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CONVOLUTIONS OF UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$. We investigate some properties of $h(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ where $f(z)$ and $g(z)$ belongs to certain subclasses of starlike and convex functions.

1. Introduction

Let S be the class consisting of the functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$. A function $f \in S$ is said to be starlike of order α, $0 \leq \alpha < 1$, if $\text{Re}\{\frac{zf'(z)}{f(z)}\} > \alpha$ for $z \in E$ and it said to be convex of order α, $0 \leq \alpha < 1$, if $\text{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > \alpha$ for $z \in E$. These classes are respectively denoted by $S^*(\alpha)$ and $K(\alpha)$. $S^*(0) = S^*$ and $K(0) = K$ are respectively the classes of starlike and convex functions in S.

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then the Hadmard product (convolution) $(f * g)(z)$ of functions $f(z)$ and $g(z)$ is defined by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

Recetly Ruscheweyh and Sheil-Small [1] proved the Polya-Schoenberg conjecture that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K$ then $h(z) = f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \in K$. Further in [2] Shild and Silverman considered convolutions of univalent functions with negative coefficients.

For $1 < \beta \leq \frac{4}{3}$ and $z \in E$ let $M(\beta) = \{f \in S : \text{Re}\frac{zf'(z)}{f(z)} < \beta\}$ and $L(\beta) = \{f \in S : \text{Re}(1 + \frac{zf''(z)}{f'(z)}) < \beta\}$. Let V be the subclass of S consisting of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. Let $V^*(\alpha) = S^*(\alpha) \cap V$, $V_k(\alpha) = K(\alpha) \cap V$ and $V(\beta) = M(\beta) \cap V$, $U(\beta) = L(\beta) \cap V$. $V^*(0) = V^*$ and $V_k(0) = V_k$ are respectively the classes of starlike and convex functions in V. The classes $V(\beta)$ and $U(\beta)$ have been studied by B. A. Uralegaddi, M. D. Ganigi and S. M. Sarangi [3]. They have shown that all functions in $V(\beta)$ are starlike and all functions in $U(\beta)$ are convex.

In this paper we investigate some properties of $f * g$ where $f, g \in V(\beta)$ or $U(\beta)$.

We need the following results [3].

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Theorem A. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in S. If $\sum_{n=2}^{\infty}(n-\beta)|a_n| \leq \beta - 1$ then $f \in M(\beta)$.*

Theorem B. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in S. If $\sum_{n=2}^{\infty} n(n-\beta)|a_n| \leq \beta - 1$ then $f \in L(\beta)$..*

Theorem C. *A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ is in $V(\beta)$ if and only if $\sum_{n=2}^{\infty}(n-\beta)\ a_n \leq \beta - 1$.*

Theorem D. *A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ is in $U(\beta)$ if and only if $\sum_{n=2}^{\infty} n(n-\beta)\ a_n \leq \beta - 1$.*

2. Convolutions of functions in $V(\beta)$ and $U(\beta)$

Theorem 1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, $b_n \geq 0$ are elements of $V(\beta)$, then $f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ is an element of $V(\gamma)$, where $\gamma = \frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}$.*

Proof. Since $f(z)$ and $g(z) \in V(\beta)$, it follows that $\sum_{n=2}^{\infty} \frac{n-\beta}{\beta-1} a_n \leq 1$ and $\sum_{n=2}^{\infty} \frac{n-\beta}{\beta-1} b_n \leq 1$. We want to show that $\sum_{n=2}^{\infty}(n-\gamma)a_n b_n \leq \gamma - 1$. Equivalently we want to show that

$$\sum_{n=2}^{\infty} \frac{n-\beta}{\beta-1} a_n \leq 1 \tag{1}$$

and

$$\sum_{n=2}^{\infty} \frac{n-\beta}{\beta-1} b_n \leq 1 \tag{2}$$

imply that

$$\sum_{n=2}^{\infty} \frac{n-\gamma}{\gamma-1} a_n b_n \leq 1 \text{ for all } \gamma = \gamma(\beta) \geq \frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}. \tag{3}$$

