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ANGULAR ESTIMATES OF CERTAIN INTEGRAL OPERATORS

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Abstract. In the present paper, we investigate certain integral preserving properties in a sector. Our results include several previous results as the special cases.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. A function f of $\mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$, the class of starlike functions of order α, if

$$Re\{\frac{zf'(z)}{f(z)}\} > \alpha(z \in U, 0 < \alpha \leq 1).$$

The class $\mathcal{S}^*$ of starlike functions is identified by $\mathcal{S}^*(0) = \mathcal{S}^*$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}(m, M)$ if

$$|\frac{zf'(z)}{f(z)} - m| < M \ (z \in U, |m-1| < M \leq m).$$

The class $\mathcal{S}(m, M)$ was introduced by Jakubowski [5]. It is clear that $m > \frac{1}{2}$ and $\mathcal{S}(m, M) \subset \mathcal{S}^*(m - M) \subset \mathcal{S}^*$.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{B}(\mu, \alpha, \beta)$ if it satisfies

$$Re\left\{\frac{zf'(z)f^{u-1}(z)}{g^{\mu}(z)}\right\} > \beta \ (z \in U)$$

for some $\mu(\mu > 0)$, $\beta(0 \leq \beta < 1)$ and $g \in \mathcal{S}^*(\alpha)$. Furthermore, we denote $\mathcal{B}_1(\mu, \alpha, \beta)$ by the subclass of $\mathcal{B}(\mu, \alpha, \beta)$ for $g(z) \equiv z \in \mathcal{S}^*(\alpha)$. The classes $\mathcal{B}(\mu, \alpha, \beta)$ and $\mathcal{B}_1(\mu, \alpha, \beta)$ are the subclasses of Bazilević functions in U [14]. We also note that $\mathcal{B}(1, \alpha, \beta) \equiv C(\alpha, \beta)$ is an important subclass of close-to-convex functions [6].

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Many authors [1, 2, 3, 4, 8] have studied the integral operators of the form

$$I_{c,\mu}(f) = \Big(\frac{c+\mu}{z^c}\int_0^z t^{c-1} f^{\mu}(t)dt\Big)^{\frac{1}{\mu}}, \tag{1.2}$$

where c and μ are suitably chosen real constants and f belong to some favoured classes of univalent functions. In particular, Kumar and Shukla [8] showed that the integral operator $I_{c,\mu}(f)$ defined by (1.2) maps $\mathcal{S}(m, M)$ into itself for $c \geq -\mu(m-M)(\mu > 0)$.

In the present paper, we give some argument properties of the integral operator defined by (1.2). We also generalize the previous results of Libera [9], Owa and Srivastava [12] and Owa and Obradović [13].

2. Main Results

In proving our main results, we shall need the following lemmas.

Lemma 1([10]). *Let $M(z)$ and $N(z)$ be regular in U with $M(0) = N(0) = 0$, and let β be real. If $N(z)$ maps U onto a (possibly many-sheedted) region which is starlike with respect to the orign, then*

$$Re\Big\{\frac{M'(z)}{N'(z)}\Big\} > \beta \ (z \in U) \Rightarrow Re\Big\{\frac{M(z)}{N(z)}\Big\} > \beta \ (z \in U)$$

and

$$Re\Big\{\frac{M'(z)}{N'(z)}\Big\} < \beta \ (z \in U) \Longrightarrow Re\Big\{\frac{M(z)}{N(z)}\Big\} < \beta \ (z \in U).$$

Lemma 2([11]). *Let $p(z)$ be analytic in U, $p(0) = 1$, $p(z) \neq 0$ in U and suppose that there exists a point $z_0 \in U$ such that*

$$\Big|\arg p(z)\Big| < \frac{\pi\beta}{2} \ for \ |z| < |z_0|$$

and

$$\Big|\arg p(z_0)\Big| = \frac{\pi\beta}{2},$$

where $0 < \beta \leq 1$. Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta,$$

where

$$k \geq \frac{1}{2}\Big(a + \frac{1}{a}\Big) \ when \ \arg p(z_0) = \frac{\pi\beta}{2}$$

and

$$k \leq -\frac{1}{2}\Big(a + \frac{1}{a}\Big) \ when \ \arg p(z_0) = -\frac{\pi\beta}{2}$$

where

$$p(z_0)^{\frac{1}{\beta}} = \pm ia \ (a > 0).$$

Lemma 3([7,8]). *The function f of the form (1.1) belongs to $\mathcal{S}(m, M)$ if and only if there exists a function w regular in U and satisfying $w(0) = 0$, $|w(z)| < 1$ for $z \in U$ such that*

$$\frac{zf'(z)}{f(z)} = \frac{1 + aw(z)}{1 - bw(z)} \qquad (z \in U), \tag{2.1}$$

where $a = (M^2 - m^2 + m)/M$ and $b = (m - 1)/M$.

