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ON INTERNAL GRAVITY WAVES

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Abstract. We are concerned with the steady wave motions in a 2-fluid system with constant densities. This is a free boundary problem in which the lighter fluid is bounded above by a free surface and is separated from the heavier fluid down below by an interface. By using a contractive mapping principle type argument, a constructive proof to the existence of some of these exact periodic internal gravity waves is provided.

1. Introduction

In this paper, we consider the steady wave motion of a 2-fluid system of immiscible, inviscid, incompressible fluids with constant densities. Suppose that the lighter fluid on the top is bounded above by a free surface and is separated from the heavier one down below by an interface. For simplicity's sake, we will assume that the bottom layer is of infinite depth. The problem is to determine the velocity fields of both layers, the free surface on the top and the interface between the two layers of fluids.

Up till now, research on the steady wave motion of an infinite ocean consisting of just one fluid of constant density has been more fruitful. See Chapter 12 of [7] and also [1] for the existence proof of the two-dimensional exact steady water waves. Existence of exact two-dimensional steady water waves resulting from localized pressure disturbances on the free surface is also established in [1]. See [8] for corresponding results on three-dimensional exact steady water waves. When the surface tension effect is taken into account on the free surface, the situation becomes more complicated. For Froude number close to 1, we may have two different kinds of two-dimensional exact periodic water waves. See [1] for the details. The resonance of these two kinds of periodic water waves, so-called the Wilton's ripples, were discussed in detail in [3]. The existence problem of these Wilton's ripples were formulated and solved by Reeder and Shinbrot in 1981; see [4] and [5]. Similar results for internal gravity waves are lacking. From Art. 231 in [2], we see that density stratification can also produce different kinds of linear periodic waves. It is the purpose of this paper to look at the linear periodic internal gravity waves

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more closely and analyze when are these linear waves in resonance. On this basis, we start an investigation on the existence of two-dimensional exact internal gravity waves. We start by assuming that the internal gravity waves sought are small perturbation of the uniform horizontal flows. Next, we use the interior equations and the streamline conditions to form two nonlinear elliptic problems. By solving these boundary value problems, we are able to express the correction terms to the horizontal flow as functions of the surface elevations of the free surface on the top and the interface separating the two fluids. Consequently, we can solve the Bernoulli equations on the free surface and on the interface as a nonlinear system for the surface elevations.

In Section 2, the governing equations for the periodic internal gravity waves sought are formulated. Linear approximations to the periodic internal gravity waves are derived in Section 3. Parameters corresponding to when these linear waves become resonanant are determined in Section 3 also. In Section 4, we introduce the function spaces used in the construction of the exact internal gravity waves and two auxiliary elliptic problems are solved. Finally in Section 5, by assuming that the internal gravity waves considered are not in resonnance, we provide a constructive proof of the existence of the exact internal gravity waves by using a contraction mapping principle type argument.

2. Formulation of the problem

In what follows, we assume that the coordinate system (X, Y) is chosen so that the progressive internal waves sought appear steady, with the Y direction pointing upward. Let $Y = S_1(X)$ and $Y = S_2(X)$ be the free surface and the interface which separates the two fluids. Let (U_1, V_1) and (U_2, V_2) be the velocity fields in the top and bottom layers. The problem is to solve the system of equations

$$U_{1,X} + V_{1,Y} = 0, \quad \text{for } S_2(X) < Y < S_1(X), \tag{2.1}$$
$$U_{1,Y} - V_{1,X} = 0, \tag{2.2}$$
$$U_{2,X} + V_{2,Y} = 0, \quad \text{for } -\infty < Y < S_2(X), \tag{2.3}$$
$$U_{2,Y} - V_{2,X} = 0, \tag{2.4}$$
$$V_1 - S_{1,X}\, U_1 = 0, \quad \text{on } Y = S_1(X), \tag{2.5}$$
$$V_i - S_{2,X}\, U_i = 0, \quad \text{on } Y = S_2(X), \text{ for } i = 1, 2, \tag{2.6}$$
$$V_2 \to 0, \quad \text{as } Y \to -\infty, \tag{2.7}$$
$$g\, S_1 + \frac{1}{2}\,({U_1}^2 + {V_1}^2) = C_1, \quad \text{on } Y = S_1(X), \tag{2.8}$$
$$g\,(\rho_2 - \rho_1)\, S_2 + \frac{\rho_2}{2}\,({U_2}^2 + {V_2}^2) - \frac{\rho_1}{2}\,({U_1}^2 + {V_1}^2) = C_2, \quad \text{on } Y = S_2(X). \tag{2.9}$$

Here ρ_1 and ρ_2 are the densities of the top and bottom layers; $\rho_1 < \rho_2$. Equations (2.1) - (2.4) correspond to the assumption that the motion in each layer is incompressible and irrotational. (2.5) and (2.6) are the streamline conditions on the free surface and on the interface. (2.7) is saying that the flow sought is horizontal at infinity; (2.8) and (2.9) are Bernoulli's conditions on the free surface and on the interface. The constant g represents the gravitational acceleration. From now on, quantities with index 1 refer to the top layer; whereas quantities with index 2 refer to the bottom layer. Note that a trivial solution of above system is the horizontal flow with

$$\begin{aligned} &(U_i, V_i) = (U_0, 0), \\ &S_1(X) = h, \quad S_2(X) = 0, \end{aligned} \tag{2.10}$$

where U_0 and h are two positive constants.

Since the internal wave motion considered is incompressible and irrotational in each layer, we can introduce stream function ψ_1, ψ_2 such that

$$\begin{aligned} (\psi_{1,Y}, -\psi_{1,X}) &= (U_1, V_1), \\ (\psi_{2,Y}, -\psi_{2,X}) &= (U_2, V_2). \end{aligned}$$

Then (2.1) – (2.4) lead to

$$\psi_{1,XX} + \psi_{1,YY} = 0, \quad \text{for } S_2(X) < Y < S_1(X), \tag{2.11}$$

$$\psi_{2,XX} + \psi_{2,YY} = 0, \quad \text{for } -\infty < Y < S_2(X). \tag{2.12}$$

Now the streamline conditions (2.5) and (2.6) are equivalent to ψ_1 and ψ_2 being constant on the free surface and on the interface, and $\psi_1 = \psi_2$ on the interface. Choose ψ_1 and ψ_2 so that

$$\begin{aligned} &\psi_1 = U_0 h, \quad \text{on } Y = S_1(X), \\ &\psi_1 = \psi_2 = 0, \quad \text{on } Y = S_2(X), \\ &\psi_{2,X} \to 0 \text{ as } Y \to -\infty. \end{aligned} \tag{2.13}$$

In what follows, we will apply the coordinate transformation

$$(X, Y) \mapsto (X, \psi_i(X, Y)),$$

in each layer. We will assume that the horizontal velocity $\psi_{i,Y}$ in each layer is positive. Under above transformations, the top and bottom layers are mapped into the horizontal strips

$$\{(X, \psi_1) : 0 < \psi_1 < U_0 h\}, \quad \{(X, \psi_2) : -\infty < \psi_2 < 0\}.$$

Let f_i be the streamline functions such that

$$\begin{aligned} &Y = f_1(X, \psi_1(X, Y)), \quad \text{for } S_2(X) < Y < S_1(X), \\ &Y = f_2(X, \psi_2(X, Y)), \quad \text{for } -\infty < Y < S_2(X). \end{aligned} \tag{2.14}$$

Let

$$\begin{aligned}\psi &= \psi_1, \quad \text{for } S_2(X) < Y < S_1(X),\\ &= \psi_2, \quad \text{for } -\infty < Y < S_2(X).\end{aligned}$$

