

EXISTENCE OF INTEGRAL MANIFOLDS FOR IMPULSIVE DIFFERENTIAL EQUATION WITH SMALL PARAMETER

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Abstract. Applying the Banach fixed-point theorem, sufficient conditions for the existence of bounded manifolds of Lipschitz type are found.

1. Introduction

Many papers and monographs devoted to impulsive differential equations have appeared in the last 10 years (for example [1], [3], [4] and [5], where further references can be found). On the other hand, the theory of the differential equations with small parameter makes rapid development because it can be successfully applied in physics, mechanics etc. By generalizing of some results from [6] the present paper is trying to connect these two important fields of study (see also [2]).

2. Statement of the problem

Consider the impulsive differential equation:

$$\begin{aligned} \frac{dx}{dt} &= F(t, x, y, \varepsilon) \quad (t \geq 0, t \neq t_n) \\ \Delta x /_{t=t_n} &= Q_n(x(t_n), y(t_n), \varepsilon) \quad (n \in N) \end{aligned} \tag{1}$$

$$\begin{aligned} \varepsilon \frac{dy}{dt} &= A(t, x)y + G(t, x, y, \varepsilon) \quad (t \geq 0, t \neq t_n) \\ \varepsilon \Delta y /_{t=t_n} &= P_n(x)y(t_n) + R_n(x(t_n), y(t_n), \varepsilon) \quad (n \in N) \end{aligned} \tag{1'}$$

where $x \in R^k$, $y \in R^m$, $\varepsilon \geq 0$ and $A(t, x)$, P_n are (m, m) -matrices ($t \geq 0, n \in N$).

We introduce the following conditions:

H1. There exist numbers $l > 0$ and $m \in N$ such that each interval on R^+ with length l contains not more than m points of the sequence $\{t_n\}_{n=0}^{+\infty}$, $t_{i+1} > t_i$ and $\inf_{i \in N} (t_{i+1} - t_i) = p > 0$.

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H2. There exist $\varepsilon_0 \geq 0$ and $C \geq 0$ such that

$$\begin{aligned} \|F(t, x_1, y_1, \varepsilon) - F(t, x_2, y_2, \varepsilon)\| &\leq C(\|x_1 - x_2\| + \|y_1 - y_2\|), \\ \|Q_n(x_1, y_1, \varepsilon) - Q_n(x_2, y_2, \varepsilon)\| &\leq C(\|x_1 - x_2\| + \|y_1 - y_2\|), \\ \|Q_n(x, y, \varepsilon)\| &\leq C_p, \\ |A(t, x_1) - A(t, x_2)| &\leq C(|t_1 - t_2| + \|x_1 - x_2\|), \\ \|G(t, x_1, y_1, \varepsilon) - G(t, x_2, y_2, \varepsilon)\| &\leq C(\|x_1 - x_2\| + \|y_1 - y_2\|)((\max\{\|y_1\|, \|y_2\|\})^2 + \varepsilon^2), \\ \|G(t, x, y, \varepsilon)\| &\leq C(\|y\|^2 + \varepsilon), \\ \|R_n(x_1, y_1, \varepsilon) - R_n(x_2, y_2, \varepsilon)\| &\leq C((\max\{\|y_1\|, \|y_2\|\})^2 + \varepsilon^2)(\|x_1 - x_2\| + \|y_1 - y_2\|), \\ \|R_n(x, y, \varepsilon)\| &\leq C(\|y\|^4 + \varepsilon^4), \\ \|P_n(x_1) - P_n(x_2)\| &\leq C\|x_1 - x_2\| \end{aligned}$$

for any $x, x_1, x_2 \in R^k$, $y, y_1, y_2 \in R^m$, $t \geq 0$, $\varepsilon \in [0, \varepsilon_0]$ and $n \in N$.

H3. The eigenvalues of $A(t, x)$ satisfy the inequation

$$\operatorname{Re}\lambda_i(t, x) \leq -2\gamma \quad i = 1, \dots, m$$

for some $\gamma > 0$ and $\prod_{i=1}^{\infty} \|I + P_i\| < \infty$.

Under the assumptions H1, H2 and H3 we are going to prove the existence of integral manifold for equation (1,1').

3. Main results

Theorem 1. *Let the conditions H1, H2, and H3 hold. Then exist $\varepsilon^* \in (0, \varepsilon_0]$ and positive constants D and Δ such that for every $\varepsilon \in (0, \varepsilon^*]$ the equation (1,1') has an unique integral manifold $y = H^*(t, x, \varepsilon)$ with the following properties:*

1. $\|H^*(t, x, \varepsilon)\| \leq D$.
2. $\|H^*(t, x_1, \varepsilon) - H^*(t, x_2, \varepsilon)\| \leq \Delta\|x_1 - x_2\|$.
3. $\frac{dx}{dt} = F(t, x, H^*(t, x, \varepsilon), \varepsilon) \quad (t \geq 0, t \neq t_n)$
 $\Delta x / t=t_n = Q_n(x(t_n), H^*(t_n, x(t_n), \varepsilon), \varepsilon) \quad (n \in N)$.

