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ON THE SPIN REAL PROJECTIVE BUNDLE

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Abstract. In this paper, we give a characterization of spin real projective bundles. We also construct special spin real projective bundls over real projective space.

Introduction

The motivation for studying spin real projective bundle comes from the existence (or non-existence) of positive scalar curvature on a given manifold with fundamental group π. Modifying a conjecture of Gromov and Lawson, Rosenberg conjectures (cf. [GrLa2], [Ro1], [Ro3]) that a connected spin manifold M of dimension $n \geq 5$ with fundamental group π admits a Riemannian metric of positive scalar curvature if and only if all KO_*-valued index obstructions associated to Dirac operators with coefficients in flat bundles vanishes. If M is a spin manifold, the indices of all the Dirac operators with coefficients in flat bundles turn out to be a single element $\alpha(M,u) \in \mathrm{KO}_n(\mathrm{C}_r^*\pi)$, where $\mathrm{C}_r^*(\pi)$ is the C^*-completion of the real group ring $\mathbb{R}\pi$ and $u: M \to B\pi$ is the classifying map of the universal covering $\widetilde{M} \to M$. It is known that the vanishing of the index α is necessary for existence of a positive scalar curvature metric on M (cf. [Ro2]). This has been proved to be a sufficient condition if π is the trivial group ([St1], Thm.A), an odd order cyclic group ([Ro2], Thm.1.3; [KwSc], Thm.1.8), $\mathbb{Z}/2$ ([RS], Thm.5.3), and more generally a finite group with periodic cohomology (cf. [BGS]). It turns out that $\alpha(M,u)$ depends only on the spin bordism class $[M,u] \in \Omega_n^{\mathrm{Spin}}(B\pi)$ and, hence, we have a homomorphism

$$\alpha\colon \Omega_n^{\mathrm{Spin}}(B\pi) \to \mathrm{KO}_n(\mathrm{C}_r^*\pi).$$

In fact, α can be factorized in the following way : (cf. [Ro2], [Ro3], [RoSt])

$$\Omega_n^{\mathrm{Spin}}(B\pi) \xrightarrow{D} \mathrm{ko}_n(B\pi) \xrightarrow{p} \mathrm{KO}_n(B\pi) \xrightarrow{A} \mathrm{KO}_n(C_r^*\pi).$$

Here the first map is induced from the orientation class D : MSpin $\to$ ko from the Thom spectrum to the connective real K-theory spectrum via Pontrjagin-Thom construction $\Omega_n^{\mathrm{Spin}}(B\pi) \cong \pi_n(\mathrm{MSpin} \wedge B\pi_+)$, p is the canonical map from connective to periodic

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KO-homology and A is the assembly map (cf. [Ro3]). It follows from results of Jung (cf. [Ju]) and Stolz (cf. [St1]) that whether M has a positive scalar curvature metric depends only on its image $D_*[M,u] \in \mathrm{ko}_n(B\pi)$; more precisely, M has a positive scalar curvature metric if and only if $D_*[M,u] = D_*[M',u'] \in \mathrm{ko}_n(B\pi)$ for some manifold M' which admits a positive scalar curvature metric.

This result is a significant improvement since the connective KO-theory group $\mathrm{ko}_n(B\pi)$ is much smaller than $\Omega_n^{\mathrm{Spin}}(B\pi)$ and it is much easier to find generators of the groups $\mathrm{ko}_n(B\pi)$ than of the bordism groups. Due to this result, calculating the connective real K-homology of $B\pi$ and representing every element in $\ker(A \circ p)$ by a positive scalar curvature manifold are two possible steps to study the Gromov-Lawson-Rosenberg Conjecture. Due to the fact that real projective spaces $\mathbb{RP}^n, n \equiv 3 \pmod 4$ are spin manifolds with positive scalar curvature, Rosenberg and Stolz showed that the images of the collection $\mathbb{RP}^n$, $n \equiv 3 \pmod 4$ under D_* known to generate $\ker(A \circ p)$ and thus proved the Gromov-Lawson-Rosenberg Conjective for $\pi = \mathbb{Z}/2$.

Note that we may regard real projective space $\mathbb{RP}^n$ as a real projective bundle of $(n+1)\epsilon$, $(n+1)$-dimensional real trivial vector bundle over a point.Here $(n+1)\epsilon$ means $\epsilon \oplus \epsilon \oplus \cdots \oplus \epsilon$, Whitney sum of $(n+1)$-copies of trivial line bundles. In general, we are interested in determining which real projective bundle $\mathbb{RP}(\alpha)$ of a real vector bundle α is spin and admits a Riemannian metric of positive scalar curvature.In fact, if the base space B of α is compact then $\mathbb{RP}(\alpha)$ always has a metric of positive scalar curvature due to the following observation, which is well known to experts in this field (cf. [GrLa1], [Mi], [Ro2] and [St1]).

