carlitz [2].

In the present paper, we obtain numerous properties of the polynomials and numbers defined above. These properties are of an algebraic nature and for the most parts are generalizations of the corresponding properties of the Bernoulli, Euler and Eulerian polynomials and numbers.

4. Addition Theorems

$$\frac{2^2(i_1+i_2+\cdots+i_k)}{m^2} \tag{4.1}$$
$$\times \sum_{cd=n} P(c,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,2x_1,2x_2,\ldots,2x_k)$$
$$\times \sum_{cd=n} P(d,h_1,h_2,\ldots,h_k,-a_1,-a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,2y_1,2y_2,\ldots,2y_k)$$
$$=P(n,2h_1,2h_2,\ldots,2h_k,a_1^2,a_2^2,\ldots,a_k^2,2i_2,\ldots,2i_k,x_1+y_1,x_2+y_2,\ldots,x_k+y_k)$$

Proof. Since

$$\frac{m^2(\frac{h_1}{m}\log\mathcal{G}(s))^{i_1}(\frac{h_2}{m}\log\mathcal{G}(s))^{i_2}\cdots(\frac{h_k}{m}\log\mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1}-a_1)((\mathcal{G}(s))^{h_2}-a_2)\cdots((\mathcal{G}(s))^{h_k}-a_k)}\times((\mathcal{G}(s))^{2h_1x_1+2h_2x_2+\cdots+2h_kx_k}$$
$$\times\frac{m^2(\frac{h_1}{m}\log\mathcal{G}(s))^{i_1}(\frac{h_2}{m}\log\mathcal{G}(s))^{i_2}\cdots(\frac{h_k}{m}\log\mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1}+a_1)((\mathcal{G}(s))^{h_2}+a_2)\cdots((\mathcal{G}(s))^{h_k}+a_k)}\times((\mathcal{G}(s))^{2h_1y_1+2h_2y_2+\cdots+2h_ky_k}$$
$$=m^2\frac{m^2(\frac{h_1}{m}\log\mathcal{G}(s))^{2i_1}(\frac{h_2}{m}\log\mathcal{G}(s))^{2i_2}\cdots(\frac{h_k}{m}\log\mathcal{G}(s))^{2i_k}}{((\mathcal{G}(s))^{2h_1}-a_1^2)((\mathcal{G}(s))^{2h_2}-a_2^2)\cdots((\mathcal{G}(s))^{2h_k}-a_k^2}\times(\mathcal{G}(s))^{2h_1(x_1+y_1)+\cdots+2h_k(x_k+y_k)}$$

In the light of (3.1), after little simplification, we get

$$\sum_{n=1}^{\infty} P(c,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,2x_1,2x_2,\ldots 2x_k)$$
$$\times\sum_{n=1}^{\infty} P(d,h_1,h_2,\ldots h_k,-a_1,-a_2,\ldots,-a_k,m,i_1,i_2,\ldots,i_k,2y_1,2y_2,\ldots,2y_k)$$
$$=\frac{m^2}{2^{2i_1+2i_2+\cdots+2i_k}}\times$$
$$\sum_{n=1}^{\infty} P(c,2h_1,2h_2,\ldots,2h_k,a_1^2,a_2^2,\ldots,a_k^2,2i_1,2i_2,\ldots,2i_k,x_1+y_1,x_2+y_2,\ldots,x_k+y_k)$$

whcih completes the proof.

$$\frac{2^{2(i_1+i_2+\cdots i_k)}}{m^2}\sum_{cd=n} P(c,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,2x_1,2x_2,\ldots,2x_k)$$
$$\times\sum_{cd=n} P(d,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,m,i_1,i_2,\ldots,i_k,y_1,y_2,\ldots,y_{k-1},0)$$
$$=P(n,2h_1,2h_2,\ldots,2h_k,a_1^2,a_2^2,\ldots,a_k^2,2i_1,2i_2,\ldots,2i_k,x_1+y_1,x_2+y_2,\ldots,x_k+0) \tag{4.2}$$

Proof is similar to the proof of (4.1)

5. Multiplication Theorems

$$\sum_{r_1=0}^{\mu_1-1}\sum_{r_2=0}^{\mu_2-1}\cdots\sum_{r_k=0}^{\mu_k-1}\frac{1}{a_1^{r_1}a_2^{r_2}\dots a_k^{r_k}}$$
$$\times P(n,h_1,\mu_1,\dots,h_k,\mu_k,a_1^{\mu_1},\dots,a_k^{\mu_k},m,i_1,i_2,\dots,i_k,x_1+\frac{r_1}{\mu_1},\dots,x_k+\frac{r_k}{\mu_k})$$
$$=\frac{\mu_1^{i_2}\cdot\mu_2^{i_2}\cdots\mu_k^{i_k}}{a_1^{\mu_1-1}\cdot a_2^{\mu_2-1}\cdots a_k^{\mu_k-1}}$$
$$\times P(n,h_1,h_2,\dots,h_k,a_1,a_2,\dots,a_k,m,i_1,i_2,\dots,i_k,\mu_1x_1,\mu_2x_2,\dots,\mu_kx_k) \tag{5.1}$$

where and throughout this investigation, $\mu_1,\mu_2,\dots,\mu_k$ are positive integers.