In advance we will prove the following lemma:

Lemma 1. *Let $A(t)$ be bounded $m \times m$ matrix for every $t \geq 0$ and Lipschitz with constant q . Besides let the eigenvalues of $A(t)$ satisfy the inequation*

$$\operatorname{Re}\lambda_i(t) \leq -2\gamma \quad i = 1, \dots, m$$

for some $\gamma > 0$ and $\prod_{i=1}^{\infty} \|I + P_i\| < \infty$. Then exist positive constants L and ε_1 such that the Cauchy operator $W(\tau, t, \varepsilon)$ of the equation

$$\begin{aligned} \varepsilon \frac{dz}{dt} &= A(t)z \quad (t \geq 0, t \neq t_n) \\ \varepsilon \Delta z / t=t_n &= P_n(z) \quad (n \in N) \end{aligned} \tag{2}$$

satisfies the inequation

$$\|W(\tau, t, \varepsilon)\| \leq Ke^{-\frac{\gamma}{\varepsilon}(\tau-t)}$$

for any $0 \leq t \leq \tau < \infty$, $\varepsilon \in [0, \varepsilon_1]$ and a constant $K > 0$.

Proof. At first let prove (see [6]) that the Cauchy operator $U(\tau, t, \varepsilon)$ of the equation

$$\varepsilon \frac{dz}{dt} = A(t)z$$

satisfies the estimate

$$\|U(\tau, t, \varepsilon)\| \leq Le^{-\frac{\gamma}{\varepsilon}(\tau-t)} e^{-[\frac{\gamma}{2\varepsilon} - \sqrt{\frac{Lq}{\varepsilon}}](\tau-t)} \tag{3}$$

for some $L \geq 0$ and $\varepsilon \in [0, \varepsilon'_1]$. For any fixed $u \in R$ the following equation holds true:

$$U(\tau, t, \varepsilon) = e^{\frac{1}{\varepsilon}A(u)(\tau-t)} + \frac{1}{\varepsilon} \int_t^\tau e^{\frac{1}{\varepsilon}A(u)(\tau-s)} [A(s) - A(u)]U(s, t, \varepsilon)ds$$

Since $\|A(t)\|$ is bounded and $\text{Re}\lambda_i(t) \leq -2\gamma$ for some $L \geq 0$, we have

$$\|e^{\frac{1}{\varepsilon}A(u)(\tau-t)}\| \leq Le^{-\frac{3\gamma}{3\varepsilon}(\tau-t)}$$

for all $0 \leq t \leq \tau < \infty$, $\varepsilon \in [0, \varepsilon_0]$ and $u \in R$. Now let $u = \tau$, then

$$\|U(\tau, t, \varepsilon)\| = Le^{-\frac{3\gamma}{2\varepsilon}(\tau-t)} + \frac{Lq}{\varepsilon} \int_t^\tau e^{\frac{3\gamma}{2\varepsilon}(s-t)} \|U(s, t, \varepsilon)\|(\tau - s)ds.$$

Hence

$$\|U(\tau, t, \varepsilon)\|e^{\frac{3\gamma}{2\varepsilon}(\tau-t)} = L + \frac{Lq}{\varepsilon} \int_t^\tau e^{\frac{3\gamma}{2\varepsilon}(s-t)} \|U(s, t, \varepsilon)\|(\tau - s)ds.$$

Let $\nu(s) = e^{\frac{3\gamma}{2\varepsilon}(s-t)} \|U(s, t, \varepsilon)\|$. It is not hard to check that

$$\nu(\tau) \leq \nu_0(\tau) = \frac{L}{2} [e^{\sqrt{\frac{Lq}{\varepsilon}}(\tau-t)} + e^{-\sqrt{\frac{Lq}{\varepsilon}}(\tau-t)}].$$

This means that

$$\|U(\tau, t, \varepsilon)\| \leq \frac{L}{2} e^{-\frac{\gamma}{\varepsilon}(\tau-t)} [e^{-(\frac{\gamma}{2\varepsilon} - \sqrt{\frac{Lq}{\varepsilon}})(\tau-t)} + e^{-(\frac{\gamma}{2\varepsilon} + \sqrt{\frac{Lq}{\varepsilon}})(\tau-t)}].$$

Consequently for $\varepsilon'_1 = \frac{\gamma^2}{4Lq}$ and $\varepsilon \in [0, \varepsilon'_1]$ (3) holds true.