From (1) and (2) using Cauchy-Schawarz inequality, we get

$$\sum_{n=2}^{\infty} \frac{n-\beta}{\beta-1} \sqrt{a_n b_n} \leq 1. \tag{4}$$

Hence it suffices to show that

$$\frac{n-\gamma}{\gamma-1} a_n b_n \leq \frac{n-\beta}{\beta-1} \sqrt{a_n b_n}, \quad n = 2, 3, \ldots.$$

$$\text{or } \sqrt{a_n b_n} \leq \frac{n-\beta}{\beta-1} \frac{\gamma-1}{n-\gamma}.$$

From (4) it follows that

$$\sqrt{a_n b_n} \leq \frac{\beta-1}{n-\beta} \text{ for each } n.$$

Therefore it is sufficient if we show that

$$\frac{\beta-1}{n-\beta} \le \frac{n-\beta}{\beta-1}\frac{\gamma-1}{n-\gamma} \quad \text{for all } n. \tag{5}$$

The inequality (5) is equivalent to

$$\gamma \ge \frac{1+n(\frac{\beta-1}{n-\beta})^2}{1+(\frac{\beta-1}{n-\beta})^2}. \tag{6}$$

The right-hand side of (6) is a decreasing function of n, $n = 2, 3, \ldots$. For $n = 2$, we get

$$\gamma \ge \frac{1+2(\frac{\beta-1}{2-\beta})^2}{1+(\frac{\beta-1}{2-\beta})^2} = \frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}.$$

The result follows.
Observe that $1 < \gamma < 4/3$.

The result is sharp, for the functins

$$f(z) = g(z) = z + (\frac{\beta-1}{2-\beta})z^2.$$

Remark 1. $V(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}) \subset V^*(\frac{2-\beta^2}{3-2\beta})$ follows from the result [3]: If $f \in V(\beta)$ then $f \in V^*(\frac{4-3\beta}{3-2\beta})$.

Remark 2. In the above theorem we have shown that if $f, g \in V(\beta)$ then $f * g \in V(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2})$. For given $h \in V(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2})$ do there exist functions $f, g \in V(\beta)$ such that $h = f * g$? We shall show by an example that the answer is no. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ be the functions in $V(\beta)$, then $a_n \le \frac{\beta-1}{n-\beta}$, $b_n \le \frac{\beta-1}{n-\beta}$. By the above theorem we have

$$f * g = z + \sum_{n=2}^{\infty} a_n b_n z^n \in V(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}).$$

Also note that $a_n b_n \le (\frac{\beta-1}{n-\beta})^2$.
Consider

$$h(z) = z + \frac{\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2} - 1}{n - \frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}} z^n \in V(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}).$$

For this function we have

$$\frac{\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2} - 1}{n - \frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}} = \frac{(\beta-1)^2}{(5-6\beta+2\beta^2)n - (6-8\beta+3\beta^2)} > \frac{(\beta-1)^2}{(n-\beta)^2} \quad \text{for } n \ge 3.$$

i.e there is no f and $g \in V(\beta)$ such that $f * g = h \in V(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2})$.

Corollary. *Let $f(z)$ and $g(z)$ be in $V(\beta)$. Then $h(z) = z + \sum_{n=2}^{\infty} \sqrt{a_n b_n} z^n \in V(\beta)$.*
This result follows from the inequality (4). It is sharp for the same functions as in Theorem 1.

Theorem 2. *If $f \in V(\beta)$ and $g \in V(\gamma)$ then $f * g \in V(\delta)$, where $\delta = \frac{6-4\beta-4\gamma+3\beta\gamma}{5-3\beta-3\gamma+2\beta\gamma}$.*

Proof. Proceeding as in the proof of Theorem 1, we obtain

$$\delta \geq \frac{1 + \frac{n(\beta-1)(\gamma-1)}{(n-\beta)(n-\gamma)}}{1 + \frac{(\beta-1)(\gamma-1)}{(n-\beta)(n-\gamma)}}. \tag{7}$$

The right-hand side of (7) is a decreasing function of n ($n = 2, 3, \ldots$). Taking $n = 2$, we get

$$\delta \geq \frac{1 + \frac{2(\beta-1)(\gamma-1)}{(2-\beta)(2-\gamma)}}{1 + \frac{(\beta-1)(\gamma-1)}{(2-\beta)(2-\gamma)}} = \frac{6 - 4\beta - 4\gamma + 3\beta\gamma}{5 - 3\beta - 3\gamma + 2\beta\gamma}.$$