Proof. By virtue of generating relation (3.1), we have

$$\sum_{n=1}^{\infty}\frac{P(n,h_1,h_2,\dots,h_k,a_1,a_2,\dots,a_k,m,i_1,i_2,\dots,i_k,\mu_1x_1,\mu_2x_2,\dots,\mu_kx_k)}{n^s}$$
$$=\frac{m^2(\frac{h_1}{m}\log\mathcal{G}(s))^{i_1}(\frac{h_2}{m}\log\mathcal{G}(s))^{i_2}\dots(\frac{h_k}{m}\log\mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1}-a_1)((\mathcal{G}(s))^{h_2}-a_2)\cdots((\mathcal{G}(s))^{h_k}-a_k)}(\mathcal{G}(s))^{h_1\mu_1x_1+h_2\mu_2\mu_2x_2+\dots+h_k\mu_kx_k}$$
$$=a_1^{\mu_1-1}a_2^{\mu_2-1}\dots a_k^{\mu_k-1}\times\frac{m^2(\frac{h_1}{m}\log\mathcal{G}(s))^{i_1}(\frac{h_2}{m}\log\mathcal{G}(s))^{i_2}\dots(\frac{h_k}{m}\log\mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1\mu_1}-a_1^{\mu_1})\cdots((\mathcal{G}(s))^{h_k\mu_k}-a_k^{\mu_k})}$$
$$\times\sum_{r_1=0}^{\mu_1-1}\frac{(\mathcal{G}(s))^{r_1h_1}}{a_1^{r_1}}\cdots\sum_{r_k=0}^{\mu_k-1}\frac{(\mathcal{G}(s))^{r_kh_k}}{a_k^{r_k}}$$
$$=a_1^{\mu_1-1}a_2^{\mu_2-1}\dots a_k^{\mu_k-1}\times\frac{m^2(\frac{h_1}{m}\log\mathcal{G}(s))^{i_1}(\frac{h_2}{m}\log\mathcal{G}(s))^{i_2}\dots(\frac{h_k}{m}\log\mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1\mu_1}-a_1^{\mu_1})\cdots((\mathcal{G}(s))^{h_k\mu_k}-a_k^{\mu_k})}$$
$$\times\sum_{r_1=0}^{\mu_1-1}(\mathcal{G}(s))^{(x_1+\frac{r_1}{\mu_1})\mu_1h_1}\cdots\sum_{r_k=0}^{\mu_k-1}(\mathcal{G}(s))^{(x_k+\frac{r_1}{\mu_k})\mu_kh_k}$$

By an appeal to (3.1), the result would follow immediatedly.

Theorem 5.2.

$$a_1^{\mu_1}a_2^{\mu_2}\cdots a_k^{\mu_k}\cdot p_1^{i_1}p_2^{i_2}\cdots p_k^{i_k}\sum_{r_1=0}^{\mu_1-1}\cdots\sum_{r_k=0}^{\mu_k-1}\frac{1}{a_1^{r_1p_1}a_2^{r_2p_2}\dots a_k^{r_kp_k}}$$
$$\times P(n,h_1\mu_1,\dots h_k\mu_k,a_1^{\mu_1},\dots a_k^{\mu_k},m,i_1,\dots i_k,\frac{x_1}{\mu_1}+\frac{r_1p_1}{\mu_1},\dots,\frac{x_k}{\mu_k}+\frac{r_kp_k}{\mu_k}$$
$$=\mu_1^{i_1}\mu_2^{i_2}\dots\mu_k^{i_k}a_1^{p_1}a_2^{p_2}\dots a_k^{p_k}\sum_{e_1=0}^{p_1-1}\cdots\sum_{e_k=0}^{p_k-1}\frac{1}{a_1^{\mu_1e_1}a_k^{\mu_ke_k}}$$
$$\times P(n,p_1h_1,\dots,p_kh_k,a_1^{p_1},\dots,a_k^{p_k},m,i_1,\dots,i_k,\frac{x_1}{p_1}+\frac{e_1\mu_1}{p_1},\dots,\frac{x_k}{p_k}+\frac{e_k\mu_k}{p_k})$$

proof is similar to the proof of (5.1)