> *Observation* Let $\pi : E \to B$ be a fiber bundle with fiber F and structure group G. If F is a compact manifold of positive scalar curvature, B is a compact manifold and G acts on F by isometries, then E also has a metric of positive scalar curvature.

In this paper, we study spin real projective bundle $\mathbb{RP}(\alpha)$ of a real vector bundle α and give the following characterization:

Theorem A. *Let α be a n-dimensional real vector bundle with projection map $\pi : E \to B$.*

(1) *Assume $n \geq 2$, $\mathbb{RP}(\alpha)$ is oriented if and only if*

$$\begin{cases} n \equiv 0 & \pmod 2 \\ w_1(\alpha) = w_1(B). \end{cases}$$

(2) *Assume $n \geq 3$, $\mathbb{RP}(\alpha)$ is spin if and only if*

$$\begin{cases} n \equiv 0 & \pmod 4 \\ w_1(\alpha) = 0 = w_1(B) \\ w_2(\alpha) = w_2(B). \end{cases}$$

Here w_1, w_2 mean first and second Stiefel Whitney classes.

Let L_0 denote the Hope line bundle over $\mathbb{RP}^n$ and let $\beta_{m,s;n}$ denote the real vector bundle $mL_0 \oplus s\epsilon$ over $\mathbb{RP}^n$. Then $\mathbb{RP}(\beta_{m,s;n})$ is a fiber bundle over $\mathbb{RP}^n$ with fiber $\mathbb{RP}^{m+s-1}$. Let $\bar{l}$ means the non-negative integers congruent to l (mod4).

Proposition B. *$\mathbb{RP}(\beta_{m,s;n})$ is spin if and only if*

$$(m,s;n) = (2,0;\bar{3}),\ (1,1;\bar{2}),\ (0,2;\bar{3}),(\bar{0},\bar{0};\bar{3}),(\bar{2},\bar{2};\bar{1}).$$

Proposition B shows spin real projective bundles $\mathbb{RP}(\beta_{m,s;n})$ over $\mathbb{RP}^n$ with fiber $\mathbb{RP}^{n_1}$ can be constructed if and only if

$$(n_1, n) = \begin{cases} (1,\bar{3}),\ (1,\bar{2}) \\ (\bar{3},\bar{1}),\ (\bar{3},\bar{3}). \end{cases}$$

Let L_1 denote the canonical line bundle over $\mathbb{RP}(\beta_{m,s;n})$ and let $\widetilde{L}_0$ denote the pull-back of Hopf line bundle L_0 by the projection map $p : \mathbb{RP}(\beta_{m,s;n}) \to \mathbb{RP}^n$. Using the similar construction, we can form real projective bundle $\mathbb{RP}(\beta_I)$ over $\mathbb{RP}(\beta_{m,s;n})$ with fiber $\mathbb{RP}^{n_2}$, where $I = (m_1, \ldots, m_4; m, s; n)$, $n_2 = \left(\sum_{i=1}^{4} m_i\right) - 1$ and β_I is the real vector bundle $m_1L_1 \otimes \widetilde{L}_0 \oplus m_2L_1 \oplus m_3\widetilde{L}_0 \oplus m_4\epsilon$ over $\mathbb{RP}(\beta_{m,s;n})$.

Proposition C. *Spin real projective bundle $\mathbb{RP}(\beta_I)$ can be constructed if and only if*

$$(n_2, n_1, n) = \begin{cases} (1,1,\bar{1}),\ \ (1,1,\bar{2}),\ \ (1,1,\bar{3}) \\ (1,\bar{2},\bar{1}),\ \ (1,\bar{2},\bar{3}) \\ (1,\bar{3},\mathit{all}) \\ (\bar{3},\bar{3},\mathit{all}),\ (\bar{3},\bar{1},\mathit{all}) \end{cases}$$

except for $(3,\bar{3},\bar{2})$, $(3,\bar{3},\bar{0})$.