Consider now the impulsive differential equation (2). In [3] is shown that

$$W(\tau, t, \varepsilon) = U(\tau, t_{j+k}, \varepsilon)(I + P_{j+k})U(t_{j+k}, t_{j+k-1}, \varepsilon)(I + P_{j+k-1}) \\ \times \cdots \times (I + P_j)U(t_j, t, \varepsilon) \quad (0 \leq t \leq t_j < \cdots < t_{j+k} \leq \tau < \infty).$$

Set

$$\mu = \frac{\gamma}{2\varepsilon} - \sqrt{\frac{Lq}{\varepsilon}}.$$

Hence

$$\begin{aligned} \|W(\tau, t, \varepsilon)\| &\leq L e^{-\frac{\gamma}{\varepsilon}(\tau-t)} e^{-\mu(\tau-t)} \left(\prod_{i=0}^k L \|I + P_{j+i}\| \right) \\ &\leq L^3 e^{-\frac{\gamma}{\varepsilon}(\tau-t)} e^{\mu(\tau-t) + (k-1) \ln L + \sum_{i=0}^k \ln \|I + P_{i+j}\|} \end{aligned}$$

There exists ε_1 such that for $\varepsilon \in [0, \varepsilon_1]$ the estimate

$$\ln L \leq \left(\frac{\gamma}{2\varepsilon} - \sqrt{\frac{Lq}{\varepsilon}} \right) p$$

holds true. Hence

$$(k - 1) \ln L < (k - 1) \mu p < \mu(\tau - t)$$

and consequently

$$\|W(\tau, t, \varepsilon)\| \leq L e^{-\frac{\gamma}{\varepsilon}(\tau-t)} \prod_{i=1}^k \|I + P_{j+i}\| \leq L e^{-\frac{\gamma}{\varepsilon}(\tau-t)} \prod_{i=1}^{\infty} \|I + P_i\| \leq K e^{-\frac{\gamma}{\varepsilon}(\tau-t)}.$$

Let $M(\Delta, D)$ be the set of all functions $f : \Omega \rightarrow R^m$, where $\Omega = \{t \geq 0, x \in R^k, \varepsilon \in [0, \varepsilon_0]\}$, f are continuous for $t \neq t_i$, left-hand continuous in $t_i (i \in N)$, and satisfy the following conditions

$$\begin{aligned} \|f(t, x, \varepsilon) - f(t, x', \varepsilon)\| &\leq \Delta \|x - x'\|, \\ \|f(t, x, \varepsilon)\| &\leq D \end{aligned}$$

Consider the function

$$\rho(f, f^*) = \sup_{(t, x, \varepsilon) \in \Omega} \|f(t, x, \varepsilon) - f^*(t, x, \varepsilon)\| = \|f - f^*\|$$

It is not hard to see that $(M(\Delta, D), \rho)$ is a complete metric space.

Proof of Theorem 1. Let $H \in M(\Delta, d)$. Then using H2 we obtain

$$\begin{aligned} &\|F(t, x, H(t, x, \varepsilon), \varepsilon) - F(t, x^*, H(t, x^*, \varepsilon), \varepsilon)\| \\ &\leq C(\|x - x^*\| + \|H(t, x, \varepsilon) - H(t, x^*, \varepsilon)\|) \leq C(1 + \Delta)\|x - x^*\| \quad (x, x^* \in R^k, t \geq 0) \end{aligned}$$

Similarly

$$\|Q_n(x, H(t, x, \varepsilon), \varepsilon) - Q_n(x^*, H(t, x^*, \varepsilon), \varepsilon)\| \leq C(1 + \Delta)\|x - x^*\| \quad (x, x^* \in R^n, n \in N)$$

The equation

$$\frac{dx}{dt} = F(t, x, H(t, x, \varepsilon), \varepsilon) \quad (t \geq 0, t \neq t_n) \tag{4}$$

$$\Delta x / t=t_n = Q_n(x(t_n), H(t_n, x(t_n), \varepsilon), \varepsilon) \quad (n \in N) \tag{4'}$$

$$x(0) = \xi \tag{4''}$$

has an unique solution $\varphi(t)$ for any $\xi \in R^m$. Let fix a number $\tau \geq 0$ and denote $\varphi(\tau) = x$. Since later on we use the abbreviation $\phi(t) = \Phi(t, \tau, x, \varepsilon/H)$. Notice that the function $\varphi(t)$ is a solution of the equation

$$\varphi(t) = x - \int_t^\tau F(s, \varphi(s), H(s, \varphi(s), \varepsilon), \varepsilon) ds - \sum_{t < t_i < \tau} Q_i(\varphi(t_i), H(t_i, \varphi(t_i), \varepsilon), \varepsilon) \quad (0 \leq t \leq \tau)$$