Corollary. *If $f(z)$, $g(z)$, $h(z) \in V(\beta)$, then $f * g * h \in V(\frac{6-6\beta+\beta^3}{7-9\beta+3\beta^2})$.*

Proof. From Theorem 1, we have $f * g \in V(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2})$. Using Theorem 2, we get

$$f * g * h \in V\left(\frac{6 - 4\beta - 4(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}) + 3\beta(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2})}{5 - 3\beta - 3(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2}) + 2\beta(\frac{6-8\beta+3\beta^2}{5-6\beta+2\beta^2})}\right)$$

i.e. $f * g * h \in V(\frac{6-6\beta+\beta^3}{7-9\beta+3\beta^2})$.
For functions in the class $U(\beta)$ we have similar result. We shall prove:

Theorem 3. *If $f \in U(\beta)$ and $g \in U(\gamma)$, then $f*g \in U(\delta)$, where $\delta = \frac{2(5-3\beta-3\gamma+2\beta\gamma)}{9-5\beta-5\gamma+3\beta\gamma}$.*

Proof. From Theorem D, we have $\sum_{n=2}^{\infty} n(n-\beta)a_n \leq \beta - 1$ and $\sum_{n=2}^{\infty} n(n-\gamma)b_n \leq \gamma - 1$. We wish to show that $\sum_{n=2}^{\infty} n(n-\delta)a_n b_n \leq \delta - 1$. It is sufficient to show that

$$\sum_{n=2}^{\infty} \frac{n(n-\beta)}{\beta - 1} a_n \leq 1$$

and

$$\sum_{n=2}^{\infty} \frac{n(n-\gamma)}{\gamma - 1} b_n \leq 1$$

imply

$$\sum_{n=2}^{\infty} \frac{n(n-\delta)}{\delta - 1} a_n b_n \leq 1 \text{ for all } \delta = \delta(\beta, \gamma) \geq \frac{2(5 - 3\beta - 3\gamma + 2\beta\gamma)}{9 - 5\beta - 5\gamma + 3\beta\gamma}.$$

Proceeding as in the proofs of Theorems 1 and 2, we get:

$$\frac{n-\delta}{\delta-1} \le \frac{n(n-\beta)(n-\gamma)}{(\beta-1)(\gamma-1)}$$

or

$$\delta \ge \frac{1+\frac{(\beta-1)(\gamma-1)}{(n-\beta)(n-\gamma)}}{1+\frac{(\beta-1)(\gamma-1)}{n(n-\beta)(n-\gamma)}}. \tag{8}$$

The right hand side of (8) is a decreasing function of $n (n = 2, 3, \ldots)$. Taking $n = 2$, we get

$$\delta \ge \frac{2(5-3\beta-3\gamma+2\beta\gamma)}{9-5\beta-5\gamma+3\beta\gamma}.$$

The result follows.

Theorem 4. *If $f \in V(\beta)$ and $g \in V(\gamma)$ then $f * g \in U(\delta)$, where $\delta = \frac{8-6\beta-6\gamma+5\beta\gamma}{6-4\beta-4\gamma+3\beta\gamma}$.*

Proof. From Theorem C, we have $\sum_{n=2}^{\infty}(n-\beta)a_n \le \beta-1$ and $\sum_{n=2}^{\infty}(n-\gamma)b_n \le \gamma-1$. It follows that

$$\sum_{n=2}^{\infty}(n-\beta)(n-\gamma)a_n b_n \le (\beta-1)(\gamma-1).$$

We wish to show that $\sum_{n=2}^{\infty} n(n-\delta)a_n b_n \le \delta - 1$.
This is satisfied if

$$\frac{n(n-\delta)}{\delta-1} \le \frac{(n-\beta)(n-\gamma)}{(\beta-1)(\gamma-1)}$$

i.e. for

$$\delta \ge \frac{1+\frac{n^2(\beta-1)(\gamma-1)}{(n-\beta)(n-\gamma)}}{1+\frac{n(\beta-1)(\gamma-1)}{(n-\beta)(n-\gamma)}}. \tag{9}$$

The right hand side of (9) is a decreasing function of $n (n = 2, 3, \ldots)$. Taking $n = 2$, we get

$$\delta \ge \frac{8-6\beta-6\gamma+5\beta\gamma}{6-4\beta-4\gamma+3\beta\gamma}.$$

The result follows.