6. Finite Difference Formula

Norlund's operator is defined by the relation

$$\underset{W}{\Delta} U(x) = \frac{U(x+w) - U(x)}{w} \tag{6.1}$$

This symbol has the advantage that

$$\lim_{w \to 0} \underset{W}{\Delta} U(x) = DU(x), D = \frac{d}{dx}$$

The definition of the operator $\underset{W}{\nabla}$ is given

$$\underset{W}{\nabla} U(x) = \frac{1}{2}[U(x) + U(x+w)] \tag{6.2}$$

If $w = 1$, it is convenient to write Δ instead of $\underset{1}{\Delta}$ and ∇ instead $\underset{1}{\nabla}$, Again, consider $f(x, y)$, where x and y are regarded as independent variables. We then define partial difference quotients with respect to X and Y by

$$\underset{W^x}{\Delta} f(x,y) = [f(x+w,y) - f(x,y)]/W \tag{6.3}$$

and

$$\underset{h^y}{\Delta} f(x,y) = [f(x,y+h) - f(x,y)]/h \tag{6.4}$$

These symbols have the advantage that

$$\lim_{W \to 0} \underset{W^x}{\Delta} f(x,y) = D_x f(x,y)$$

and

$$\lim_{h \to 0} \underset{h^y}{\Delta} f(x,y) = D_y f(x,y)$$

Likewise

$$\underset{W^x}{\nabla} f(x,y) = \frac{1}{2}[f(x,y) + f(x+w,y)] \tag{6.5}$$

and

$$\underset{h^y}{\nabla} f(x,y) = \frac{1}{2}[f(x,y) + f(x,y+h)] \tag{6.6}$$

Similarly, the above operators can be defined for multivariate function $f(x_1, x_2, \ldots, x_k)$ as under

$$\underset{W_1^{x_1}}{\Delta} f(x_1, x_2, \ldots, x_k) = [[f(x_1 + w_1, x_2, \ldots, x_k) - f(x_1, x_2, \ldots, x_k)]/W_1 \tag{6.7}$$

$$\underset{W_2^{x_2}}{\Delta} f(x_1, x_2, \ldots, x_k) = [f(x_1, x_2 + w_2, x_3, \ldots, x_k) - f(x_1, x_2, \ldots, x_k)]1/W_2 \quad (6.8)$$

and similarly

$$\underset{W_k^{x_k}}{\Delta} f(x_1, x_2, \ldots, x_k) = [[f(x_1, x_2, \ldots, x_k + w_k) - f(x_1, x_2, \ldots, x_k)]/W_k \quad (6.9)$$

These symbols have advantage that

$$\operatorname*{Lim}_{W_1 \to 0} \underset{W_2^{x_2}}{\Delta} f(x_1, x_2, \ldots, x_k) = D_{x_1} f(x_1, x_2, \ldots, x_k)$$

$$\operatorname*{Lim}_{W_2 \to 0} \underset{W_2^{x_2}}{\Delta} f(x_1, x_2, \ldots, x_k) = D_{x_2} f(x_1, x_2, \ldots, x_k)$$

and

$$\operatorname*{Lim}_{W_k \to 0} \underset{W_k^{x_k}}{\Delta} f(x_1, x_2, \ldots, x_k) = D_{x_k} f(x_1, x_2, \ldots, x_k)$$

As above, we can define the ∇ for $f(x_1, x_2, \ldots, x_k)$

$$\underset{W_1^{x_1}}{\nabla} f(x_1, x_2, \ldots, x_k) = \frac{1}{2}[f(x_1, x_2, \ldots, x_k) + f(x_1 + w_1, x_2, x_3, \ldots, x_k)] \quad (6.10)$$

$$\underset{W_2^{x_2}}{\nabla} f(x_1, x_2, \ldots, x_k) = \frac{1}{2}[f(x_1, x_2 \ldots, x_l) + f(x_1, x_2 + w_2, x_3, \ldots, x_k)] \quad (6.11)$$