If $\varphi(t)$ and $\varphi^*(t)$ are two solutions of (4)-(4'') and $\phi(\tau) = x$, $\varphi^*(\tau) = x^*$, then the following inequality is valid

$$\begin{aligned} & \|\varphi(t) - \varphi^*(t)\| \leq \|x - x^*\| \\ & + \int_t^\tau \|F(s, \varphi(s), H(s, \varphi(s), \varepsilon), \varepsilon) - F(s, \varphi^*(s), H^*(s, \varphi^*(s), \varepsilon), \varepsilon)\| ds \\ & + \sum_{t < t_i < \tau} \|Q_i(\varphi(t_i), H(t, \varphi(t_i), \varepsilon), \varepsilon) - Q_n(\varphi^*(t_i), H^*(t_i, \varphi^*(t_i), \varepsilon), \varepsilon)\| \\ & \leq \|x - x^*\| + C \int_t^\tau ((1 + \Delta)\|\varphi(s) - \varphi^*(s)\| + \| |H - H^*| \|) ds \\ & + C \sum_{t < t_i < \tau} ((1 + \Delta)\|\varphi(t_i) - \varphi^*(t_i)\| + \| |H - H^*| \|) \quad (0 \leq t \leq \tau) \end{aligned}$$

Applying the integral inequality from [3], we obtain the estimate

$$\|\varphi(t) - \varphi^*(t)\| \leq (\|x - x^*\| + \frac{\| |H - H^*| \|}{1 + \Delta}) (1 + a)^{k(t, \tau)} - \frac{\| |H - H^*| \|}{1 + \Delta}$$

where $k(t, \tau)$ is the number of the impulse points in $[t, \tau)$, and a is an arbitrary constant fulfilling $a \geq C(1 + \Delta)$.

Consider the equation

$$\begin{aligned} \varepsilon \frac{dy}{dt} &= A(t, \varphi(t))y + G(t, \varphi(t), y, \varepsilon) \quad (t \geq 0, t \neq t_n) \\ \varepsilon \Delta y /_{t=t_n} &= P_n(\varphi(t_n))y + R_n(\varphi(t_n), y(t_n)) \quad (n \in N) \end{aligned}$$

At first we are going to prove that the Cauchy matrix $U = U_\varphi(\tau, t, \varepsilon)$ of the equation

$$\begin{aligned} \varepsilon \frac{dy}{dt} &= A(t, \varphi(t))y \quad (t \geq 0, t \neq t_n) \\ \varepsilon \Delta y /_{t=t_n} &= P_n(\varphi(t_n))y \quad (n \in N) \end{aligned}$$

for some $K \geq 0$ satisfies the inequality

$$\|U - \varphi(\tau, t, \varepsilon)\| \leq K e^{-\frac{K}{\varepsilon}(\tau-t)} \quad (0 \leq \tau \leq t < \infty) \tag{5}$$

Using H_2 and H_1 we obtain

$$\|A(t, \varphi(t))\| \leq C,$$

and for $t_2 > t_1$:

$$\begin{aligned} & \|A(t_2, \varphi(t_2)) - A(t_1, \varphi(t_1))\| \leq C(t_2 - t_1 + \|\varphi(t_2) - \varphi(t_1)\|) \\ & \leq C(t_2 - t_1 + \int_{t_1}^{t_2} \|F(s, \varphi(s), H(s, \varphi(s), \varepsilon), \varepsilon)\| ds + \sum_{t_1 < t_i < t_2} \|Q_i(\varphi(t_i), \varphi(t_i), \varepsilon), \varepsilon)\|) \\ & \leq C(t_2 - t_1 + C(t_2 - t_1)) + C \sum_{t_1 < t_i < t_2} p \leq C(1 + 3C)(t_2 - t_1). \end{aligned}$$

From this estimates and from H3 follows that Lemma 1 can be applied and consequently exist $K \geq 0$ and $\varepsilon_1 \geq 0$ such that (5) is satisfied in $[0, \varepsilon_1]$.

If $\varphi(t)$ and $\varphi^*(t)$ are two solution of (4)-(4''), then $U_\varphi(\tau, t, \varepsilon) - U_{\varphi^*}(\tau, t, \varepsilon)$ satisfies the equation

$$\begin{aligned} \varepsilon \frac{d}{ds} (U_\varphi(\tau, s, \varepsilon) - U_{\varphi^*}(\tau, s, \varepsilon)) &= A(s, \varphi(s))(U_\varphi(\tau, s, \varepsilon) - U_{\varphi^*}(\tau, s, \varepsilon)) \\ &\quad + (A(s, \varphi(s)) - A(s, \varphi^*(s)))U_{\varphi^*}(s) (U_\varphi(\tau, s, \varepsilon) - U_{\varphi^*}(\tau, s, \varepsilon)). \\ \varepsilon \Delta (U_\varphi(\tau, t_n, \varepsilon) - U_{\varphi^*}(\tau, t_n, \varepsilon)) / t=t_n &= P_n(\varphi(t_n))(U_\varphi(\tau, t_n, \varepsilon) - U_{\varphi^*}(\tau, t_n, \varepsilon)) \\ &\quad + (P_n(\varphi(t_n)) - P_n(\varphi^*(t_n)))U_{\varphi^*}(\tau, t_n, \varepsilon) \end{aligned}$$

