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A NOTE ON HILBERT TYPE INEQUALITY

B. G. PACHPATTE

Abstract. In the present note we establish a new Hilbert type inequality involving sequences of real numbers. An integral analogue of the main result is also given.

1. Introduction

One of the many fundamental mathematical discoveries of Hilbert is the following inequality (see [2, p.226]).

Theorem A. *If $p > 1$, $p' = p/(p-1)$ and $\Sigma a_m^p \leq A$, $\Sigma b_n^{p'} \leq B$, the summations running from 1 to ∞, then*

$$\Sigma\Sigma \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} A^{1/p} B^{1/p'}, \tag{1}$$

unless the sequence $\{a_m\}$ or $\{b_n\}$ is null.

The integral analogue of the inequality given in Theorem A can be stated as follows (see [2, p. 226]).

Theorem B. *If $p > 1$, $p' = p/(p-1)$ and*

$$\int_0^\infty f^p(x)dx \leq F, \int_0^\infty g^{p'}(y)dy \leq G,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} F^{1/p} G^{1/p'}, \tag{2}$$

unless $f \equiv 0$ or $g \equiv 0$.

A great deal of attention has been given to the above inequalities and many papers dealing with numerious variants, generalizations and extensions have appeared in the literature, see [1-5, 7,10] and the references given therein. The main purpose of the present

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note is to establish a new inequality of Hilbert type involving sequences of nonnegative real numbers. The integral analogue of our main result is also given. The analysis used in the proofs is elementary and our results provide new estimates on this type of inequalities.

2. Main results

Our main result is given in the following theorem.

Theorem 1. *Let $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2 \ldots, r$ with $a_0 = b_0 = 0$ and let $\{p_m\}$ and $\{q_n\}$ be two positive sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$, where k, r are natural numbers and define $P_m = \sum_{s=1}^{m} p_s$ and $Q_n = \sum_{t=1}^{n} q_t$. Let ϕ and ψ be two real-valued nonnegative, convex and submultiplicative functions defined on $R_+ = [0, \infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(a_m)\psi(b_n)}{m+n} \le M(k,r)\Big(\sum_{m=1}^{k}(k-m+1)[p_m\phi(\frac{\nabla a_m}{p_m})]^2\Big)^{1/2}$$
$$\times\Big(\sum_{n=1}^{r}(r-n+1)[q_n\psi(\frac{\nabla b_n}{q_n})]^2\Big)^{1/2}, \tag{3}$$

where

$$M(k,r) = \frac{1}{2}\Big(\sum_{m=1}^{k}[\frac{\phi(P_m)}{P_m}]^2\Big)^{1/2}\Big(\sum_{n=1}^{r}[\frac{\psi(Q_n)}{Q_n}]^2\Big)^{1/2}, \tag{4}$$

and $\nabla a_m = a_m - a_{m-1}$, $\nabla b_n = b_n - b_{n-1}$.

Proof. From the hypotheses, it is easy to observe that the following identities hold

$$a_m = \sum_{s=1}^{m}\nabla a_s, \quad m = 1, 2, \ldots, k, \tag{5}$$

$$b_n = \sum_{t=1}^{n}\nabla b_t, \quad n = 1, 2, \ldots, r. \tag{6}$$

From (5) and (6) and using Jensen's inequality (see [6]) we observe that

$$\phi(a_m) = \phi\Bigg(\frac{P_m\sum\limits_{s=1}^{m} p_s\frac{\nabla a_s}{p_s}}{\sum\limits_{s=1}^{m} p_s}\Bigg)$$
$$\le \phi(P_m)\phi\Bigg(\frac{\sum\limits_{s=1}^{m} p_s\frac{\nabla a_s}{p_s}}{\sum\limits_{s=1}^{m} p_s}\Bigg)$$

$$\leq \phi(P_m)\frac{\sum_{s=1}^{m} p_s\phi(\frac{\nabla a_s}{p_s})}{P_m}, \tag{7}$$

and similarly

$$\psi(b_n) \leq \psi(Q_n)\frac{\sum_{t=1}^{n} q_t\psi(\frac{\nabla b_t}{q_t})}{Q_n}. \tag{8}$$