With the help of Lemma 1 and Lemma 2, we now derive

Theorem 1. *Let c and μ be real numbers with $c \geq 0$, $\mu > 0$ and let $f \in \mathcal{A}$. If*

$$\left|\arg\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)} - \beta\right)\right| < \frac{\pi\delta}{2} \quad (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in \mathcal{S}^*(m, M)$, then

$$\left|\arg\left(\frac{z(I_{c,\mu}(f))' I_{c,\mu}^{\mu-1}(f)}{I_{c,\mu}^{\mu}(g)} - \beta\right)\right| < \frac{\pi\delta}{2},$$

where $I_{c,\mu}$ is the integral operator defined by (1.2) and $\eta (0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\mathrm{Tan}^{-1}\left(\frac{\eta \sin\frac{\pi}{2}(1 - \frac{2}{\pi}\sin^{-1}(\frac{M}{c+m}))}{c + m + M + \eta\cos\frac{\pi}{2}(1 - \frac{2}{\pi}\sin^{-1}(\frac{M}{c+m}))}\right). \tag{2.2}$$

Proof. Let

$$p(z) = \frac{M(z)}{N(z)},$$

where

$$M(z) = \frac{1}{1-\beta}\left\{z^c f^{\mu}(z) - c\int_0^z t^{c-1} f^{\mu}(t)dt - \beta\mu\int_0^z t^{c-1} g^{\mu}(t)dt\right\}$$

and

$$N(z) = \mu\int_0^z t^{c-1} g^{\mu}(t)dt.$$

Then $p(z)$ is analytic in U with $p(0) = 1$. By a simple calculation, we have

$$\begin{aligned}\frac{M'(z)}{N'(z)} &= p(z)\left(1 + \frac{N(z)}{zN'(z)}\frac{z'p(z)}{p(z)}\right)\\ &= \frac{1}{1-\beta}\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)} - \beta\right).\end{aligned}$$

Since $g \in \mathcal{S}(m, M)$, $I_{c,\mu}(g) \in \mathcal{S}(m, M)$ [8] and hence $N(z)$ is (possibly many sheeted) starlike function with respect to the orign. Therefore, from our assumption and Lemma 1, $p(z) \neq 0$ in U.

If there exists a point $z_0 \in U$ such that

$$\left|\arg p(z)\right| < \frac{\pi\eta}{2} \text{ for } |z| < |z_0|$$

and

$$\left|\arg p(z_0)\right| = \frac{\pi\eta}{2},$$

then, from Lemma 2, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta,$$

where

$$k \geq \frac{1}{2}\Big(a + \frac{1}{a}\Big) \text{ when } \arg p(z_0) = \frac{\pi\eta}{2}$$

and

$$k \leq \frac{1}{2}\Big(a + \frac{1}{a}\Big) \text{ when } \arg p(z_0) = -\frac{\pi\eta}{2}$$

where

$$p(z_0)^{\frac{1}{\eta}} = \pm ia \; (a > 0).$$

Since $I_{c,\mu}(g) \in \mathcal{S}(m, M)$, we have

$$\frac{zN'(z)}{N(z)} = \frac{z(I_{c,\mu}(g))'}{I_{c,\mu}(g)} + c = \rho e^{i\frac{\pi\phi}{2}},$$

where

$$\begin{cases} c + m - M < \rho < c + m + M, \\ -\frac{2}{\pi}\sin^{-1}(\frac{M}{c+m}) < \phi < \frac{2}{\pi}\sin^{-1}(\frac{M}{c+m}). \end{cases}$$