Hence

$$\begin{aligned} & \|U_\varphi(\tau, t, \varepsilon) - U_{\varphi^*}(\tau, t, \varepsilon)\| \\ & \leq \frac{1}{\varepsilon} \int_t^\tau \|U_\varphi(\tau, u, \varepsilon)(A(u, \varphi(u)) - A(u, \varphi^*(u)))U_{\varphi^*}(u, t, \varepsilon)\| du \\ & \quad + \frac{1}{\varepsilon} \sum_{t < t_n < \tau} \|U_\varphi(\tau, t_n, \varepsilon)(P_n(\varphi^*(t_n)))U_{\varphi^*}(t_n, t, \varepsilon)\| \\ & \leq \frac{K^2 C}{\varepsilon} e^{-\frac{\gamma}{\varepsilon}(\tau-t)} \int_t^\tau \|\varphi(u) - \varphi^*(u)\| du + \frac{K^2 C F}{\varepsilon} e^{-\frac{\gamma}{\varepsilon}(\tau-t)} \sum_{t, t_n < \tau} \|\varphi(t_n) - \varphi^*(t_n)\| \\ & \leq \frac{K^2 C}{\varepsilon} (\|x - x^*\| + \frac{\|H - H^*\|}{1 + \Delta}) e^{-\frac{\gamma}{\varepsilon}(\tau-1)} (\int_t^\tau (1 + a)^{k(u, \tau)} e^{a(\tau-u)} du \\ & \quad + \sum_{t < t_i < \tau} \|\varphi(t_n) - \varphi^*(t_n)\| \\ & \leq \frac{K^2}{\varepsilon} (\|x - x^*\| + \frac{\|H - H^*\|}{1 + \Delta}) e^{(a-\frac{\gamma}{\varepsilon})(\tau-t)} (\int_t^{t_k} (1 + a)^{n-k+1} e^{-a(t_i-t)} du \\ & \quad + \sum_{i=k}^{n-1} (1 + a)^{n-i} \int_{t_i}^{t_{i+1}} e^{-a(u-t)} du + \int_{t_n}^\tau e^{-a(u-t)} du + \sum_{i=k}^n (1 + a)^{n-i+1} e^{-a(t_i-t)}) \\ & \leq \frac{K^2 C}{\varepsilon} (\|x - x^*\| + \frac{\|H - H^*\|}{1 + \Delta}) e^{(a-\frac{\gamma}{\varepsilon})(\tau-t)} ((1 + a)^{n-k+1} \frac{1}{a} + a \sum_{i=k}^n (1 + a)^{n-1} e^{-a(t_i-t)}) \end{aligned}$$

$$\begin{aligned} &\leq \frac{K^2C}{\varepsilon} (\|x - x^*\| + \frac{\|H - H^*\|}{1 + \Delta}) e^{(a - \frac{\gamma}{\varepsilon})(\tau - t)} (1 + a)^{n-k} (1 + a)^{n-k} \\ &\quad \cdot (\frac{1 + a}{a} + am \sum_{i=0}^{\infty} (1 + a)^{-i} e^{-ai}) \end{aligned}$$

Notice that $(n - k)p \leq \tau - t$ and consequently

$$(1 + a)^{n-k} < e^{a(n-k) < e^{\frac{a}{p}(\tau - t)},}$$

hence

$$\begin{aligned} &\|U_{\varphi}(\tau, t, \varepsilon) - U_{\varphi^*}(\tau, t, \varepsilon)\| \\ &\leq \frac{K^2C}{\varepsilon} (\|x - x^*\| + \frac{\|H - H^*\|}{1 + \Delta}) e^{((1 + \frac{1}{p})a - \frac{\gamma}{\varepsilon})(\tau - t)} (\frac{1 + a}{a} + \frac{am}{1 - (1 + a)e^{-at}}) \end{aligned}$$

Now we are in position to define a mapping T for every $H \in M(\Delta, D)$.

$$\begin{aligned} T(H)(\tau, x, \varepsilon) &= \frac{1}{\varepsilon} \int_0^{\tau} U_{\varphi}(\tau, t, \varepsilon) G(t, \varphi(t), H(t, \varphi(t_0), \varepsilon), \varepsilon) dt \\ &\quad + \frac{1}{\varepsilon} \sum_{0 < t_n < \tau} U_{\varphi}(\tau, t_n, \varepsilon) R_n(\varphi(t_n), H(t_n), \varphi(t_n), \varepsilon), \varepsilon) \end{aligned} \tag{6}$$