From (7) and (8) and using Schwarz inequality and the elementary inequality $c^{1/2}d^{1/2} \leq \frac{1}{2}(c+d)$, (for c, d nonnegative reals) we observe that

$$\begin{aligned}
\phi(a_m)\psi(b_n) &\leq [\frac{\phi(P_m)}{P_m}\sum_{s=1}^{m} p_s\phi(\frac{\nabla a_s}{p_s})][\frac{\psi(Q_n)}{Q_n}\sum_{t=1}^{n} q_t\psi(\frac{\nabla b_t}{q_t})] \\
&\leq [\frac{\phi(P_m)}{P_m}]\{m\}^{1/2}\{\sum_{s=1}^{m}[p_s\phi(\frac{\nabla a_s}{p_s})]^2\}^{1/2} \\
&\quad\times[\frac{\psi(Q_n)}{Q_n}]\{n\}^{1/2}\{\sum_{t=1}^{n}[q_t\psi(\frac{\nabla b_t}{q_t})]^2\}^{1/2} \\
&\leq \frac{1}{2}(m+n)[[\frac{\phi(P_m)}{P_m}]\{\sum_{s=1}^{m}[p_s\phi(\frac{\nabla a_s}{p_s})]^2\}^{1/2}] \\
&\quad\times[[\frac{\psi(Q_n)}{Q_n}]\{\sum_{t=1}^{n}[q_t\psi(\frac{\nabla b_t}{q_t})]^2\}^{1/2}].
\end{aligned} \tag{9}$$

Dividing both sides of (10) by $m+n$ and then taking the sum over n from 1 to r first and then the sum over m from 1 to k and using Schwarz inequality and then interchanging the order of summations (see [8,9]) we observe that

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(a_m)\psi(b_n)}{m+n} &\leq \frac{1}{2}\sum_{m=1}^{k}[[\frac{\phi(P_m)}{P_m}]\{\sum_{s=1}^{m}[p_s\phi(\frac{\nabla a_s}{p_s})]^2\}^{1/2}] \\
&\quad\times\sum_{n=1}^{r}[[\frac{\psi(Q_n)}{Q_n}]\{\sum_{t=1}^{n}[q_t\psi(\frac{\nabla b_t}{q_t})]^2\}^{1/2}] \\
&\leq \frac{1}{2}(\sum_{m=1}^{k}[\frac{\phi(P_m)}{P_m}]^2)^{1/2}(\sum_{m=1}^{k}(\sum_{s=1}^{m}[p_s\phi(\frac{\nabla a_s}{p_s})]^2))^{1/2} \\
&\quad\times(\sum_{n=1}^{r}[\frac{\phi(Q_n)}{Q_n}]^2)^{1/2}(\sum_{n=1}^{r}(\sum_{t=1}^{n}[q_t\psi(\frac{\nabla b_t}{q_t})]^2))^{1/2} \\
&= M(k,r)(\sum_{s=1}^{k}[p_s\phi(\frac{\nabla a_s}{p_s})]^2(\sum_{m=s}^{k}1))^{1/2}(\sum_{t=1}^{r}[q_t\psi(\frac{\nabla b_t}{q_t})]^2(\sum_{n=t}^{r}1))^{1/2} \\
&= M(k,r)(\sum_{s=1}^{k}[p_s\phi(\frac{\nabla a_s}{p_s})]^2(k-s+1))^{1/2}
\end{aligned}$$

$$\times(\sum_{t=1}^{r}[q_t\psi(\frac{\nabla b_t}{q_t})]^2(r-t+1))^{1/2}$$

$$= M(k,r)(\sum_{m=1}^{k}(k-m+1)[p_m\phi(\frac{\nabla a_m}{p_m})]^2)^{1/2}$$

$$\times(\sum_{n=1}^{r}(r-n+1)[q_n\psi(\frac{\nabla b_n}{q_n})]^2)^{1/2}.$$

This completes the proof.