In what follows, we will use (X, ψ) as our new independent variables. As a result, (2.1) – (2.9) are transformed into

$$(1 + (f_{1,X})^2)\, f_{1,XX} - 2f_{1,X}\, f_{1,\psi}\, f_{1\,X\psi} + (f_{1,\psi})^2\, f_{1,\psi\psi} = 0, \quad \text{for } 0 < \psi < U_0 h, \tag{2.15}$$

$$(1 + (f_{2,X})^2)\, f_{2,XX} - 2f_{2,X}\, f_{2,\psi}\, f_{2\,X\psi} + (f_{2,\psi})^2\, f_{2,\psi\psi} = 0, \quad \text{for } -\infty < \psi < 0, \tag{2.16}$$

$$f_1(X,\, U_0 h) = S_1(x), \tag{2.17}$$

$$f_1(X,\, 0) = f_2(X,\, 0) = S_2(X), \tag{2.18}$$

$$f_{2,X}(X,\, \psi) \to 0 \quad \text{as } \psi \to -\infty \tag{2.19}$$

$$gS_1 + \frac{1}{2}\left(\frac{1 + (f_{1,X})^2}{(f_{1,\psi})^2}\right) = C_1, \quad \text{on } \psi = U_0 h, \tag{2.20}$$

$$(1-\rho)\, gS_2 + \frac{1}{2}\left(\frac{1 + (f_{2,X})^2}{(f_{2,\psi})^2}\right)_{\psi=0-} - \frac{\rho}{2}\left(\frac{1 + (f_{1,X})^2}{(f_{1,\psi})^2}\right)_{\psi=0+} = C_2, \quad \text{on } \psi = 0. \tag{2.21}$$

Here, in (2.21), $\rho = \rho_1/\rho_2$; $0 < \rho < 1$. In (2.20) and (2.21), C_1 and C_2 are two fixed constants.

In terms of the streamline functions, the uniform horizontal flow defined in (2.10) can be rewritten as

$$\begin{aligned}&f_i(X, \psi) = \psi/U_0, \quad \text{for } i = 1, 2,\\ &S_1(X) = h, \quad S_2(X) = 0.\end{aligned} \tag{2.22}$$

Assume that the exact internal waves sought are small perturbations of the above uniform horizontal flow. For this reason, we consider the following change of corrdinates and variables

$$\begin{aligned}&x = X/h, \quad \xi = \psi/U_0 h,\\ &f_i = h(\xi + \epsilon w_i\,(x, \xi)), \quad \text{for } i = 1, 2,\\ &S_1 = h(1 + \epsilon\eta_1(x)), \quad S_2 = \epsilon\eta_2(x).\end{aligned} \tag{2.23}$$

Now the top and bottom layers of fluid occupy the horizontal strips

$$\Omega_1 = \{\, (x, \xi) \,:\, 0 < \xi < 1 \,\} \text{ and } \Omega_2 = \{\, (x, \xi) \,:\, -\infty < \xi < 0 \,\}.$$

And equations (2.15) – (2.21) are transformed into

$$(1+\epsilon^2 w_{i,x}{}^2)\, w_{i,\xi\xi} - 2\epsilon w_{i,x}\,(1+\epsilon w_{i,\xi})\, w_{i,x\xi} + (1+\epsilon w_{i,\xi})^2\, w_{i,xx} = 0, \quad \text{in } \Omega_i\,, \quad \text{for } i = 1, 2, \tag{2.24}$$

$$w_1 = \eta_1, \quad \text{on } \xi = 1, \tag{2.25}$$

$$w_1 = w_2 = \eta_2, \quad \text{on } \xi = 0, \tag{2.26}$$

$$w_{2,x} \to 0, \quad \text{as } \xi \to -\infty, \tag{2.27}$$

$$\eta_1 + \frac{\gamma}{2}\left[\frac{-2w_{1,\xi} + \epsilon\, w_{1,x}{}^2 - \epsilon\, w_{1,\xi}{}^2}{(1+\epsilon\, w_{1,\xi})^2}\right] = 0, \quad \text{on } \xi = 1, \tag{2.28}$$

$$(1-\rho)\eta_2 + \frac{\gamma}{2}\left[\frac{-2w_{2,\xi} + \epsilon\, w_{2,x}{}^2 - \epsilon\, w_{2,\xi}{}^2}{(1+\epsilon\, w_{2,\xi})^2}\right]_{\xi=0-} - \frac{\rho\gamma}{2}\left[\frac{-2w_{1,\xi} + \epsilon\, w_{1,x}{}^2 - \epsilon\, w_{1,\xi}{}^2}{(1+\epsilon\, w_{1,\xi})^2}\right]_{\xi=0+} = 0, \quad \text{on } \xi = 0. \tag{2.29}$$

Here γ is the Froude number $U_0{}^2/gh$. In particular, when $\epsilon = 0$, we obtain the linearized system

$$w_{i,xx} + w_{i,\xi\xi} = 0, \quad \text{in } \Omega_i\,, \text{ for } i = 1, 2, \tag{2.30}$$

$$w_1 = \eta_1, \quad \text{on } \xi = 1, \tag{2.31}$$

$$w_1 = w_2 = \eta_2, \quad \text{on } \xi = 0, \tag{2.32}$$

$$w_{2,x} \to 0, \quad \text{as } \xi \to -\infty, \tag{2.33}$$

$$\eta_1 - \gamma\, w_{1,\xi} = 0, \quad \text{on } \xi = 1, \tag{2.34}$$

$$(1-\rho)\,\eta_2 - \gamma\,[w_{2,\xi}]_{\xi=0-} + \rho\gamma\,[w_{1,\xi}]_{\xi=0+} = 0, \quad \text{on } \xi = 0. \tag{2.35}$$

3. Linearized internal gravity waves

In this section, we solve linear system (2.30) – (2.35) for linearized internal waves with η_i of the form

$$\eta_i = \sum_{m=-\infty}^{+\infty} \eta_{im}\, e^{imkx}, \quad \text{for } i = 1, 2. \tag{3.1}$$

We will assume that the linearized internal waves sought have surface elevations $\eta_i(x)$ being even functions of x; i. e., in (3.1), $\eta_{i\,m} = \eta_{i\,(-m)}$, for all integer m. The wave number k is yet to be determined.

Assume that, for $i = 1, 2$, w_i can be expressed as

$$w_i(x,\, \xi) = \sum_{m=-\infty}^{+\infty} w_{i\,m}(\xi)\, e^{ikmx}. \tag{3.2}$$

For each integer m, (2.30) – (2.33) lead to

$$
\begin{aligned}
&w_{1\,m}'' = m^2k^2 w_{1\,m}, \quad 0 < \xi < 1, \\
&w_{1\,m} = \eta_{1\,m}, \quad \text{on } \xi = 1, \\
&w_{1,\,m} = \eta_{2\,m}, \quad \text{on } \xi = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&w_{2\,m}'' = m^2k^2 w_{2\,m}, \quad -\infty < \xi < 0, \\
&w_{2\,m} = \eta_{2\,m}, \quad \text{on } \xi = 0, \\
&w_{2\,m} \to 0, \quad \text{as } \xi \to -\infty.
\end{aligned}
$$

Here $'$ represents derivative with respect to ξ. We obtain

$$w_{1\,m} = \eta_{2\,m} \cosh mk\xi + \frac{\eta_{1\,m} - \eta_{2\,m} \cosh mk}{\sinh mk} \sinh mk\xi, \tag{3.3}$$

$$w_{2\,m} = \eta_{2\,m}\, e^{|m|\,k\xi}, \tag{3.4}$$

for $m \neq 0$, and

$$
\begin{aligned}
&w_{1\,0} = \eta_{2\,0} + (\eta_{1\,0} - \eta_{2\,0})\,\xi, \\
&w_{2\,0} = \eta_{2\,0}.
\end{aligned} \tag{3.5}
$$

Note that we have $w_{i\,m} = w_{i\,(-m)}$ for each integer m; that is to say, $w_i(x,\,\xi)$ are also even functions of the x variable. Substitute (3.3),(3.4) into (2.34) and (2.35). We obtain linear system

$$
\begin{aligned}
&(1 - \gamma mk \coth mk)\, \eta_{1\,m} - \gamma mk \sinh mk\, (1 - \coth^2 mk)\, \eta_{2\,m} = 0, \\
&\frac{\rho\gamma mk}{\sinh mk}\, \eta_{1\,m} - [\,(1-\rho) - \gamma mk - \rho\gamma mk \coth mk\,]\, \eta_{2\,m} = 0,
\end{aligned} \tag{3.6}
$$

for each integer $m > 0$. When $m = 0$, we have

$$
\begin{aligned}
&(1 - \gamma)\, \eta_{1\,0} + \gamma\, \eta_{2\,0} = 0, \\
&\rho\gamma\, \eta_{1\,0} - (1 - \rho - \rho\gamma)\, \eta_{2\,0} = 0.
\end{aligned} \tag{3.7}
$$

Because of the symmetries in the x direction, it suffices to consider those linear systems corresponding to $m \geq 0$.