1. Outline of the proof of Theorem A

1.1. H-structure

Let G be a Lie group. A principal G-bundle P is a bundle with a G-action on P preserving fibers whose restriction to a fiber F is free and transitive. An isomorphism $f : P \to P'$ between principal G-bundle is a fiber-preserving map which is G-equivalent. Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation and $P \to X$ principal G-bundle. The associated vector bundle is a vector bundle $P \times_G V := (P \times V)/G \to P/G = X$. Let E be an oriented vector bundle and let

$$\begin{aligned} \mathrm{O}(E) \quad &:= \{(v_1, \ldots, v_n, x) | \{v_1, \ldots, v_n\} \text{is an orthonormal basis of } E_x\} \\ \mathrm{SO}(E) &:= \{(v_1, \ldots, v_n, x) | \{v_1, \ldots, v_n\} \text{is an oriented orthonormal basis of } E_x\}. \end{aligned}$$

In fact, the principal $\mathrm{O}(n)$-bundle $\mathrm{O}(E)$ and the principal $\mathrm{SO}(n)$-bundle $\mathrm{SO}(E)$ are related by the $\mathrm{O}(n)$-bundle isomorphism $\mathrm{SO}(E) \times_{\mathrm{SO}(n)} \mathrm{O}(n) \cong \mathrm{O}(E)$.

Definition1.1.1. Let $\rho : H \to G$ be a representation. A H-structure on a principal G-bundle $P_G \to X$ is an H-bundle P_H together with an isomorphism

$$P_H \times_H G \cong P_G.$$

Remark 1.1.2. An orientation on a vector bundle E^n is a $\mathrm{SO}(n)$-structure on $\mathrm{O}(E)$.

Definition 1.1.3. A spin-structure on E is a $\mathrm{Spin}(n)$-structure on $\mathrm{O}(E)$.

The following characterization of orientation-structure and spin-structure on vector bundle is well known to experts in this field (cf. [LaMi]).

Theorem 1.1.4.

(1) *Vector bundle E is orientable if and only if* $w_1(E) = 0$.
(2) *Vector bundle E has a spin-structure if and only if* $w_1(E) = 0$, $w_2(E) = 0$.

Definition 1.1.5. A spin manifold is an oriented Riemannian manifold with a spin structure on its tangent bundle.

The Stiefel-Whitney classes $w_i(X)$ of a manifold X are defined to be the Stiefel-Whitney classes of its tangent bundle TX. Hence, we have the following.

Theorem 1.1.6.

(1) *Manifold X is orientable if and only if* $w_1(X) = 0$.
(2) *Manifold X has a spin-structure if and only if* $w_1(X) = 0, w_2(X) = 0$.

1.2. The Splitting Principle

Our proof of Theorem A then amounts to calculating first and second Stiefel Whitney classes of $\mathbb{RP}(\alpha)$. For this purpose, we need the splitting principle (cf. [BoTu]). Let α be a n-dimensional real vector bundle with projection map $\pi : E \to B$. There exists a manifold $F(E)$, called a split manifold of E, and a map $\sigma : F(E) \to B$ such that $\sigma^* E = l_1 \oplus \cdots \oplus l_n$ the pullback of E to $F(E)$ splits into a Whitney sum of line bundles and the homomorphism $\sigma^* : \mathrm{H}^*(B) \to \mathrm{H}^*(F(E))$ is injective.

> *The Splitting Principle* To prove a polynomial identity in the Stiefel-Whitney classes of real vector bundles, it suffices to prove it under the assumption that the vector bundles are Whitney sums of line bundles.

Let $p : \mathbb{RP}(\alpha) \to B$ denote the projectivization of the n-dimensional real vector bundle α. It is a fiber bundle whose fiber at b is the real projective space $\mathbb{RP}(E_b) = \mathbb{RP}^{n-1}$ of all lines in E_b. $p^*\alpha$ contains a line bundle L, called canonical line bundle, defined tautologically at a line $l \subset E_b$ to be l. We have the splitting $p^*\alpha = L \oplus L^\perp$. Note that $\mathbb{RP}^{n-1}$ can be regard as $\mathbb{RP}(n\epsilon)$ real projective bundle of n-dimensional trivial bundle $n\epsilon$ over point, the tangent bundle of $\mathbb{RP}^{n-1}$ is stably isomorphic to the bundle $\mathrm{Hom}(L_0, L_0^\perp)$, where L_0 is the Hopf line bundle over $\mathbb{RP}^{n-1}$ and $p^*(n\epsilon) = L_0 \oplus L_0^\perp$. The tangent bundle of $\mathbb{RP}(\alpha)$ is stably isomorphic to $p^*TB \oplus \mathrm{Hom}(L, L^\perp)$. Due to the fact that $\mathrm{Hom}(L, L)$ has a nowhere-vanishing cross section, $\mathrm{Hom}(L, L)$ is trivial. Hence we have the following stable isomorphism

$$\begin{aligned} T\mathbb{RP}(\alpha) &\simeq_s p^*TB \oplus \mathrm{Hom}(L, L^\perp) \\ &\simeq_s p^*TB \oplus \mathrm{Hom}(L, L^\perp) \oplus \mathrm{Hom}(L, L^\perp) \\ &\cong p^*TB \oplus \mathrm{Hom}(L, L \oplus L^\perp) \\ &\cong p^*TB \oplus L \otimes p^*\alpha \end{aligned}$$