At first, suppose that $p(z_0)^{\frac{1}{\eta}} = ia(a > 0)$. We obtain

$$\begin{aligned}
\arg\Big(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\Big) &= \arg\frac{(1-\beta)M'(z_0)}{N'(z_0)} \\
&= \arg p(z_0) + \arg\Big(1 + \frac{1}{\frac{z(I_{c,\mu}(g))'}{I_{c,\mu}(g)} + c}\frac{z_0 p'(z_0)}{p(z_0)}\Big) \\
&= \frac{\pi\eta}{2} + \arg\Big(1 + (\rho e^{i\frac{\pi\phi}{2}})^{-1} i\eta k\Big) \\
&= \frac{\pi\eta}{2} + \mathrm{Tan}^{-1}\Big(\frac{\eta k \sin\frac{\pi}{2}(1-\phi)}{\rho + \eta k \cos\frac{\pi}{2}(1-\phi)}\Big) \\
&\geq \frac{\pi\eta}{2} + \mathrm{Tan}^{-1}\Big(\frac{\eta\sin\frac{\pi}{2}(1 - \frac{2}{\pi}\sin^{-1}(\frac{M}{c+m}))}{c + m + M + \eta\cos\frac{\pi}{2}^{-1} 1 - \frac{2}{\pi}\sin^{-1}(\frac{M}{c+m}))}\Big) \\
&= \frac{\pi}{2}\delta,
\end{aligned}$$

where δ are given by (2.2). This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\eta}} = -ia (a > 0)$. Applying the same method as the above, we have

$$\arg\left(\frac{z_0 f'(z_0) f^{\mu-1}(z_0)}{g^{\mu}(z_0)} - \beta\right) \leq -\frac{\pi\eta}{2} - \mathrm{Tan}^{-1}\left(\frac{\eta \sin\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}(\frac{M}{c+m}))}{c+m+M+\eta\cos\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}(\frac{M}{c+m}))}\right)$$
$$= -\frac{\pi}{2}\delta,$$

where δ are given by (2.2), which contracts the assumption. Therefore we complete the proof of our theorem.

Let us choose $m = N - \alpha(N-1)$ and $M = N(1-\alpha)$, where $N \geq 1$ and $0 \leq \alpha < 1$. Then $|m-1| < M \leq m$, $a = \alpha/N + (1-2\alpha)$ and $b = 1 - 1/N$ in Lemma 3. Now as $N \to \infty$, $a \to 1-2\alpha$ and $b \to 1$. In this case, the relation (2.1) reduces to

$$\frac{zf'(z)}{f(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)} (z \in U),$$

which is a necessary and sufficient condition for f to be in $\mathcal{S}^*(\alpha)$. Hence we have the following

Corollary 1. *Let $c \geq 0$, $\mu > 0$ and $f \in \mathcal{A}$. If*

$$\left|\arg\left(\frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)} - \beta\right)\right| < \frac{\pi\delta}{2} \quad (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in \mathcal{S}^*(\alpha)$, then

$$\left|\arg\left(\frac{z(I_{c,\mu}(f))' I_{c,\mu}^{\mu-1}(f)}{I_{c,\mu}^{\mu-1}(g)} - \beta\right)\right| < \frac{\pi\delta}{2},$$

where $I_{c,\mu}$ is the integral operator defined by (1.2).

Remark 1. For $\delta = 1$, Corollary 1 is the result obtained by Owa and Obradović [13].

Letting $\mu = 1$ in Theorem 1, we have

Corollary 2. *Let $c \geq 0$ and let $f \in \mathcal{A}$. If*

$$\left|\arg\left(\frac{zf'(z)}{g(z)} - \beta\right)\right| < \frac{\pi\delta}{2} \quad (0 \leq \beta < 1, 0 < \delta \leq 1)$$

for some $g \in \mathcal{S}(m, M)$, then

$$\left|\arg\left(\frac{z(I_{c,1}(f))'}{I_{c,1}(g)} - \beta\right)\right| < \frac{\pi\eta}{2},$$

where $I_{c,1}$ is the integral operator defined by (1.2) and $\eta (0 < \eta \leq 1)$ is the solution of the equation (2.2).

Taking $m = N-\alpha(N-1)$, $M = N(1-\alpha)(0 \leq \alpha < 1)$, $N \to \infty$ and $\delta = 1$ in Corollary 2, we have the result of Owa and Srivastave [12].