The result is sharp for the functions

$$f(z) = z + \frac{\beta-1}{2-\beta}z^2 \in V(\beta) \quad \text{and} \quad g(z) = z + \frac{\gamma-1}{2-\gamma}z^2 \in V(\gamma).$$

Putting $\beta = \gamma = 4/3$ in Theorem 4, we get the following

Corollary. *If f, $f \in V(4/3)$, then $f * g \in U(4/3)$.*
Since $U(4/3) \subset V_k$ [3], the convolution of any two functions in $V(4/3)$ is convex.

Theorem 5. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$, $\in V(\beta)$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, *with* $|b_i| \leq 1$, $i = 2, 3, \dots$, *then* $f * g \in M(\beta)$.

Proof. $\sum_{n=2}^{\infty}(n-\beta)|a_n b_n| = \sum_{n=2}^{\infty}(n-\beta)a_n|b_n| \leq \sum_{n=2}^{\infty}(n-\beta)a_n \leq \beta - 1$.

Corollary. *If* $f(z) \in V(\beta)$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, *with* $0 \leq b_i \leq 1$, $i = 2, 3, \dots$, *then* $f * g \in V(\beta)$.

Theorem 6. *let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in U(\beta)$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ *with* $|b_i| \leq 1$, $i = 2, 3 \dots$, *then* $f * g \in L(\beta)$.

Proof.

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-\beta)|a_n b_n| \\
&= \sum_{n=2}^{\infty} n(n-\beta)a_n|b_n| \\
&\leq \sum_{n=2}^{\infty} n(n-\beta)a_n \leq \beta - 1.
\end{aligned}$$

Corollary. *If* $f(z) \in U(\beta)$ *and* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, *with* $0 \leq b_i \leq 1$, $i = 2, 3, \dots$, *then* $f * g \in U(\beta)$.

Theorem 7. *If* $f,\ g \in V(\beta)$, *then* $h(z) = z + \sum_{n=2}^{\infty}(a_n^2 + b_n^2)z^n \in V(\delta)$ *where* $\delta = \frac{8-12\beta+5\beta^2}{6-8\beta+3\beta^2}$.

Proof. Since $\sum_{n=2}^{\infty}(n-\beta)a_n \leq \beta - 1$, we have

$$\sum_{n=2}^{\infty}\left(\frac{n-\beta}{\beta-1}\right)^2 a_n^2 < \left\{\sum_{n=2}^{\infty}\frac{n-\beta}{\beta-1}a_n\right\}^2 \leq 1. \tag{10}$$

Simillarly

$$\sum_{n=2}^{\infty}\left(\frac{n-\beta}{\beta-1}\right)^2 b_n^2 \leq 1. \tag{11}$$

From (10) and (11), we get

$$\sum_{n=2}^{\infty}\frac{1}{2}\left(\frac{n-\beta}{\beta-1}\right)^2(a_n^2 + b_n^2) \leq 1. \tag{12}$$

We want to find $\delta = \delta(\beta)$ such that

$$\sum_{n=2}^{\infty}\left(\frac{n-\delta}{\delta-1}\right)(a_n^2 + b_n^2) \leq 1. \tag{13}$$

Comparing this with (12), we see that (13) will be satisfied if

$$\frac{n-\delta}{\delta-1} \le \frac{1}{2}(\frac{n-\beta}{\beta-1})^2.$$

or

$$\delta \ge \frac{(\frac{n-\beta}{\beta-1})^2 + 2n}{(\frac{n-\beta}{\beta-1})^2 + 2} \tag{14}$$

Right-hand side of (14) is a decreasing function of $n (n = 2, 3, \ldots)$. Let $n = 2$,

$$\delta \ge \frac{8 - 12\beta + 5\beta^2}{6 - 8\beta + 3\beta^2}.$$

The result is sharp for the function

$$f(z) = g(z) = z + \frac{\beta - 1}{2 - \beta} z^2.$$

Note that if in Theorem 4, we let $\gamma = \beta$, we get the same value for δ as here.

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