An integral analogue of Theorem 1 is given in the following theorem.

Theorem 2. *Let $f \in C^1[[0,x),R_+]$, $g \in C^1[[0,y),R_+]$ with $f(0) = g(0) = 0$ and let $p(\sigma)$ and $q(\tau)$ be two positive functions defined for $\sigma \in [0,x)$ and $\tau \in [0,y)$, and $P(s) = \int_0^s p(\sigma)d\sigma$ and $Q(t) = \int_0^t q(\tau)d\tau$ for $s \in [0,x)$ and $t \in [0,y)$, where x,y are positive real numbers. Let ϕ and ψ be as in Theorem 1. Then*

$$\int_0^x\int_0^y \frac{\phi(f(s)\psi(g(t))}{s+t}dsdt \le L(x,y)(\int_0^x (x-s)[p(s)\phi(\frac{f'(s)}{p(s)})]^2ds)^{1/2}$$
$$\times(\int_0^y (y-t)[q(t)\psi(\frac{g'(t)}{q(t)})]^2dt)^{1/2}, \tag{10}$$

where

$$L(x,y) = \frac{1}{2}(\int_0^x [\frac{\phi(P(s))}{P(s)}]^2ds)^{1/2}(\int_0^y [\frac{\psi(Q(t))}{Q(t)}]^2dt)^{1/2}, \tag{11}$$

and ′ denotes the derivative of a function.

Proof. From the hypotheses we have the following identities

$$f(s) = \int_0^s f'(\sigma)d\sigma, \quad s \in [0,x), \tag{12}$$

$$g(t) = \int_0^t g'(\tau)d\tau, \quad t \in [0,y). \tag{13}$$

From (12) and (13) and using the Jensen's integral inequality (see [6]) we observe that

$$\phi(f(s)) = \phi(\frac{P(s)\int_0^s p(\sigma)\frac{f'(\sigma)}{p(\sigma)}d\sigma}{\int_0^s p(\sigma)d\sigma}) \le \phi(P(s))\phi(\frac{\int_0^s p(\sigma)\frac{f'(\sigma)}{p(\sigma)}d\sigma}{\int_0^s p(\sigma)d\sigma})$$
$$\le [\frac{\phi(P(s))}{P(s)}]\int_0^s p(\sigma)\phi(\frac{f'(\sigma)}{p(\sigma)})d\sigma, \tag{14}$$

and similarly

$$\psi(g(t)) \le [\frac{\psi(Q(t))}{Q(t)}]\int_0^t q(\tau)\psi(\frac{g'(\tau)}{q(\tau)})d\tau. \tag{15}$$

From (14) and (15) and using Schwarz inequality and the elementary inequality $c^{1/2}d^{1/2} \leq \frac{1}{2}(c+d)$, (for c, d nonnegative reals) we observe that

$$
\begin{aligned}
\phi(f(s))\psi(g(t)) &\leq [[\frac{\phi(P(s))}{P(s)}]\int_0^s p(\sigma)\phi(\frac{f'(\sigma)}{p(\sigma)})d\sigma][[\frac{\psi(Q(t))}{Q(t)}]\int_0^t q(\tau)\psi(\frac{g'(\tau)}{q(\tau)})d\tau] \\
&\leq [[\frac{\phi(P(s))}{P(s)}]\{s\}^{1/2}\{\int_0^s [p(\sigma)\phi(\frac{f'(\sigma)}{p(\sigma)})]^2 d\sigma\}^{1/2}] \\
&\quad \times [[\frac{\psi(Q(t))}{Q(t)}]\{t\}^{1/2}\{\int_0^t [q(\tau)\psi(\frac{g'(\tau)}{q(\tau)})]^2 d\tau\}^{1/2}] \\
&\leq \frac{1}{2}(s+t)[[\frac{\phi(P(s))}{P(s)}]\{\int_0^s [p(\sigma)\phi(\frac{f'(\sigma)}{p(\sigma)})]^2 d\sigma\}^{1/2}] \\
&\quad \times [[\frac{\psi(Q(t))}{Q(t)}]\{\int_0^t [q(\tau)\psi(\frac{g'(\tau)}{q(\tau)})]^2 d\tau\}^{1/2}]. \qquad (16)
\end{aligned}
$$