Note that linear system (3.7) has determinant d_0 given by

$$d_0 = (1 - \rho) - \gamma.$$

From now on, we will assume that the internal gravity waves sought have Froude number

$$\gamma \neq (1 - \rho). \tag{3.8}$$

As a result, linear system (3.7) has solution $(\eta_{1\,0},\, \eta_{2\,0}) = (0,0)$. Next, for integer $m > 0$, linear system (3.6) has determinant d_m given by

$$d_m = (\,1 - \gamma\, mk\,)\,(\,1 - \rho\,(1 + \gamma\, mk) - \gamma\, mk \coth mk\,). \tag{3.9}$$

In order to determine the coefficients $\eta_{1\,m}$, $\eta_{2\,m}$, it is customary in classical literaure to have the wave number k chosen so that the determinants $d_1 = 0$ and $d_m \neq 1$, for all integer $m > 0$. This causes $\eta_{1m} = \eta_{2m} = 0$ for all integer $\mid m \mid > 1$. The coefficients $\eta_{1\,\pm 1}$ and $\eta_{2,\,\pm 1}$ are then determined by imposing an appropriate normalization. For this reason, besides system (2.15) – (2.21), we will impose an additional normalization

$$\int_0^{U_0 h} \int_{-L}^{L} e^{-i\pi \frac{X}{L}}\, f_1(X,\, \psi)\, dX\, d\psi = \epsilon C^*. \tag{3.10}$$

Here $2L$ is the wavelength of the internal gravity wave considered; C^* is some fixed constant. This is equivalent to, in terms of the (x, ξ) coordinates,

$$\frac{2\pi U_0 h^2}{k} \int_0^1 w_{11}(\xi)\, d\xi = \epsilon C^*. \tag{3.11}$$

For convenience's sake, we choose

$$C^* = \frac{2\pi U_0 h^2}{k^2}\, \frac{(\cosh k - 1)}{\sinh k} \left[1 + \frac{\gamma k \cosh k - \sinh k}{\gamma k} \right].$$

Consequently, we have

$$\eta_{11} = 1, \quad \eta_{21} = \frac{\gamma k \cosh k - \sinh k}{\gamma k}.$$

Now let $b(\tau)$ be the function

$$b(\tau) = (\,1 - \gamma\, \tau\,)\,(\,1 - \rho\,(1 + \gamma\, \tau) - \gamma\, \tau\, \coth \tau\,). \tag{3.12}$$

It is easy to see that $b(\tau) = 0$ if and only if τ satisfies $\gamma = 1/\tau$, or

$$\begin{aligned} \gamma &= \frac{1 - \rho}{\rho\, \tau + \tau\, \coth \tau} \\ &= \frac{(1 - \rho)\, \tanh \tau}{\tau\,(1 + \rho\, \tanh \tau)}. \end{aligned} \tag{3.13}$$

It is not hard to find out that the function

$$\frac{(1 - \rho)\, \tanh \tau}{\tau\,(1 + \rho\, \tanh \tau)}, \quad \tau > 0,$$

is decreasing in τ when $0 < \rho < 1$, with

$$\frac{(1 - \rho)\, \tanh \tau}{\tau\,(1 + \rho\, \tanh \tau)} \to (1 - \rho), \quad \text{as } \tau \to 0+\,.$$

When $\gamma > 1 - \rho$, (3.13) can not hold. As a result, equation $b(\tau) = 0$ has only one root; namely, $k_s = 1/\gamma$. Thus, for a fixed density ratio $0 < \rho < 1$, there is only one linearized internal gravity wave for each Froude number $\gamma > 1 - \rho$. This linearized internal gravity wave has wave number $k = k_s$, and is given by

$$\begin{aligned} w_1(x,\, \xi;\, k) &= 2\Big(\frac{\gamma k \cosh k - \sinh k}{\gamma k} \cosh k\xi \\ &\quad + \frac{\gamma k - \gamma k \cosh^2 k - \sinh k \cosh k}{\gamma k \sinh k} \sinh k\xi \Big) \cos kx, \\ w_2(x,\, \xi;\, k) &= 2 \frac{\gamma k \cosh k - \sinh k}{\gamma k} e^{k\xi} \cos kx, \\ \eta_1(x;\, k) &= 2 \cos kx, \\ \eta_2(x;\, k) &= 2 \frac{(\gamma k \coth k - 1) \sinh^2 k}{\gamma k} \cos kx. \end{aligned} \tag{3.14}$$

When $0 < \gamma < 1 - \rho$, the situation is more complicated. For $0 < \gamma < 1 - \rho$, there exists a unique $k_i > 0$ such that

$$\gamma = \frac{(1-\rho) \tanh k_i}{k_i\,(1 + \rho \tanh k_i)}. \tag{3.15}$$

Then eqaution $b(\tau) = 0$ has two positive roots; they are k_i and k_s, where $k_s = 1/\gamma$. Both k_i and k_s are functions of the Froude number γ and the density ratio ρ.

Note that, when $0 < \gamma < 1 - \rho$, we have

$$k_s = \frac{\rho\, k_i + k_i \coth k_i}{1 - \rho}. \tag{3.16}$$

It is not hard to see that $\frac{dk_s}{dk_i} > 0$ for positive k_i. Moreover, the ratio k_s/k_i tends to $+\infty$ as k_i goes to 0. In particular, when the Froude number γ approaches $1 - \rho$ from below, the wave number k_i decreases to 0 and the ratio the ratio k_s/k_i increases without bound. For each integer $n \geq 1$, there exists a Froude number γ_n such that

$$\frac{k_s}{k_i} = n. \tag{3.17}$$

Now (3.16) and (3.17) lead to

$$\tanh k_i = \frac{1}{n(1-\rho) - \rho}. \tag{3.18}$$

From (3.18), we see that a sufficient and necessary condition for $k_s/k_i = n$ to occur is to have

$$\frac{1}{n(1-\rho) - \rho} < 1.$$

That is to say,

$$\frac{1+\rho}{1-\rho} < n. \tag{3.19}$$

Now, for each integer n which satisfies (3.19), let

$$k_n^* = \tanh^{-1}\left(\frac{1}{n(1-\rho)-\rho}\right), \tag{3.20}$$

and let

$$\gamma_n^* = \frac{(1-\rho)\tanh k_n^*}{k_n^*\,(1+\rho\tanh k_n^*)} \tag{3.21}$$

with k_n^* in place of k_i in (3.15). Suppose that the Froude number γ of the flow is less than $1-\rho$ and is different from all of the γ_n^*, $n = 1, 2, 3, \ldots$. Then the function $b(\tau)$ mentioned above has two positive zeros k_s and k_i, with ratio k_s/k_i nonintegeral. This gives two different types of linearized internal gravity waves. One of them has wave number k_s. When γ is small, this linearized internal gravity wave has surface elevation on the free surface much larger than that of the interface separating the two fluids. The other linearized internal gravity wave has wave number k_i, whose surface elevation on the free surface is much smaller than that of the interface when γ is close to 0. See Art. 231 in [2]. These two nonresonant linearized internal gravity waves all have the form (3.14). If the Froude number γ is equal to one of the γ_n^*, $n = 1, 2, 3, \ldots$, then the two positive zeros k_s, k_i of the function $b(\xi)$ are given by

$$k_s = nk_n^*, \quad k_i = k_n^*.$$

In this case, the general solutions of linear system (2.30) – (2.35) are the resonant of two different families of linear internal gravity waves.