First we show that T maps $M(\Delta, D)$ into $M(\Delta, D)$:

$$\begin{aligned} \|T(H)(\tau, x, \varepsilon)\| &\leq \frac{KC}{\varepsilon} (D^2 + \varepsilon^2) \int_0^{\tau} e^{-\frac{\gamma}{\varepsilon}(\tau - t)} dt + \frac{KC}{\varepsilon} (D^4 + \varepsilon^4) \sum_{0 < t_n < \tau} e^{-\frac{\gamma}{\varepsilon}(\tau - t_n)} \\ &\leq \frac{KC}{\gamma} (D^2 + \varepsilon^2) \int_0^{\tau} e^{-\frac{\gamma}{\varepsilon}(\tau - t)} dt + \frac{KC}{\varepsilon} (D^4 + \varepsilon^4) \frac{m}{1 - e^{-\frac{\gamma t}{\varepsilon}}}. \end{aligned}$$

Now we suppose that

$$(1 + \frac{1}{p})a - \frac{\gamma}{\varepsilon} < 0$$

Then

$$\begin{aligned} &\|T(H)(\tau, x, \varepsilon) - T(\bar{H})(\tau, \bar{x}, \varepsilon)\| \\ &\leq \frac{1}{\varepsilon} \int_0^{\tau} \|U_{\varphi}(\tau, t, \varepsilon) - U_{\bar{\varphi}}(\tau, t, \varepsilon)\| \|G(t, \bar{\varphi}(t), \bar{H}(t, \bar{\varphi}(t), \varepsilon), \varepsilon)\| dt \\ &\quad + \frac{1}{\varepsilon} \int_0^{\tau} \|U_{\varphi}(\tau, t, \varepsilon) - U_{\bar{\varphi}}(\tau, t, \varepsilon)\| \|G(t, \bar{\varphi}(t), \bar{H}(t, \bar{\varphi}(t), \varepsilon), \varepsilon)\| dt \\ &\quad + \sum_{0 < t_n < \tau} \|U_{\varphi}(\tau, t_n, \varepsilon)\| \|R_n(\varphi(t_n), H(t_n), H(t_n), \varphi(t_n), \varepsilon), \varepsilon \\ &\quad - R_n(\bar{\varphi}(t_n), \bar{H}(t_n), \bar{\varphi}(t_n), \varepsilon), \varepsilon)\| \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\varepsilon} \sum_{0 < t_n < \tau} \|U_\varphi(\tau, t_n, \varepsilon) - U_{\bar{\varphi}}(\tau, t, \varepsilon)\| \|R_n(\bar{\varphi}(t_n), \bar{H}(t_n, \bar{\varphi}(t_n), \varepsilon, \varepsilon))\| \\
& \leq \frac{K}{\varepsilon} (D^2 + \varepsilon^2)(1 + \Delta) (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \int_0^\tau (1 + a)^{k(t, \tau)} e^{(a - \frac{\gamma}{\varepsilon})(\tau - t)} dt \\
& + \sum_{\frac{K^2 C}{\varepsilon}} (D^2 + \varepsilon^2) (\frac{1 + a}{a} + \frac{am}{1 - (1 + a)e^{-at}}) (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \frac{1}{\gamma - \frac{p+1}{p} a \varepsilon} \\
& + \frac{K}{\varepsilon} (D^2 + \varepsilon^2)(1 + \Delta) (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \sum_{0 < t_n < \tau} (1 + a)^{k(t_n, t)} e^{(a - \frac{\gamma}{\varepsilon})(\tau - t_n)} \\
& + \frac{K^2 C}{\varepsilon^2} (D^4 + \varepsilon^4) (\frac{1 + a}{a} + \frac{am}{1 - (1 + a)e^{-at}}) (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \sum_{0 < t_n < \tau} e^{((1 + \frac{1}{p})a - \frac{\gamma}{\varepsilon})(\tau - t_n)} \\
& \leq \frac{Ka}{\gamma - a\varepsilon} (D^2 + \varepsilon^2)(1 + \Delta) (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \sum_{i=1}^n (1 + a)^{n+i-1} e^{(a - \frac{\gamma}{\varepsilon})(\tau - t_i)} \\
& + \frac{K^2 C}{\varepsilon} (D^2 + \varepsilon^2) (\frac{1 + a}{a} + \frac{am}{1 - (1 + a)e^{-at}}) (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \frac{1}{\gamma - \varepsilon \frac{p+1}{p} a} \\
& + \frac{K}{\varepsilon} (D^2 + \varepsilon^2)(1 + \Delta) (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \sum_{0 < t_n < \tau} e^{((1 + \frac{1}{p})a - \frac{\gamma}{\varepsilon})(\tau - t_n)} \\
& + \frac{K^2 C}{\varepsilon^2} (D^4 + \varepsilon^4) (\frac{1 + a}{a} + \frac{am}{1 - (1 + a)e^{-at}}) (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \frac{m}{1 - e^{((1 + \frac{1}{p})a - \frac{\gamma}{\varepsilon})l}} \\
& \leq (\|x - \bar{x}\| + \frac{\|H - \bar{H}\|}{1 + \Delta}) \left\{ \frac{Ka}{\gamma - a\varepsilon} (D^2 + \varepsilon^2)(1 + \Delta) \frac{m}{1 - e^{((1 + \frac{1}{p})a - \frac{\gamma}{\varepsilon})l}} \right. \\
& + \frac{K^2 C}{\varepsilon} (D^2 + \varepsilon^2) (\frac{1 + a}{a} + \frac{am}{1 - (1 + a)e^{-at}}) \frac{1}{\gamma - \varepsilon \frac{p+1}{p} a} \\
& + \frac{K}{\varepsilon} (D^2 + \varepsilon^2)(1 + \Delta) \frac{m}{1 - e^{((1 + \frac{1}{p})a - \frac{\gamma}{\varepsilon})l}} \\
& \left. + \frac{K^2 C}{\varepsilon^2} (D^4 + \varepsilon^4) (\frac{1 + a}{a} + \frac{am}{1 - (1 + a)e^{-at}}) \frac{m}{1 - e^{((1 + \frac{1}{p})a - \frac{\gamma}{\varepsilon})l}} \right\}
\end{aligned}$$