Dividing both sides of (16) by $s+t$ and then integrating over t from 0 to y first and then integrating the resulting inequality over s from 0 to x and using Schwarz inequality we observe that

$$
\begin{aligned}
\int_0^x\int_0^y \frac{\phi(f(s))\psi(g(t))}{s+t}dsdt &\leq \frac{1}{2}\int_0^x [[\frac{\phi(P(s))}{P(s)}]\{\int_0^s [p(\sigma)\phi(\frac{f'(\sigma)}{p(\sigma)})]^2 d\sigma\}^{1/2}]ds \\
&\quad \times \int_0^y [[\frac{\psi(Q(t))}{Q(t)}]\{\int_0^t [q(\tau)\psi(\frac{g'(\tau)}{q(\tau)})]^2 d\tau\}^{1/2}]dt \\
&\leq \frac{1}{2}(\int_0^x [\frac{\phi(P(s))}{P(s)}]^2 ds)^{1/2}(\int_0^x(\int_0^s [p(\sigma)\phi(\frac{f'(\sigma)}{p(\sigma)})]^2 d\sigma)ds)^{1/2} \\
&\quad \times (\int_0^y [\frac{\psi(Q(t))}{Q(t)}]^2 dt)^{1/2}(\int_0^y(\int_0^t [q(\tau)\psi(\frac{g'(\tau)}{q(\tau)})]^2 d\tau)ds)^{1/2} \\
&= L(x,y)(\int_0^x (x-s)[p(s)\phi(\frac{f'(s)}{p(s)})]^2 ds)^{1/2} \\
&\quad \times (\int_0^y (y-t)[q(t)\psi(\frac{g'(t)}{q(t)})]^2 dt)^{1/2}.
\end{aligned}
$$

This completes the proof.

It is easy to observe that the bounds obtained on the right sides in (3) and (10) are different from those of the bounds given in (1) and (2) by taking $\phi(u) = u$ and $\psi(v) = v$. Further, if we apply the elementary inequality $c^{1/2}d^{1/2} \leq \frac{1}{2}(c+d)$, (for c, d nonnegative reals) on the right sides of (3) and (10) we get respectively the following inequalities

$$
\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(a_m)\psi(b_n)}{m+n} &\leq \frac{1}{2}M(k,r)[\sum_{m=1}^{k}(k-m+1)[p_m\phi(\frac{\nabla a_m}{p_m})]^2 \\
&\quad + \sum_{n=1}^{r}(r-n+1)[q_n\psi(\frac{\nabla b_n)}{q_n})]^2], \qquad (17)
\end{aligned}
$$

and

$$\int_0^x \int_0^y \frac{\phi(f(s))\psi(g(t))}{s+t} dsdt \leq \frac{1}{2}L(x,y)[\int_0^x (x-s)[p(s)\phi(\frac{f'(s)}{p(s)})]^2 ds + \int_0^y (y-t)[q(t)\psi(\frac{g'(t)}{q(t)})]^2 dt]. \quad (18)$$

For a number of generalizations, variants and extensions of the inequalities given in Theorems A and B, we refer the interested readers to [1-5,7,10] and the references given therein.

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Department of Mathematics, Marathwada University, Aurangabad 431 004, (Maharashtra) India.