Using the splitting principle, we may assume $\alpha = l_1 \oplus \cdots \oplus l_n$. By abuse of notation we write $p^*\alpha = l_1 \oplus \cdots \oplus l_n$ for the pullback of α. Using the fact that p^* is injective, we write $w(B)$ for $p^*(w(B))$.It follows from the Whitney Product Formula, the tatal Stiefel-Whitney class of $\mathbb{RP}(\alpha)$ can calculated as follows.

$$\begin{aligned} w(\mathbb{RP}(\alpha)) &= w(p^*TB)w(L \otimes p^*\alpha) \\ &= w(B)w(L \otimes l_1 \oplus \cdots L \otimes l_n) \\ &= w(B)w(L \otimes l_1) \cdots w(L \otimes l_n) \\ &= w(B)(1 + y + x_1) \cdots (1 + y + x_n), \end{aligned}$$

where $y = w_1(L)$, $x_i = w_1(l_i)$, $i = 1, \ldots, n$.

1.3. Cohomology of $\mathbb{RP}(\alpha)$

Leray-Hirsch Theorem 1.3.1. *Let E be a fiber bundle over B with fiber F. If there is a global cohomology classes $e_1, \ldots, e_r$ on E which when restricted to each fiber freely generate the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(B)$ with basis $e_1, \ldots, e_r$.*

Since the restriction of the canonical line bundle L to a fiber $\mathbb{RP}(E_b)$ is the Hopf line bundle L_0 of the projective space $\mathbb{RP}(E_b)$, by the naturality property of the Stiefel-Witney class, $w_1(L_0)$ is the restriction of $y = w_1(L)$ to $\mathbb{RP}(E_b)$. Hence the cohomology classes $1, y, \ldots, y^{n-1}$ are global classes on $\mathbb{RP}(\alpha)$ which when restricted to each fiber $\mathbb{RP}(E_b)$ freely generate the cohomology of the fiber. The Leray-Hirsch Theorem implies the cohomology of $\mathbb{RP}(\alpha)$ is a free module over $\mathrm{H}^*(B)$ with basis $1, y, \ldots, y^{n-1}$.

Proposition 1.3.2.

$$\mathrm{H}^*(\mathbb{RP}(\alpha)) = \mathrm{H}^*(B)[y]/(y^n + w_1(\alpha)y^{n-1} + \cdots + w_n(\alpha)).$$

Proof. Since L is a subbundle of $p^*(\alpha)$, $\mathrm{Hom}(L, p^*(\alpha))$ has a nowhere-vanishing cross section and hence, $w_n(\mathrm{Hom}(L, p^*(\alpha)) = 0$. By direction computation,

$$\begin{aligned}
0 &= w_n(L \otimes p^*(\alpha)) \\
&= w_n(L \otimes (l_1 \oplus \cdots \oplus l_n)) \\
&= w_n(L \otimes l_1 \oplus \cdots \oplus L \otimes l_n) \\
&= w_1(L \otimes l_1) \cdots w_1(L \otimes l_n) \\
&= (y + x_1) \cdots (y + x_n) \\
&= \sum_{i=0}^{n} y^{n-i} \sigma_i(x_1, \dots, x_n),
\end{aligned}$$

where $\sigma_i(x_1, \dots, x_n)$ is the i-th elementary symmetric function of $x_1, \dots, x_n$. Using the assumption $\alpha = l_1 \oplus \cdots \oplus l_n$, $w_i(\alpha) = \sigma_i(x_1, \dots, x_n)$ and hence complete the proof.

It follows from the fact that $w_i(\alpha) = \sigma_i(x_1, \dots, x_n), i = 1, \dots, n$,

$$\begin{aligned}
w(\mathbb{RP}(\alpha)) &= w(B)(1 + y + x_1) \cdots (1 + y + x_n) \\
&= w(B)(1 + (ny + w_1(\alpha)) + (\frac{n(n-1)}{2}y^2 + (n-1)w_1(\alpha)y + w_2(\alpha)) + \cdots) \\
&= 1 + (w_1(B) + ny + w_1(\alpha)) + (w_2(B) + w_1(B)(ny + w_1(\alpha)) \\
&\quad + (\frac{n(n-1)}{2}y^2 + (n-1)w_1(\alpha)y + w_2(\alpha)) + \cdots,
\end{aligned}$$

and hence

$$\begin{cases} w_1(\mathbb{RP}(\alpha)) = w_1(B) + ny + w_1(\alpha) \\ w_2(\mathbb{RP}(\alpha)) = w_2(B) + w_1(B)(ny + w_1(\alpha)) + (\frac{n(n-1)}{2}y^2 + (n-1)w_1(\alpha)y + w_2(\alpha)). \end{cases}$$