4. An auxiliary linear problem

In the following section, we will solve the nonlinear system (2.24) – (2.29) by iterations. In order to do that, we need to solve the following linear problems.

$$(\mathbf{P_1}) \qquad w_{1,xx} + w_{1,\xi\xi} = f_1, \quad \text{in } \Omega_1, \tag{4.1}$$

$$w_1 = g_1, \quad \text{on } \xi = 1, \tag{4.2}$$

$$w_1 = g_2, \quad \text{on } \xi = 0. \tag{4.3}$$

$$(\mathbf{P_2}) \qquad w_{2,xx} + w_{2,\xi\xi} = f_2, \quad \text{in } \Omega_2, \tag{4.4}$$

$$w_2 = g_2, \quad \text{on } \xi = 0, \tag{4.5}$$

$$w_{2,x} \to 0, \quad \text{as } \xi \to -\infty. \tag{4.6}$$

Here $f_i : \Omega_i \to \mathbf{R}$, and $g_j : \mathbf{R} \to \mathbf{R}$ are given functions which are periodic in x of period $2\pi/k$.

Before we proceed to solve (4.1) – (4.6), we now define the function spaces used from here on. We will call any terminating Fourier series of the form

$$f(x,\,\xi)=\sum_m f_m(\xi)\,e^{imkx},\quad (x,\,\xi)\in\Omega$$

a trigonometric polynomial. Given a smooth, bounded trigonometric polynomial $f:\Omega_i\to\mathbf{R}$, we take

$$\|\,f\,\|_{\Omega,0}=\sum_{m=-\infty}^{+\infty}\sup_{\xi}|\,f_m(\xi)\,|=\sum_{m=-\infty}^{+\infty}\|\,f_m\,\|_{\infty}\,. \tag{4.7}$$

For $n\geq 1$, let

$$\|\,f\,\|_{\Omega,n}=\|\,f_0\,\|_{\infty}+\sum_{j=0}^{n}\|\,D_x^{n-j}\,D_\xi^j\,f\,\|_{\Omega_i,0}\,. \tag{4.8}$$

We denote by $\Lambda^n(\Omega)$ the completion of these trigonometric polynomials in the norm $\|\cdot\|_{\Omega,n}$. If $g:\mathbf{R}\to\mathbf{R}$, then we say g is in $\Lambda^n(\mathbf{R})$ if the extension $\tilde{g}(x,\,\xi)=g(x)$ is in by $\Lambda^n(\Omega)$. The norm of g in $\Lambda^n(\mathbf{R})$ is denoted as $\|\,g\,\|_n$. A subscript e or o is used to indicate subspaces consisting of functions that are even or odd in x. For example, $\Lambda_e^2(\Omega_1)$ is the subspace of $\Lambda^2(\Omega_1)$ consisting of functions which are even in the x direction. Several lemmas of the function spaces defined above will be listed below. See [6] for their detailed proof.

Lemma 1. *Let u, v are smooth trigonometric polynomials defined on Ω; let integer $n\geq 0$. Then*

$$\|\,D_x^n(uv)\,\|_{\Omega,0}\leq\sum_{j=0}^{n}C_j^n\;\|\,D_x^{n-j}u\,\|_{\Omega,0}\,\|\,D_x^jv\,\|_{\Omega,0}\,.$$

Here $C_j^n=n!/j!\,(n-j)!$.

From Lemma 1, we can prove

Lemma 2. *Let $u,\,v\in\Lambda^n(\Omega)$, where integer $n\geq 0$. Then $uv\in\Lambda^n(\Omega)$, with*

$$\|\,uv\,\|_{\Omega,n}\leq C\sum_{j=0}^{n}C_j^n\;\|\,u\,\|_{\Omega,n-j}\,\|\,v\,\|_{\Omega,j}\,.$$

Here C is some constant >0.

The following lemma says that the imbedding from $\Lambda^{n+1}(\Omega)$ to $\Lambda^n(\Omega)$ is continuous.

Lemma 3. *Let n be an integer ≥ 0. If $u\in\Lambda^{n+1}(\Omega)$, then $u\in\Lambda^n(\Omega)$ and*

$$\|\,u\,\|_{\Omega,n}\leq C\;\|\,u\,\|_{\Omega,n+1}$$

for some constant $C>1$. Furthermore, $\mathcal{D}^1u$ is in $\Lambda^n(\Omega)$, with

$$\|\,\mathcal{D}^1u\,\|_{\Omega,n}\leq C'\;\|\,u\,\|_{\Omega,n+1}$$

for some constant $C' > 1$. Here $\mathcal{D}^1 u$ represents a first order derivative of u either in the x direction or in the ξ direction..

The following lemma concerns the compositions of funcions in $\Lambda^n(\Omega)$ with analytic functions.

Lemma 4. *Let B be a ball in $\mathbf{R}^p$, centered at the point $\tilde{g} = (\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_p)$. Let $f : B \to \mathbf{R}$ be analytic. Let $g = (g_1, g_2, \cdots, g_p) \in [\Lambda^n(\Omega)]^p$, where integer $n \geq 0$. If $\| g - \tilde{g} \|^2_{\Omega, n} = \sum_{j=1}^p \| g_j - \tilde{g}_j \|^2_{\Omega, n}$ is small enough, then the composition $f(g(x, \xi))$ is in $\Lambda^n(\Omega)$.*

Finally, if $f : \mathbf{R} \to \mathbf{R}$ is a trigonometric polynomial, we define

$$\| f \|_{-1} = \sum_{m \neq 0} \left| \frac{f_m}{m} \right|, \tag{4.9}$$

and we denote by $\Lambda^{-1}(\mathbf{R})$ the completion of the trigonometric polynomials in this semi-norm. In what follows, we will have $\Omega = \Omega_1$, or Ω_2. Then the supremum in (4.7) is taken over $0 \leq \xi \leq 1$ for $i = 1$ and $-\infty < \xi \leq 0$ for $i = 2$. In (4.8), D_x and D_ξ represent derivatives in the x and ξ directions respectively.

Theorem 5. *Let $f_1 \in \Lambda^0_e(\Omega_1)$ and let $g_1, g_2 \in \Lambda^2_e(\mathbf{R})$. There exists a unique solution w_1 of the linear problem (P_1) in $\Lambda^2_e(\Omega_1)$, with estimate*

$$\| w_1 \|_{\Omega_1, 2} \leq C_1 (\| g_1 \|_2 + \| g_2 \|_2 + \| f_1 \|_{\Omega_1, 0}), \tag{4.10}$$

for some constant $C_1 > 0$.

Proof. Expand w_1, f_1, g_j formally in Fourier series

$$w_1(x, \xi) = \sum_{m=-\infty}^{+\infty} w_{1m}(\xi)\, e^{imkx},$$

$$f_1(x, \xi) = \sum_{m=-\infty}^{+\infty} f_{1m}(\xi)\, e^{imkx},$$

$$g_j = \sum_{m=-\infty}^{+\infty} g_{jm}(\xi)\, e^{imkx}, \quad \text{for } j = 1, 2.$$

For each integer m, problem (P_1) leads to

$$w''_{im} = m^2 k^2 w_{im} + f_{im}, \quad 0 < \xi < 1,$$
$$w_{1m}(1) = g_{1m}, \quad w_{1m}(0) = g_{2m}.$$