Corollary 3. *If the function f defined by (1.1) is in the class $\mathcal{C}(\alpha,\beta)$, then the integral operator $I_{c,1}(f)(c \geq 0)$ defined by (1.2) is also in the class $\mathcal{C}(\alpha,\beta)$.*

Remark 2. Putting $\alpha = \beta = 0$ and $c = 1$ in Corollary 3, we obtain the result given by Libera [9].

By using the same technique as in proving Theorem 1, we have

Theorem 2. *Let c and μ be real numbers with $c \geq 0$, $\mu > 0$ and let $f \in \mathcal{A}$. If*

$$\left|\arg\left(\beta - \frac{zf'(z)f^{\mu-1}(z)}{g^{\mu}(z)}\right)\right| < \frac{\pi\delta}{2} \qquad (\beta > 1, 0 < \delta \leq 1)$$

for some $g \in \mathcal{S}(m,M)$, then

$$\left|\arg\left(\beta - \frac{z(I_{c,\mu}(f))' I_{c,\mu}^{\mu-1}(f)}{I_{c,\mu}^{\mu}(g)}\right)\right| < \frac{\pi\eta}{2},$$

where $I_{c,\mu}$ is the integral operator defined by (1.2) and $\eta(0 < \eta \leq 1)$ is the solution of the equation (2.2).

Letting $m = N - \alpha(N-1)$, $M = N(1-\alpha)(0 \leq \alpha < 1)$, $N \to \infty$, $\mu = 1$ and $\delta = 1$ in Theorem 2, we have the following result by Owa and Srivastava [12].

Corollary 4. *Let $c \geq 0$ and $f \in \mathcal{A}$. If*

$$Re\left\{\frac{zf'(z)}{g(z)}\right\} < \beta \quad (\beta > 1)$$

for some $g \in \mathcal{S}^*(\alpha)$, then

$$Re\left\{\frac{z(I_{c,1}(f))'}{I_{c,1}(g)}\right\} < \beta,$$

where $I_{c,1}$ is the integral operator defined by (1.2).

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References

[1] G. P. Bhargave and S. L. Shukla, "On the classes of regular univalent functions," *Jnanabha*, 12(1982), 33-39.

[2] W. M. Cauchy and W. L. White, "Starlikeness of certain functions with integral representations," *J. Math. Anal. Appl.*, 64(1978), 458-466.

[3] R. M. Goel and B. S. Mehrok, "Some invarance properties of a subclass of close-to-convex functions," *Indian J. Pure. Appl. Math.*, 12(1981), 1240-1249.

[4] V. P. Gupta and P. K. Jain, "On starlike functions," *Rend. Mat.*, 9(6)(1976), 433-437.

[5] Z. J. Jakubowski, "On the coefficients of starlike functions of some classes," *Ann. Polon. Math.*, 26(1972), 305-313.

[6] W. Kaplan, "Close-to-convex schlicht functions," *Michigan Math. J.*, 1(1952), 169-185.

[7] V. Kumar and S. L. Shukla, "Certain integrals for classes of p-valent meromorphic functions," *Bull. Austral. Math. Soc.*, 25(1982), 85-97.

[8] V. Kumar and S. L. Shukla, "Jakubowski integral operators," *J. Austral. Math. Soc.* (series A), 37(1984), 117-127.

[9] R. J. Libera, "Some classes of regular univalent functions," *Proc. Amer. Math. Soc.*, 16(1965), 755-758.

[10] S. S. Miller and P. T. Mocanu, "Second order differential inequalities in the complex plane," *J. Math. Anal. Appl.*, 65(1978), 289-305.

[11] M. Nunokawa, "On the order of strongly starlikeness of strongly convex functions," *Proc. Japan Acad.*, 69, Ser. A(1993), 234-237.

[12] S. Owa and H. M. Srivastava, "Some applications of the generalized Libera integral operator," *Proc. Japan Acad.*, 62, Ser. A(1986), 125-128.

[13] S. Owa and M. Obradović, "Certain subclasses of Bazilević functions of type α," *Internat. J. Math. & Math. Sci.*, 9(1986), 347-359.

[14] R. Singh, "On Bazilević functions," *Proc. Amer. Math. Soc.*, 1(1952), 169-185.

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