Therefore, $\mathbb{RP}(\alpha)$ is oriented if and only if $0 = w_1(\mathbb{RP}(\alpha)) = w_1(B) + ny + w_1(\alpha)$, or equivalently,

$$\begin{cases} n \equiv 0 & (\mathrm{mod} 2) \\ w_1(B) = w_1(\alpha), \end{cases}$$

provided $n \geq 2$. Similarly, for $n \geq 3$, $\mathbb{RP}(\alpha)$ is spin if and only if

$$\begin{cases} 0 = w_1(\mathbb{RP}(\alpha)) = w_1(B) + ny + w_1(\alpha) \\ 0 = w_2(\mathbb{RP}(\alpha)) = w_2(B) + w_1(B)(ny + w_1(\alpha)) + (\frac{n(n-1)}{2}y^2 + (n-1)w_1(\alpha)y + w_2(\alpha)), \end{cases}$$

or equivalently,

$$\begin{cases} n \equiv 0 & (\mathrm{mod} 2) \\ w_1(B) = w_1(\alpha) \\ 0 = (w_2(B) + w_1(B)w_1(\alpha) + w_2(\alpha)) + (n-1)w_1(\alpha)y + \frac{n(n-1)}{2}y^2, \end{cases}$$

or equivalently,

$$\begin{cases} n \equiv 0 & \pmod 4 \\ w_1(B) = 0 = w_1(\alpha) \\ w_2(B) = w_2(\alpha)). \end{cases}$$

This completes the proof of Theorem A.

2. Construction of spin real projective bundle over $\mathbb{RP}^n$

2.1. Construction of $\mathbb{RP}(\beta_{m,s;n})$

Let L_0 denote the Hope line bundle over $\mathbb{RP}^n$ and let $\beta_{m,s;n}$ denote the real vector bundle $mL_0 \oplus s\epsilon$ over $\mathbb{RP}^n$ and L_1 the canonical line bundle over $\mathbb{RP}(\beta_{m,s;n})$. Then

$$w(\mathbb{RP}(\beta_{m,s;n})) = (1+y_0)^{n+1}(1+y_o+y_1)^m(1+y_1)^s,$$

where $y_0 = w_1(L_0)$, $y_1 = w_1(L_1)$.

2.1.1. $\mathbb{RP}(\beta_{m,s;n})$, $m+s \geq 3$

For $m+s \geq 3$, $\mathbb{RP}(\beta_{m,s;n})$ is spin if and only if

$$\begin{cases} m+s \equiv 0 & \pmod 4 \\ w_1(\mathbb{RP}^n) = 0 = w_1(\beta_{m,s;n}) \\ w_2(\mathbb{RP}^n) = w_2(\beta_{m,s;n}). \end{cases}$$

Note that $w(\mathbb{RP}^n) = (1+y_0)^{n+1} = 1+(n+1)y_0 + \frac{(n+1)n}{2}y_0^2 + \cdots$ and $w(\beta_{m,s;n}) = (1+y_0)^m = 1+my_0+\frac{m(m-1)}{2}y_0^2+\cdots$. Hence, for $m+s\geq 3$, $\mathbb{RP}(\beta_{m,s;n})$ is spin if and only if

$$\begin{cases} m+s \equiv 0 & \pmod 4 \\ n+1 \equiv 0 & \pmod 2 \\ m \equiv 0 & \pmod 2 \\ n+1 \equiv m & \pmod 4, \end{cases}$$

or equivalently,

$$m \equiv s \equiv n+1 \equiv 0,2 \pmod 4,$$

or equivalently,

$$(m,s;n) = (\bar{0},\bar{0},\bar{3}),\ (\bar{2},\bar{2};\bar{1}),$$

where $\bar{l}$ means the non-negative integers congruent to l (mod4).