Here $'$ represents derivative with respect to ξ. Thus we obtain, for integer $m \neq 0$

$$w_{1m}(\xi) = g_{1m} \frac{\sinh mk\xi}{\sinh mk} + g_{2m} \frac{\sinh mk(1-\xi)}{\sinh mk} + \frac{\sinh mk\xi}{mk} (F_m(\xi) - F_m(1)), \tag{4.11}$$

where

$$F_m(\xi) = \int_0^\xi \frac{\sinh mk(\xi - t)}{\sinh mk\xi} f_{1m}(t)\, dt.$$

When $m = 0$, we have

$$w_{10}(\xi) = g_{10}\xi + g_{20} + \xi\,(\, F_0(\xi) - F_0(1)\,), \tag{4.12}$$

where

$$F_0(\xi) = \int_0^\xi \int_0^t \frac{f_{10}(\tau)}{\xi}\, d\tau\, dt.$$

From (4.11), it is easy to see that we have

$$w_{1m} = w_{1(-m)}, \quad \text{for all } m \neq 0.$$

Thus w_1 is even in x. It remains to estimate $\| w_1 \|_{\Omega_1,2}$. Note that, for $| m |$ large enough,

$$\begin{aligned}
| \sinh mk\xi\,(F_m(\xi) - F_m(1)) | &= | \int_0^\xi \frac{\sinh mk\tau \sinh mk(\xi - 1)}{\sinh mk} f_{1m}(\tau)\, d\tau \,| \\
&\quad + | \sinh mk\xi \int_\xi^1 \frac{\sinh mk(1-\tau)}{\sinh mk} f_{1m}(\tau)\, d\tau \,| \\
&\leq \| f_{1m} \|_\infty | \int_0^\xi \frac{\sinh mk\tau \sinh mk(\xi - 1)}{\sinh mk}\, d\tau \,| \\
&\quad + \| f_{1m} \|_\infty | \sinh mk\xi \int_\xi^1 \frac{\sinh mk(1-\tau)}{\sinh mk}\, d\tau \,| \\
&\leq \| f_{1m} \|_\infty | \frac{\sinh mk(\xi - 1)}{mk \sinh mk} (\cosh mk\xi - 1) \,| \\
&\quad + \| f_{1m} \|_\infty | \frac{\sinh mk\xi}{mk \sinh mk} (\cosh mk(1-\xi) - 1) \,| \\
&\leq C \frac{\| f_{1m} \|_\infty}{| mk |},
\end{aligned} \tag{4.13}$$

for some constant $C > 0$ independent of m. Similarly, we can show that

$$| \cosh mk\xi\,(F_m(\xi) - F_m(1)) \leq C \frac{\| f_{1m} \|_\infty}{| mk |}. \tag{4.14}$$

Next, observe that

$$\begin{aligned}
| F_m'(\xi) | &= | -mk \int_0^\xi \frac{\sinh mk\tau}{\sinh^2 mk\xi} f_{1m}(\tau)\, d\tau \,| \\
&\leq \| f_{1m} \|_\infty | -mk \int_0^\xi \frac{\sinh mk\tau}{\sinh^2 mk\xi}\, d\tau \,| \\
&\leq C \frac{\| f_{1m} \|_\infty}{| \sinh mk\xi |},
\end{aligned} \tag{4.15}$$

for some constant $C > 0$ independent of m.

As a result, (4.13) leads to

$$\| w_{1m} \|_\infty \le C\,(\,| g_{1m} | + | g_{2m} | + \frac{1}{m^2k^2} \| f_{1m} \|_\infty\,), \tag{4.16}$$

whereas (4.14) and (4.15) give us

$$\| w_{1m}' \|_\infty \le C\,(\,| mk || g_{1m} | + | mk || g_{2m} | + \frac{1}{| mk |} \| f_{1m} \|_\infty\,), \tag{4.17}$$

for $| m |$ sufficient large.

Now, in order to finish the proof, we estimate $\| w_{1m}'' \|_\infty$. What we need now is an estimation of the term $| \; F_m''(\xi) \; |$. Note that

$$\begin{aligned} F_m''(\xi) = &-\frac{mk}{\sinh mk\xi} f_{1m}(\xi) \\ &+ 2m^2k^2 \int_0^\xi \frac{\cosh mk\xi}{\sinh^3 mk\xi} \sinh mk\tau \, f_{1m}(\tau)\, d\tau. \end{aligned}$$

Hence

$$\begin{aligned} | \, F_m''(\xi) \, | \le &\frac{| mk |}{| \sinh mk\xi |} \| f_{1m} \|_\infty \\ &+ 2 \, | mk | \, \frac{| \cosh mk\xi |}{| \sinh^3 mk\xi |} \| f_{1m} \|_\infty \left| \int_0^\xi mk \sinh mk\tau \, d\tau \right| \\ \le &\, C \frac{| mk |}{| \sinh mk\xi |} \| f_{1m} \|_\infty, \end{aligned}$$

for some constant $C > 0$ independent of m. Consequently, we obtain

$$\| w_{1m}'' \|_\infty \le C\,(\,| mk |^2 \; | g_{1m} | + | mk |^2 \; | g_{2m} | + \| f_{1m} \|_\infty\,). \tag{4.18}$$

Now, by (4.16) – (4.18),

$$\begin{aligned} \| w_1 \|_{\Omega_1,2} &= \| w_{10} \|_\infty + \| D_x^2 w_1 \|_{\Omega_1,0} + \| D_x D_\xi w_1 \|_{\Omega_1,0} + \| D_\xi^2 w_1 \|_{\Omega_1,0} \\ &= \| w_{10} \|_\infty + \sum_{m=-\infty}^{+\infty} [\, | mk |^2 \; \| w_{1m} \|_\infty + | mk | \; \| w_{1m}' \|_\infty + \| w_{1m}'' \|_\infty \,] \\ &\le C\,(\| g_1 \|_0 + \| g_2 \|_0 + \| f_1 \|_{\Omega_1,0}). \end{aligned} \tag{4.19}$$

The uniqueness of the solution w_1 in $\Lambda^2(\Omega_1)$ is immediate from above inequality. This completes the proof of Theorem 5.

Now we turn to linear problem (P_2). We have

Theorem 6. *Let $f_2 \in \Lambda_e^0(\Omega_2)$, $g_2 \in \Lambda_e^2(\mathbf{R})$. There exists a unique solution of the linear problem (P_2) in $\Lambda_e^2(\Omega_2)$, with estimate*

$$\| w_2 \|_{\Omega_2,2} \le C_2\,(\| g_2 \|_0 + \| f_2 \|_{\Omega_2,0}), \tag{4.20}$$

for some constant $C_2 > 0$.

The proof of Theorem 6 is similar to that of Theorem 5 and is thus obmitted here.

5. Existence of exact internal gravity waves

In this section, we assume that the periodic internal gravity waves considered are not in resonance with other internal gravity waves. For example, we may assume that the Froude number γ is larger than $1-\rho$ where ρ is the density ratio of the two layers. Then there is only one kind of periodic linear internal gravity waves and no resonnance occurs. Or, when $0 < \gamma < 1-\rho$, we assume that the Froude number γ is different from any of the critical values γ_n^*, $n = 1, 2, \cdots$. Then there exists a wave number $k > 0$ which is a zero of the function

$$b(\tau) = (1-\gamma\tau)(1-\rho(1+\gamma\tau)-\gamma\tau\coth\tau),$$

with $b(mk) \neq 0$, for all integer $m > 1$. Although now we have two different kind of linear internal gravity waves, their resonance is avoided. See Section 3 for the detailed discussion.

The existense of the exact internal gravity waves is carried out in the following steps. Frist we rewrite systems (2.24) – (2.27) as

$$w_{i,xx} + w_{i,\xi\xi} = \epsilon N_i, \quad \text{in } \Omega_i, \text{ for } i = 1,2, \tag{5.1}$$

$$w_1 = \eta_1, \quad \text{on } \xi = 1, \tag{5.2}$$

$$w_1 = w_2 = \eta_2, \quad \text{on } \xi = 0, \tag{5.3}$$

$$w_{2,x} \to 0, \quad \text{as } \xi \to -\infty, \tag{5.4}$$

where

$$\begin{aligned} N_i &= N_i(\mathcal{D}^1 w_i, \mathcal{D}^2 w_i, \epsilon) \\ &= 2w_{i,x}(1+\epsilon w_{i,\xi})w_{i,x\xi} - 2w_{i,\xi}w_{i,xx} - \epsilon {w_{i,x}}^2 w_{i,\xi\xi} - \epsilon {w_{i,\xi}}^2 w_{i,xx}, \quad \text{for } i = 1,2 \end{aligned} \tag{5.5}$$