Now set $D = \varepsilon D_0$ and $\Delta = \varepsilon \Delta_0$. Hence

$$\|T(H)(\tau, x, \varepsilon)\| \leq \varepsilon \left(\frac{KC}{\gamma} (D_0^2 + 1) + \varepsilon KC (D_0^4 + 1) \frac{m}{1 - e^{-\frac{\gamma l}{\varepsilon}}} \right). \quad (7)$$

If $x = \bar{x}$ then

$$\|T(H)(\tau, x, \varepsilon) - T(\bar{H})(\tau, x, \varepsilon)\| \leq \frac{Ka\varepsilon}{\gamma - a\varepsilon} (D_0^2 + 1)(1 + \varepsilon \Delta) \frac{m}{1 - e^{((1 + \frac{1}{p})a - \gamma\varepsilon)l}}$$

$$\begin{aligned}
 &+K^2C\varepsilon(D_0^2+1)\left(\frac{1+a}{a}+\frac{am}{1-(1+a)e^{-al}}\right)-\frac{1}{\gamma-\varepsilon\frac{p+1}{p}a} \\
 &+K\varepsilon(D_0^2+1)(1+\varepsilon\Delta_0)\frac{m}{1-e^{((1+\frac{1}{p})a-\frac{\gamma}{\varepsilon})l}} \\
 &+K^2C\varepsilon(D_0^4)\left(\frac{1+a}{a}+\frac{am}{1-(1+a)e^{-al}}\right)\frac{m}{1-e^{((1+\frac{1}{p})a-\frac{\gamma}{\varepsilon})l}}\left\|\frac{H-\bar{H}}{1+\varepsilon\Delta_0}\right\|. \tag{8}
 \end{aligned}$$

If $H = \bar{H}$ then

$$\begin{aligned}
 \|T(H)(\tau, x, \varepsilon) - T(H)(\tau, \bar{x}, \varepsilon)\| &\leq \left\{ \frac{Ka\varepsilon^2}{\gamma - a\varepsilon} (D_0^2 + 1)(1 + \varepsilon\Delta_0) \frac{m}{1 - e^{((1+\frac{1}{p})a - \gamma\varepsilon)l}} \right. \\
 &+K^2C\varepsilon(D_0^2+1)\left(\frac{1+a}{a}+\frac{am}{1-(1+a)e^{-al}}\right)-\frac{1}{\gamma-\varepsilon\frac{p+1}{p}a} \\
 &+K\varepsilon(D_0^2+1)(1+\varepsilon\Delta_0)\frac{m}{1-e^{((1+\frac{1}{p})a-\frac{\gamma}{\varepsilon})l}} \\
 &\left.+K^2C\varepsilon^2(D_0^4)\left(\frac{1+a}{a}+\frac{am}{1-(1+a)e^{-al}}\right)\frac{m}{1-e^{((1+\frac{1}{p})a-\frac{\gamma}{\varepsilon})l}} \right\} \|x - \bar{x}\|. \tag{9}
 \end{aligned}$$