2.1.2. $\mathbb{RP}(\beta_{m,s;n})$, $m+s=2$

For the case $m+s=2$, we have the relation $(y_0+y_1)^m y_1^s = 0$ coming from $w_2(\mathrm{Hom}(L_1, p^*\beta_{m,s;n})) = 0$. It follows from $w(\mathbb{RP}(\beta_{m,s;n}) = (1+y_0)^{n+1}(1+y_0+y_1)^m(1+y_1)^s = (1+y_0)^{n+1}(1+my_0)$ that $\mathbb{RP}(\beta_{m,s;n}$ is spin if and only if

$$\begin{cases} m+s=2 & \\ n+1 \equiv m & \pmod 2 \\ \frac{(n+1)n}{2} + (n+1)m \equiv 0 & \pmod 2, \end{cases}$$

or equivalently,

$$\begin{cases} n+1 \equiv 0 \pmod 2 & \text{for } (m,s) = (0,2),\ (2,0) \\ n+1 \equiv 3 \pmod 2 & \text{for } (m,s) = (1,1), \end{cases}$$

or equivalently,

$$(m,s,;n) = (0,2;\bar{3}),\ (1,1;\bar{2}),\ (2,0;\bar{3}).$$

This finishes the proof of Proposition B.

2.2. Construction of $\mathbb{RP}(\beta_I)$

Let $\widetilde{L}_0$ denote the pull-back of Hopf line bundle L_0 by the projection map p : $\mathbb{RP}(\beta_{m,s;n}) \to \mathbb{RP}^n$. Using the similar construction, we can form real projective bundle $\mathbb{RP}(\beta_I)$ over $\mathbb{RP}(\beta_{m,s;n})$ with fiber $\mathbb{RP}^{n_2}$, where $I = (m_1,\ldots,m_4;m,s;n)$, $n_2 = \left(\sum_{i=1}^{4} m_i\right) - 1$ and β_I is the real vector bundle $m_1 L_1 \otimes \widetilde{L}_0 \oplus m_2 L_1 \oplus m_3 \widetilde{L}_0 \oplus m_4 \epsilon$ over $\mathbb{RP}(\beta_{m,s;n})$. Let L_2 denote the canonical line bundle over$\mathbb{RP}(\beta_I)$. Then the tatal Stiefel-Whitney class of $\mathbb{RP}(\beta_I)$ is

$$\begin{aligned} w(\mathbb{RP}(\beta_I)) = (1+y_0)^{n+1}(1+y_0+y_1)^m(1+y_1)^s \\ (1+y_0+y_1+y_2)^{m_1}(1+y_1+y_2)^{m_2}(1+y_0+y_2)^{m_3}(1+y_2)^{m_4}, \end{aligned}$$

where $y_2 = w_1(L_2)$.

2.2.1. $\mathbb{RP}(\beta_I)$, $\sum_{i=1}^{4} m_i \geq 3$ and $m+s \geq 3$

For the case $\sum_{i=1}^{4} m_i \geq 3$ and $m+s \geq 3$, $\mathbb{RP}(\beta_I)$ is spin if and only if

$$\begin{cases} \sum_{i=1}^{4} m_i \equiv 0 & \pmod 4 \\ w_1(\mathbb{RP}(\beta_{m,s;n})) = 0 = w_1(\beta_I) & \\ w_2(\mathbb{RP}(\beta_{m,s;n})) = w_2(\beta_I). & \end{cases}$$

Note that

$$\begin{cases} w(\mathbb{RP}(\beta_{m,s;n})) = (1+y_0)^{n+1}(1+y_0+y_1)^m(1+y_1)^s \\ w((\beta_I) \qquad\quad = (1+y_0+y_1)^{m_1}(1+y_1)^{m_2}(1+y_0)^{m_3}, \end{cases}$$

$\mathbb{RP}(\beta_{m,s;n})$ is spin if and only if

$$\begin{cases} \sum_{i=1}^{4} m_i \equiv 0 & \pmod 4 \\ n+1+m \equiv 0 & \pmod 2 \\ m+s \equiv 0 & \pmod 2 \\ m_1+m_3 \equiv 0 & \pmod 2 \\ m_1+m_2 \equiv 0 & \pmod 2 \\ \binom{n+1}{2}+\binom{m}{2}+(n+1)m \equiv \binom{m_1}{2}+\binom{m_3}{2}+m_1m_3 & \pmod 2 \\ \binom{m}{2}+\binom{s}{2}+ms \equiv \binom{m_1}{2}+\binom{m_2}{2}+m_1m_3 & \pmod 2 \\ ms \equiv m_1m_3 & \pmod 2, \end{cases}$$

or equivalently,

$$\begin{cases} \sum_{i=1}^{4} m_i \equiv 0 & \pmod 4 \\ n+1 \equiv m \equiv s & \pmod 2 \\ m_1 \equiv m_2 \equiv m_3 & \pmod 2 \\ \binom{n+1}{2}+\binom{m}{2}+(n+1)m \equiv \binom{m_1}{2}+\binom{m_3}{2}+m_1m_3 & \pmod 2 \\ \binom{m}{2}+\binom{s}{2}+ms \equiv \binom{m_1}{2}+\binom{m_2}{2}+m_1m_3 & \pmod 2 \\ ms \equiv m_1m_3 & \pmod 2. \end{cases}$$