Here $\mathcal{D}^1 w_i$ and $\mathcal{D}^2 w_i$ represent the first and second derivatives of w_i respectively. Note that, for each choice of (η_1, η_2) in $[\Lambda_e^2(\mathbf{R})]^2$, (5.1) – (5.4) give us two nonlinear elliptic boundary value problems for w_1 and w_2 over domains Ω_1 and Ω_2 respectively. By solving these nonlinear boundary value problems, we can regard both w_1 and w_2 as nonlinear functions of η_1, η_2, and ϵ. The solvability of these nonlinear elliptic problems can be achieved by using Theorem 5 and 6 in the previous section and the contraction mapping principle. This enables us to treat w_1 and w_2 as nonlinear functions of η_1, η_2 and ϵ. Next we rewrite (2.28), (2,29) as

$$\eta_1 + F(\eta_1, \eta_2, \epsilon) = 0, \quad \text{on } \xi = 1, \tag{5.6}$$

$$(1-\rho)\eta_2 + G(\eta_1, \eta_2, \epsilon) = 0, \quad \text{on } \xi = 0. \tag{5.7}$$

Here

$$F(\eta_1, \eta_2, \epsilon) = \frac{\gamma}{2}\left[\frac{-2w_{1,\xi} + \epsilon\, w_{1,x}{}^2 - \epsilon\, w_{1,\xi}{}^2}{(1+\epsilon\, w_{1,\xi})^2}\right], \tag{5.8}$$

$$G(\eta_1, \eta_2, \epsilon) = \frac{\gamma}{2}\left[\frac{-2w_{2,\xi} + \epsilon\, w_{2,x}{}^2 - \epsilon\, w_{2,\xi}{}^2}{(1+\epsilon\, w_{2,\xi})^2}\right]_{\xi=0-} - \frac{\rho\gamma}{2}\left[\frac{-2w_{1,\xi} + \epsilon\, w_{1,x}{}^2 - \epsilon\, w_{1,\xi}{}^2}{(1+\epsilon\, w_{1,\xi})^2}\right]_{\xi=0+} \tag{5.9}$$

(5.6) and (5.7) will be solved together with normalization (3.11) as a system of nonlinear functional equations for η_1 and η_2 by again using the contraction mapping principle. In the normalization (3.11), the Fourier mode w_{11} of the function w_1 is treated as a nonlinear function of η_1, η_2, and ϵ.

Now let $(\eta_1, \eta_2) \in [\Lambda_e^2(\mathbf{R})]^2$ be arbitrary. Consider the following linear elliptic boundary value problems

$$\begin{aligned} &w_{1,xx} + w_{1,\xi\xi} = 0, \quad 0 < \xi < 1, \\ &w_1 = \eta_1, \quad \text{on } \xi = 1, \\ &w_1 = \eta_2, \quad \text{on } \xi = 0, \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} &w_{2,xx} + w_{2,\xi\xi} = 0, \quad -\infty < \xi < 0, \\ &w_2 = \eta_2, \quad \text{on } \xi = 0, \\ &w_{2,x} \to 0, \quad \text{as } \xi \to -\infty. \end{aligned} \tag{5.11}$$

By Theorem 5 and 6, systems (5.10) and (5.11) have solutions $w_1^{(0)}(x, \xi)$ and $w_2^{(0)}(x, \xi)$ in $\Lambda_e^2(\Omega_1)$ and $\Lambda_e^2(\Omega_2)$ respectively, with

$$\| w_1^{(0)} \|_{\Omega_1, 2} + \| w_2^{(0)} \|_{\Omega_2, 2} \leq \tilde{C}(\| \eta_1 \|_1 + \| \eta_2 \|_2). \tag{5.12}$$

Here the constant $\tilde{C} > 0$ is the sum of the constants C_1 and C_2 in Theorem 5 and 6. Now, for each integer $n \geq 0$, let $w_1^{(n+1)}$ and $w_2^{(n+1)}$ be the solution of the linear problems

$$\begin{aligned} &w_{1,xx}^{(n+1)} + w_{1,\xi\xi}^{(n+1)} = \epsilon N_1^{(n)}, \quad 0 < \xi < 1, \\ &w_1^{(n+1)} = \eta_1, \quad \text{on } \xi = 1, \\ &w_1^{(n+1)} = \eta_2, \quad \text{on } \xi = 0, \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} &w_{2,xx}^{(n+1)} + w_{2,\xi\xi}^{(n+1)} = \epsilon N_2^{(n)}, \quad -\infty < \xi < 0, \\ &w_2^{(n+1)} = \eta_2, \quad \text{on } \xi = 0, \\ &w_{2,x}^{(n+1)} \to 0, \quad \text{as } \xi \to -\infty, \end{aligned} \tag{5.14}$$

where

$$N_i^{(n)} = N_i(\mathcal{D}^1 w_i^{(n)},\ \mathcal{D}^2 w_i^{(n)},\ \epsilon), \quad \text{for } i = 1, 2. \tag{5.15}$$

By Lemma 2, Lemma 3, and the the definition of the nonlinear terms N_i, it is easy to see that, since $w_1^{(n)} \in \Lambda_e^2(\Omega_1)$ and $w_2^{(n)} \in \Lambda_e^2(\Omega_2)$, we have $N_i^{(n)} \in \Lambda_e^0(\Omega_i)$ for $i = 1, 2$. Thus, by Theorem 5 and 6, $w_i^{(n+1)} \in \Lambda(\Omega_i)$ for $i = 1, 2$; and we obtain the sequence $\{(w_1^{(n)},\ w_2^{(n)})\}$ in the Banach space $\Lambda_e^2(\Omega_1) \times \Lambda_e^2(\Omega_2)$. To prove the convergence of above sequence, take

$$R = 2\tilde{C}\,(\,\|\,\eta_1\,\|_2 + \|\,\eta_2\,\|_2\,).$$

Let B_R be the ball in $\Lambda_e^2(\Omega_1) \times \Lambda_e^2(\Omega_2)$

$$B_R = \{\,(w_1,\ w_2) \in \Lambda_e^2(\Omega_1) \times \Lambda_e^2(\Omega_2)\ : \|\,w_1\,\|_{\Omega_1,2} + \|\,w_2\,\|_{\Omega_2,2} \le\ R\,\}.$$

Note that, by solving (5.13) and (5.14) with $n = 0$, we obtain $w_1^{(1)}$ and $w_2^{(1)}$ with

$$\|\,w_1^{(1)}\,\|_{\Omega_1,2} + \|\,w_2^{(1)}\,\|_{\Omega_2,2} \le\ \tilde{C}(\,\|\,\eta_1\,\|_2 + \|\,\eta_2\,\|_2 + \epsilon\,\|\,N_1^{(0)}\,\|_{\Omega_1,0} + \epsilon\,\|\,N_2^{(0)}\,\|_{\Omega_2,0}\,).$$

Clearly, $(w_1^{(1)},\ w_2^{(1)})$ is in B_R if ϵ is sufficiently close to 0. Furthermore, if $(w_1^{(n)},\ w_2^{(n)})$ and $(w_1^{(n-1)},\ w_2^{(n-1)})$ are in B_R, then we have

$$\begin{aligned}
&\|\,N_i^{(n)} - N_i^{(n-1)}\,\|_{\Omega_i,0}\\
&= \|\,N_i(\mathcal{D}^1 w_i^{(n)},\ \mathcal{D}^2 w_i^{(n)},\ \epsilon) - N_i(\mathcal{D}^1 w_i^{(n-1)},\ \mathcal{D}^2 w_i^{(n-1)},\ \epsilon)\,\|_{\Omega_i,0}\\
&= \|\int_0^1 \frac{d}{d\tau} N_i(\,\mathcal{D}^1 w_i^{(n-1)} + \tau(\,\mathcal{D}^1 w_i^{(n)} - \mathcal{D}^1 w_i^{(n-1)}\,),\\
&\qquad \mathcal{D}^2 w_i^{(n-1)} + \tau(\,\mathcal{D}^2 w_i^{(n)} - \mathcal{D}^2 w_i^{(n-1)}\,),\ \epsilon\,)\,d\tau\,\|_{\Omega_i,0}\\
&\le \max_{\|\bar{w}\|_{\Omega_1,2} \le R} \|\,N_{i,\mathcal{D}^1\bar{w}}(\,\mathcal{D}^1\bar{w},\ \mathcal{D}^2\bar{w},\ \epsilon)\,\|_{\Omega_i,0}\,\|\,\mathcal{D}^1 w_i^{(n)} - \mathcal{D}^1 w_i^{(n-1)}\,\|_{\Omega_i,0}\\
&\quad + \max_{\|\bar{w}\|_{\Omega_1,2} \le R} \|\,N_{i,\mathcal{D}^2\bar{w}}(\,\mathcal{D}^1\bar{w},\ \mathcal{D}^2\bar{w},\ \epsilon)\,\|_{\Omega_i,0}\,\|\,\mathcal{D}^2 w_i^{(n)} - \mathcal{D}^2 w_i^{(n-1)}\,\|_{\Omega_i,0}\\
&\le C_i(R)\ \|\,w_i^{(n)} - w_i^{(n-1)}\,\|_{\Omega_i,2}
\end{aligned} \tag{5.16}$$

for each integer $i = 1, 2$. In above estimation, we have made use of the fact that N_1, N_2 are analytic in their arguments. The constants $C_i(R)$ are continuous functions of R, independent of integer n.