Set $a = \frac{\gamma p}{2\varepsilon(p+1)}$. There exists ε_3 such that for every $\varepsilon \in (0, \varepsilon_3]$ the inequation

$$C(1 + \varepsilon\Delta_0) \leq \frac{\gamma p}{2\varepsilon(p + 1)}$$

holds. Next we find D_0 such that inequation

$$\varepsilon^2\left(\frac{KC}{\gamma}(D_0^2+1)+\varepsilon KC(D_0^4+1)\frac{m}{1-e^{-\frac{\gamma l}{\varepsilon}}}\right) \leq \varepsilon D_0$$

holds true for every $\varepsilon \in (0, \varepsilon_3]$. Then from (7) follows that $\|T(H)(\tau, x, \varepsilon)\| \leq D$ in Ω . Finally we find Δ_0 such that inequation

$$\begin{aligned}
 &\varepsilon\frac{Kp}{p+2}(D_0^2+1)(1+\varepsilon\Delta_0)\frac{m}{1-e^{-\frac{\gamma l}{2\varepsilon}}}+K^2C\varepsilon(D_0^2+1)\left(\frac{2\varepsilon^2(p+1)+\gamma p\varepsilon}{\gamma(p+1)}\right. \\
 &+\frac{\gamma mp}{2(p+1)\left(1-\frac{\gamma p+2\varepsilon(p+1)}{2\varepsilon(p+1)}e^{-\frac{\gamma p}{2\varepsilon(p+1)}l}\right)}\frac{2}{\gamma}+K\varepsilon(D_0^2+1)(1+\varepsilon\Delta_0)\frac{m}{1-e^{-2\frac{\gamma}{\varepsilon}l}} \\
 &+K^2C\varepsilon(D_0^4+1)\left(\frac{2\varepsilon^2(p+1)+\gamma p\varepsilon}{\gamma(p+1)}\right. \\
 &\left.+\frac{\gamma mp}{2(p+1)\left(1-\frac{\gamma p+2\varepsilon(p+1)}{2\varepsilon(p+1)}e^{-\frac{\gamma p}{2\varepsilon(p+1)}l}\right)}\right)\left\}\frac{m}{1-e^{-2\frac{\gamma}{\varepsilon}l}} \leq \varepsilon\Delta_0
 \end{aligned}$$

holds true for every $\varepsilon \in (0, \varepsilon_3]$. Then from (9) follows that

$$\|T(H)(\tau, x, \varepsilon) - T(H)(\tau, \bar{x}, \varepsilon)\| \leq \Delta \|x - \bar{x}\|$$

in Ω and consequently $T(H) \in M(\Delta, D)$. Now let show that $T(H)$ is a contraction on $M(\Delta, D)$ providing a unique fixed point H^* . In fact, from (8) we receive the estimate

$$\begin{aligned} & \|T(H)(\tau, x, \varepsilon) - T(\bar{H})(\tau, x, \varepsilon)\| \leq \left\{ \varepsilon \frac{Kp}{p+2} (D_0^2 + 1)(1 + \varepsilon\Delta_0) \frac{m}{1 - e^{-\frac{\gamma}{2\varepsilon}l}} \right. \\ & + K^2 C\varepsilon (D_0^2 + 1) \left(\frac{2\varepsilon^2(p+1) + \gamma p\varepsilon}{\gamma(p+1)} + \frac{\gamma mp}{2(p+1)(1 - \frac{\gamma p + 2\varepsilon(p+1)}{2\varepsilon(p+1)} e^{-\frac{\gamma p}{2\varepsilon(p+1)}l})} \right) \frac{2}{\gamma} \\ & + K\varepsilon (D_0^2 + 1)(1 + \varepsilon\Delta_0) \frac{m}{1 - e^{-2\frac{\gamma}{\varepsilon}l}} + K^2 C\varepsilon (D_0^4 + 1) \left(\frac{2\varepsilon^2(p+1) + \gamma p\varepsilon}{\gamma(p+1)} \right. \\ & \left. + \frac{\gamma mp}{2(p+1)(1 - \frac{\gamma p + 2\varepsilon(p+1)}{2\varepsilon(p+1)} e^{-\frac{\gamma p}{2\varepsilon(p+1)}l})} \right) \frac{m}{1 - e^{-2\frac{\gamma}{\varepsilon}l}} \left. \right\} \frac{\|H - \bar{H}\|}{1 + \varepsilon\Delta_0} \\ & \leq \frac{\varepsilon\Delta_0}{1 + \varepsilon\Delta_0} \|H - \bar{H}\| \leq \frac{\varepsilon_3\Delta_0}{1 + \varepsilon_3\Delta_0} \|H - \bar{H}\|. \end{aligned}$$

This completes the proof.

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