In the case $m_1 \equiv m_2 \equiv m_3 \equiv 0 \pmod 2$, we have

$$\begin{cases} \sum_{i=1}^{4} m_i \equiv 0 & \pmod 4 \\ m_i \equiv 0, \ i=1,2,3,4 & \pmod 2 \\ n+1 \equiv m \equiv s \equiv 0 & \pmod 2 \\ \binom{n+1}{2}+\binom{m}{2} \equiv \binom{m_1}{2}+\binom{m_3}{2} & \pmod 2 \\ \binom{m}{2}+\binom{s}{2} \equiv \binom{m_1}{2}+\binom{m_2}{2} & \pmod 2, \end{cases}$$

or equivalently,

$$\left\{\begin{array}{l} \begin{cases} m_1+m_2 \equiv 0 \equiv m_3+m_4 & \pmod 4 \\ m_i \equiv 0, \ i=1,2,3,4 & \pmod 2 \qquad \text{for } m+s \equiv 0 \pmod 4 \\ n+1+m \equiv m_1+m_3 & \pmod 4 \end{cases} \\ \begin{cases} m_1+m_2 \equiv 2 \equiv m_3+m_4 & \pmod 4 \\ m_i \equiv 0, \ i=1,2,3,4 & \pmod 2 \qquad \text{for } m+s \equiv 2 \pmod 4 \\ n+1+m \equiv m_1+m_3 & \pmod 4 \end{cases} \end{array}\right.$$

In this case, $(m_1, m_2, m_3, m_4; m, s; n+1)$ can be presented as follows.

m_1	m_2	m_3	m_4	m	s	$n+1$	(n_2, n_1, n)
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$(\bar{3}, \bar{3}, \bar{3})$
$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$(\bar{3}, \bar{3}, \bar{1})$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$(\bar{3}, \bar{3}, \bar{1})$
$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$([\bar{3}]_7, \bar{3}, \bar{3})$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$(\bar{3}, \bar{3}, \bar{1})$
$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$(\bar{3}, \bar{3}, \bar{3})$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$(\bar{3}, \bar{3}, \bar{3})$
$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$([\bar{3}]_7, \bar{3}, \bar{1})$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$(\bar{3}, [\bar{1}]_5, \bar{1})$
$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$(\bar{3}, [\bar{1}]_5, \bar{3})$
$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$(\bar{3}, [\bar{1}]_5, \bar{3})$
$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$(\bar{3}, [\bar{1}]_5, \bar{1})$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$(\bar{3}, [\bar{1}]_5, \bar{3})$
$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$(\bar{3}, [\bar{1}]_5, \bar{1})$
$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$(\bar{3}, [\bar{1}]_5, \bar{1})$
$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$(\bar{3}, [\bar{1}]_5, \bar{3})$,

where $[\bar{l}]_r$ means the positigers $\geq r$ congruent to l.

In the other case $m_1 \equiv m_2 \equiv m_3 \equiv 1 \pmod{2}$, we have

$$\begin{cases} \sum_{i=1}^{4} m_i \equiv 0 & \pmod{4} \\ m_i \equiv 1,\ i = 1,2,3,4 & \pmod{2} \\ n+1 \equiv m \equiv s \equiv 1 & \pmod{2} \\ \binom{n+1}{2} + \binom{m}{2} \equiv \binom{m_1}{2} + \binom{m_3}{2} & \pmod{2} \\ \binom{m}{2} + \binom{s}{2} \equiv \binom{m_1}{2} + \binom{m_2}{2} & \pmod{2}, \end{cases}$$

or equivalently,

$$\begin{cases} \begin{cases} m_1 + m_2 \equiv 0 \equiv m_3 + m_4 & \pmod{4} \\ m_i \equiv 1,\ i = 1,2,3,4 & \pmod{2} \\ n+1+m \equiv m_1 + m_3 & \pmod{4} \end{cases} & \text{for } m + s \equiv 0 \pmod{4} \\ \begin{cases} m_1 + m_2 \equiv 2 \equiv m_3 + m_4 & \pmod{4} \\ m_i \equiv 1,\ i = 1,2,3,4 & \pmod{2} \\ n+1+m \equiv m_1 + m_3 & \pmod{4} \end{cases} & \text{for } m + s \equiv 2 \pmod{4} \end{cases}$$