For each integer $n \ge 0$, note that

$$\begin{aligned}
&(\,w_1^{(n+1)} - w_1^{(n)}\,)_{xx} + (\,w_1^{(n+1)} - w_1^{(n)}\,)_{\xi\xi} = \epsilon(\,N_1^{(n)} - N_1^{(n-1)}\,), \quad 0 < \xi < 1,\\
&w_1^{(n+1)} - w_1^{(n)} = 0, \quad \text{on } \xi = 1,\\
&w_1^{(n+1)} - w_1^{(n)} = 0, \quad \text{on } \xi = 0,
\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}
&(w_2^{(n+1)} - w_2^{(n)})_{xx} + (w_2^{(n+1)} - w_2^{(n)})_{\xi\xi} = \epsilon(N_2^{(n)} - N_2^{(n-1)}), \quad -\infty < \xi < 0,\\
&w_2^{(n+1)} - w_2^{(n)} = 0, \quad \text{on } \xi = 0,\\
&(w_2^{(n+1)} - w_2^{(n)})_x \to 0, \quad \text{as } \xi \to -\infty. \qquad (5.18)
\end{aligned}$$

Hence, by the estimates in Theorem 5 and 6, we have

$$\begin{aligned}
\| w_1^{(n+1)} - w_1^{(n)} \|_{\Omega_1, 2} &+ \| w_2^{(n+1)} - w_2^{(n)} \|_{\Omega_2, 2}\\
&\leq \epsilon\tilde{C}\{\| N_1^{(n)} - N_1^{(n-1)} \|_{\Omega_1, 0} + \| N_2^{(n)} - N_2^{(n-1)} \|_{\Omega_2, 0}\}\\
&\leq \epsilon\tilde{C}\max(C_1(R), C_2(R))\{\| w_1^{(n)} - w_1^{(n-1)} \|_{\Omega_1, 2}\\
&\qquad + \| w_2^{(n)} - w_2^{(n-1)} \|_{\Omega_2, 2}\}. \qquad (5.19)
\end{aligned}$$

Thus, if $\epsilon > 0$ is sufficiently close to 0, then we have $0 < \epsilon\tilde{C}\max(C_1(R), C_2(R)) < 1$. As a result, it is straightforward to prove that $\{(w_1^{(n)}, w_2^{(n)})\}$ is a Cauchy sequence in the closed ball of the Banach space $\Lambda_e^2(\Omega_1) \times \Lambda_e^2(\Omega_2)$. The limit (w_1, w_2) of this sequence provides us the solution of the equations (5.1) – (5.4). From now on, we can regard all the terms in (5.6), (5.7) and the normalization (3.11) which involve w_1 and w_2 as nonlinear functions of η_1, η_2, ϵ.

Now, in order to solve (5.6) and (5.7), we rewrite them as

$$\eta_1 - F(\eta_1, \eta_2, 0) = \epsilon N_3, \qquad (5.20)$$

$$\eta_2 - G(\eta, \eta_2, 0) = \epsilon N_4 \qquad (5.21)$$

where

$$\begin{aligned}
N_3(\eta_1, \eta_2, \epsilon) &= \frac{1}{\epsilon}F(\eta_1, \eta_2, \epsilon) - F(\eta_1, \eta_2, 0)\\
&= -\frac{\gamma}{2}\left[\frac{w_{1,x}{}^2 + 3w_{1,\xi}{}^2 + 2\epsilon w_{1,\xi}{}^3}{(1+\epsilon w_{1,\xi})^2}\right]_{\xi=1},\\
N_4(\eta_1, \eta_2, \epsilon) &= \frac{1}{\epsilon}G(\eta_1, \eta_2, \epsilon) - G(\eta_1, \eta_2, 0)\\
&= -\left[\frac{\gamma}{2}\frac{w_{2,x}{}^2 + 3w_{2,\xi}{}^2 + 2\epsilon w_{2,\xi}{}^3}{(1+\epsilon w_{2,\xi})^2}\right]_{\xi=0-} + \frac{\rho\gamma}{2}\left[\frac{w_{1,x}{}^2 + 3w_{1,\xi}{}^2 + 2\epsilon w_{1,\xi}{}^3}{(1+\epsilon w_{1,\xi})^2}\right]_{\xi=0+}
\end{aligned}$$

Here

$$F(\eta_1, \eta_2, 0) = -\gamma[w_{1,\xi}]_{\xi=1}, \qquad (5.22)$$

$$G(\eta_1, \eta_2, 0) = -\gamma[w_{2,\xi}]_{\xi=0-} + \rho\gamma[w_{1,\xi}]_{\xi=0+}; \qquad (5.23)$$

where (w_1, w_2) saitisfies (5.10) and (5.11). In order for an iteration to solve (5.20) and (5.21), we need to solve the following linear problem.

Lemma 7. *Let* $(g_1, g_2) \in [\Lambda_e^1(\mathbf{R})]^2$ *be given. Then the linear system*

$$\eta_1 - F(\eta_1, \eta_2, 0) = g_1, \tag{5.24}$$

$$(1-\rho)\eta_2 - G(\eta_1, \eta_2, 0) = g_2, \tag{5.25}$$

has a unique solution (η_1, η_2) *in* $[\Lambda_e^2(\mathbf{R})]^2$ *with*

$$\| \eta_1 \|_2 + \| \eta_2 \|_2 \leq C\,(\| g_1 \|_1 + \| g_2 \|_1),$$

for some constant $C > 0$.

Proof. Suppose that $g_1, g_2 \in \Lambda_e^0(\mathbf{R})$ are given, which can be represented by the the Fourier series

$$g_i = \sum_{m=-\infty}^{+\infty} g_{im} e^{imkx}, \quad i = 1, 2.$$

Assume that $\eta_1, \eta_2 \in \Lambda_e^2(\mathbf{R})$ are known for the time being and have Fourier series expansion

$$\eta_i = \sum_{m=-\infty}^{+\infty} \eta_{im} e^{imkx}, \quad i = 1, 2.$$

Let w_1 and w_2 be the solutions of linear problems (5.10) and (5.11) respectively. Recall that, as in shown in Section 3, w_1 and w_2 are given by the Fourier series

$$w_i = \sum_{m=-\infty}^{+\infty} w_{im}(\xi)\, e^{imkx}, \quad i = 1, 2,$$

where

$$w_{1\,m}(\xi) = \eta_{2\,m} \cosh mk\xi + \frac{\eta_{1\,m} - \eta_{2\,m} \cosh mk}{\sinh mk} \sinh mk\xi,$$
$$w_{2\,m}(\xi) = \eta_{2\,m}\, e^{|m|\,k\xi},$$

for $m \neq 0$, and

$$w_{1\,0}(\xi) = \eta_{2\,0} + (\eta_{1\,0} - \eta_{2\,0})\,\xi,$$
$$w_{2\,0}(\xi) = \eta_{2\,0}.$$

Hence

$$F(\eta_1, \eta_2, 0) = -\gamma [w_{1,\xi}]_{\xi=1},$$

$$= -\gamma(\eta_{1\,0} - \eta_{2\,0}) - \gamma \sum_{m\neq 0} mk \frac{\eta_{1m} \cosh mk - \eta_{2m}}{\sinh mk} e^{imkx}, \tag{5.26}$$

$$G(\eta_1, \eta_2, 0) = -\gamma [w_{2,\xi}]_{\xi=0-} + \rho\gamma [w_{1,\xi}]_{\xi=0+}$$

$$= \rho\gamma(\eta_{1\,0} - \eta_{2\,0}) + \sum_{m\neq 0} \{ -\gamma\eta_{2m} \mid mk \mid$$

$$+ \rho\gamma \frac{mk(\eta_{1m} - \eta_{2m} \cosh mk)}{\sinh mk} \} e^{imkx}. \tag{5.27}$$