Similarly,

$$\underset{W_k^{x_k}}{\Delta} f(x_1, x_2, \ldots, x_k) = \frac{1}{2}[f(x_1, x_2, \ldots, x_k) + f(x_1, x_2, \ldots, x_k + w_k)].$$

operating on (3.1) with Δ_{x_1} we find that

$$\sum_{n=1}^{\infty} \Delta_x \frac{P(n, h_1, h_2, \ldots, h_k, a_1, \ldots, a_k, m, i_1, i_2, \ldots, i_l, x_1, x_2, \ldots, x_k)}{n^s}$$

$$\frac{m^2(\frac{h_1}{m}\log \mathcal{G}(s))^{i_1}(\frac{h_2}{m}\log \mathcal{G}(s))^{i_2}\ldots(\frac{h_k}{m}\log \mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1} - a_1)((\mathcal{G}(s))^{h_2} - a_2 \ldots ((\mathcal{G}(s))^{h_k} - a_k)} \times ((\mathcal{G}(s))^{h_1x_1 + h_2x_2 + \cdots + h_kx_k}(\mathcal{G}(s)^{h_1}$$

$$= \frac{m^2(\frac{h_1}{m}\log \mathcal{G}(s))^{i_1}(\frac{h_2}{m}\log \mathcal{G}(s))^{i_2}\ldots(\frac{h_k}{m}\log \mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1} - a_1)((\mathcal{G}(s))^{h_2} - a_2)\cdots((\mathcal{G}(s))^{h_k} - a_k)} \times (\mathcal{G}(s))^{h_1x_1 + h_2x_2 + \cdots + h_kx_k}$$

$$= \frac{((\mathcal{G}(s))^{h_1} - 1)m^2(\frac{h_1}{m}\log \mathcal{G}(s))^{i_1}\ldots(\frac{h_k}{m}\log \mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1} - a_1)((\mathcal{G}(s)^{h_2} - a_2)\ldots((\mathcal{G}(s))^{h_k} - a_k)} \times (\mathcal{G}(s))^{h_1x_1 + h_2x_2 + \cdots + h_kx_k}$$

In view of (3.1) and the definition of $\tau_x(n)$ in [2], the above expression immediately yields

$$\Delta_{x_1} P(n, h_1, h_2, \ldots, h_k, a_1, \ldots, a_2, \ldots, a_k, m, i_1, i_2, \ldots, i_k, x_1, x_2, \ldots, x_k)$$

$$= \sum_{cd=n} \tau_{h_1}(c) P(d, h_1, h_2, \ldots, h_k, a_1, a_2, \ldots, a_k, m, i_1, i_2, \ldots, i_k, x_1, x_2, \ldots, x_k)$$

$$P(n, h_1, h_2, \ldots, h_k, a_1, a_2, \ldots, a_k, m, i_1, i_2, \ldots, i_k, x_1, x_2, \ldots, x_k)$$

Similarly, operting on (3.1) with Δ_{x_k} we intuitively obtain

$$\Delta_{x_k} P(n,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)$$
$$=\sum_{cd=n}\tau_{h_k}(c)P(d,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)$$
$$-P(n,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots x_k)$$

Furthermore, operating ∇_{x_1} on the generating relation (3.1) we have

$$\sum_{n=1}^{\infty}\nabla_{x_1}\frac{P(n,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots x_k)}{n^s}$$
$$\frac{\frac{1}{2}(\mathcal{G}(s)^{h_k}+1)m^2(\frac{h_1}{m}\log\mathcal{G}(s))^{i_1}\ldots(\frac{h_k}{m}\log\mathcal{G}(s))^{i_k}}{((\mathcal{G}(s))^{h_1}-a_1)((\mathcal{G}(s))^{h_2}-a_2)\cdots((\mathcal{G})(s))^{h_k}-a_k)}\times((\mathcal{G}(s))^{h_1x_1+h_2x_2+\cdots+h_kx_k}$$
$$=\sum_{n=1}^{\infty}\frac{1}{2}(\frac{\tau_{h_1}(n)}{n^s}+1)$$
$$\times\sum_{n=1}^{\infty}\frac{P(n,h_1,h_2,\ldots,h_k,a_1,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)}{n^s}$$

In the light of generatin relation (3.1) and Carlitz [2]. Finally, we arrive at

$$\nabla_{x_1}P(n,h_1,h_2,\ldots,h_k,a_1,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)$$
$$=\frac{1}{2}\sum_{cd=n}\tau_{h_1}(c)P(d,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)$$
$$+\frac{1}{2}P(n,h_1,h_2,\ldots,h_k,a_1,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)$$

Now finally we obtain the expression for $\nabla_{x_k}P$ which is similar to the expression $\nabla_{x_1}P$ by operating ∇_{x_k} on the generating relation (3.1) as

$$\nabla_{x_k}P(n,h_1,h_2,\ldots,h_k,a_1,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)$$
$$=\frac{1}{2}\sum_{cd=n}\tau_{h_1}(c)P(d,h_1,h_2,\ldots,h_k,a_1,a_2,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)$$
$$+\frac{1}{2}P(n,h_1,h_2,\ldots,h_k,a_1,\ldots,a_k,m,i_1,i_2,\ldots,i_k,x_1,x_2,\ldots,x_k)$$

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