Thus (5.24) and (5.25) lead to, for integer $m \neq 0$,

$$(1 - \gamma mk \coth mk)\, \eta_{1\,m} - \gamma mk \sinh mk\, (1 - \coth^2 mk)\, \eta_{2\,m} = g_{1m}, \tag{5.28}$$

$$\frac{\rho\gamma mk}{\sinh mk}\, \eta_{1\,m} - [\,(1-\rho) - \gamma \mid mk \mid - \rho\gamma mk \coth mk\,]\, \eta_{2\,m} = g_{2m}, \tag{5.29}$$

and

$$(1-\gamma)\, \eta_{1\,0} + \gamma\, \eta_{2\,0} = g_{10}, \tag{5.30}$$

$$\rho\gamma\, \eta_{1\,0} - (1 - \rho - \rho\gamma)\, \eta_{2\,0} = g_{20}, \tag{5.31}$$

when $m = 0$. By hypothesis, g_1 and g_2 are even in x. Thus we have

$$g_{im} = g_{i(-m)}, \quad i = 1, 2,$$

for all integer m. Next note that all the coefficients in the linear system (5.24), (5.25) for η_{1m} and η_{2m} are even in m. As a result, it suffices to solve (5.24), (5.25) for integer $m \geq 0$ and the resulting functions η_1, η_2 are clearly even in x. For each integer $m > 0$, we obtain

$$\eta_{1m} = \frac{g_{1m}[(1-\rho) - \gamma mk - \rho\gamma mk \coth mk] + g_{2m}\gamma mk/\sinh mk}{d_m},$$

$$\eta_{2m} = \frac{-g_{1m}\rho\gamma mk/\sin mk + (1 - \gamma mk \coth mk)}{d_m}$$

where determinant

$$d_m = (\,1 - \gamma\, mk\,)\,(\,1 - \rho\,(1 + \gamma\, mk) - \gamma\, mk \coth mk\,).$$

Note that, for large $\mid m \mid$, the dominant term in d_m is

$$\mid (\rho - \coth mk)\, \gamma^2 m^2 k^2 \mid .$$

Thus we have, for large $\mid m \mid$,

$$\mid d_m \mid \geq C' \mid mk \mid^2,$$

for some constant $C' > 0$. Consequently, by choosing a large enough constant $C > 0$, we obtain

$$\mid \eta_{1m} \mid + \mid \eta_{2m} \mid \leq \frac{C}{\mid mk \mid}(\mid g_{1m} \mid + \mid g_{2m} \mid);$$

for all integer m. Thus, for $i = 1, 2$, we have

$$\begin{aligned}
\| \eta_i \|_2 &= \mid \eta_{i0} \mid + \| \mathcal{D}_x^2 \eta_i \|_0 \\
&= \mid \eta_{i0} \mid + \sum_{m \neq 0} \mid mk \mid^2 \mid \eta_{im} \mid \\
&\leq C\,(\mid g_{10} \mid + \mid g_{20} \mid + \sum_{m \neq 0} \mid mk \mid (\mid g_{1m} \mid + \mid g_{2m} \mid) \\
&\leq C\,(\| g_1 \|_1 + \| g_2 \|_1).
\end{aligned}$$

This proves the lemma.

Now we are ready to solve system (5.20), (5.21) by an iteration. Let

$$\eta_1^{(0)}(x) = 2\cos kx,$$
$$\eta_2^{(0)}(x) = 2\frac{(\gamma k \coth k - 1)\sinh^2 k}{\gamma k}\cos kx,$$

where wave number k is a zero of the function

$$b(\tau) = (1-\gamma\tau)(1-\rho(1+\gamma\tau)-\gamma\tau\coth\tau),$$

with $b(mk) \neq 0$ for all integer $m > 1$. For each integer $n \geq 1$, let

$$N_3^{(n)} = N_3(\eta_1^{(n)}, \eta_2^{(n)}, \epsilon),$$
$$N_4^{(n)} = N_4(\eta_1^{(n)}, \eta_2^{(n)}, \epsilon).$$

By Lemma 4, $N_3^{(n)}$ and $N_4^{(n)}$ are in $\Lambda^1(\mathbf{R})$ for ϵ sufficiently close to 0 and are even in x, provided $\eta_1^{(n)}$ and $\eta_2^{(n)}$ are in $\Lambda_e^2(\mathbf{R})$. Now solves the linear system

$$\eta_1^{(n)} - F(\eta_1^{(n)}, \eta_2^{(n)}, 0) = \epsilon N_3^{(n-1)},$$
$$(1-\rho)\,\eta_2^{(n)} - G(\eta_1^{(n)}, \eta_2^{(n)}, 0) = \epsilon N_4^{(n-1)}.$$

This gives us a sequence $\{(\eta_1^{(n)}, \eta_2^{(n)})\}$ in $[\Lambda_e^2(\mathbf{R})]^2$. To show the convergence of above sequence, we need to estimate

$$\| \eta_1^{(n+1)} - \eta_1^{(n)} \|_2 + \| \eta_2^{(n+1)} - \eta_2^{(n)} \|_2 .$$

By Lemma 7, we have

$$\| \eta_1^{(n+1)} - \eta_1^{(n)} \|_2 + \| \eta_2^{(n+1)} - \eta_2^{(n)} \|_2 \leq \epsilon C\,(\| N_3^{(n)} - N_3^{(n-1)} \|_1 + \| N_4^{(n)} - N_4^{(n-1)} \|_1). \tag{5.32}$$

Note that N_3 and N_4 are analytic in their arguments. We can estimate the right hand side of (5.32) as in (5.16) and conclude that $\{(\eta_1^{(n)}, \eta_2^{(n)})\}$ is a Cauchy sequence in $[\Lambda_e^2(\mathbf{R})]^2$. Then convergence of the sequence $\{(\eta_1^{(n)}, \eta_2^{(n)})\}$ is straightforward. In conclusion, we have

Theorem 8. *Suppose that the Froude number γ is larger than $1-\rho$, or when $0 < \gamma < 1-\rho$, γ is different from γ_n^*, for $n = 1, 2, 3, \cdots$. Here γ_n^* are the critical Froude numbers defined in* (3.21). *The constant ρ is the density ratio of the two layers of fluids; $0 < \rho < 1$. Then nonlinear equations (2.24) - (2.29) have an exact solution $(w_1, w_2, \eta_1, \eta_2)$ in*

$$\Lambda_e^2(\Omega_1) \times \Lambda_e^2(\Omega_2) \times [\Lambda_e^2(\mathbf{R})]^2.$$

References

[1] J. T. Beale, "Water waves generated by a pressure disturbance on a steady stream," *Duke Math. J.*, 47(1980), 297-323.
[2] Sir Horace Lamb, *Hydrodynamics*, Dover Publications, New York 1945.
[3] Willard J. Pierson, Jr. and Paul Fife, "Some nonlinear properties of long-crested periodic waves with lengths near 2.44 centimeters," *J. of Geophysical Res.*, 66(1961), 163-179.
[4] J. Reeder and M. Shinbrot, "On Wilton ripples, I : formal derivation of the phenomenon," *Wave Motion*, 3(1981), 115-135.
[5] J. Reeder and M. Shinbrot, "On Wilton ripples, II : rigorous results," *Arch. Rational Mech. Anal.*, 77(1981), 321-347.
[6] M. Shinbrot, "Water waves over periodic bottoms in three dimensions," *J. Inst. Maths Applics*, 25(1980), 367-385.
[7] J. J. Stoker, *Water Waves*, Interscience, New York, 1953.
[8] T. Y. Sun, "Three-dimensional steady water waves generated by partially localized pressure disturbances," *SIAM J. Math. Anal.*, 24(5)(1993), 1153-1178.

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