In this case, $(m_1, m_2, m_3, m_4; m, s; n+1)$ can be presented in a similar way as above by replacing $\bar{0}$ by $\bar{1}$ and $\bar{2}$ by $\bar{3}$.

m_1	m_2	m_3	m_4	m	s	$n+1$	(n_2, n_1, n)
$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$(\bar{3}, [\bar{1}]_5, \bar{0})$
$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$([\bar{3}]_7, [\bar{1}]_5, \bar{2})$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$([\bar{3}]_7, [\bar{1}]_5, \bar{2})$
$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$([\bar{3}]_{11}, [\bar{1}]_5, \bar{0})$
$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$(\bar{3}, [\bar{1}]_5, \bar{2})$
$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$([\bar{3}]_7, [\bar{1}]_5, \bar{0})$
$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$([\bar{3}]_7, [\bar{1}]_5, \bar{0})$
$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$([\bar{3}]_{11}, [\bar{1}]_5, \bar{2})$
$\bar{3}$	$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{3}$	$([\bar{3}]_7, \bar{3}, \bar{2})$
$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$([\bar{3}]_7, \bar{3}, \bar{0})$
$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$([\bar{3}]_7, \bar{3}, \bar{0})$
$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{3}$	$([\bar{3}]_7, \bar{3}, \bar{2})$
$\bar{3}$	$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{1}$	$([\bar{3}]_7, \bar{3}, \bar{0})$
$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{3}$	$\bar{3}$	$([\bar{3}]_7, \bar{3}, \bar{2})$
$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	$\bar{3}$	$([\bar{3}]_7, \bar{3}, \bar{2})$
$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{3}$	$\bar{1}$	$\bar{3}$	$\bar{1}$	$([\bar{3}]_7, \bar{3}, \bar{0})$.

2.2.2. $\mathbb{RP}(\beta_I)$, $\sum_{i=1}^{4} m_i \geq 3$ and $m+s=2$

For the case $\sum_{i=1}^{4} m_i \geq 3$ and $m+s=2$,we have the relation $(y_0+y_1)^m y_1^s$ coming from $w_2(\mathrm{Hom}(L_1, p^*\beta_{m,s;n})) = 0$. It follows from

$$\begin{aligned}
w(\mathbb{RP}(\beta_I)) &= (1+y_0)^{n+1}(1+y_0+y_1)^m(1+y_1)^s \\
&\quad (1+y_0+y_1+y_2)^{m_1}(1+y_1+y_2)^{m_2}(1+y_0+y_2)^{m_3}(1+y_2)^{m_4} \\
&= (1+y_0)^{n+1}(1+my_0) \\
&\quad (1+y_0+y_1+y_2)^{m_1}(1+y_1+y_2)^{m_2}(1+y_0+y_2)^{m_3}(1+y_2)^{m_4} \\
&= \begin{cases} (1+y_0)^{n+1}(1+y_0+y_1+y_2)^{m_1}(1+y_1+y_2)^{m_2}(1+y_0+y_2)^{m_3}(1+y_2)^{m_4} \\ \qquad \text{for } (m,s)=(2,0),\ (0,2) \\ (1+y_0)^{n+2}(1+y_0+y_1+y_2)^{m_1}(1+y_1+y_2)^{m_2}(1+y_0+y_2)^{m_3}(1+y_2)^{m_4} \\ \qquad \text{for } (m,s)=(1,1). \end{cases}
\end{aligned}$$

In the case $(m,s)=(2,0),\ (0,2)$, $\mathbb{RP}(\beta_I)$ is spin if and only if

$$\begin{cases}
\sum_{i=1}^{4} m_i \equiv 0 & (\mathrm{mod}4) \\
n+1 \equiv 0 & (\mathrm{mod}2) \\
m_1+m_3 \equiv 0 & (\mathrm{mod}2) \\
m_1+m_2 \equiv 0 & (\mathrm{mod}2) \\
\binom{n+1}{2} \equiv \binom{m_1}{2} + \binom{m_3}{2} + m_1m_3 & (\mathrm{mod}2) \\
\binom{m_1}{2} + \binom{m_2}{2} + m_1m_2 \equiv 0 & (\mathrm{mod}2) \\
m_1m_3 \equiv 0 & (\mathrm{mod}2),
\end{cases}$$

or equivalently,

$$\begin{cases} \sum_{i=1}^{4} m_i \equiv 0 & (\text{mod}4) \\ m_i \equiv n+1 \equiv 0, \quad i=1,2,3,4 & (\text{mod}2) \\ n+1 \equiv m_1+m_3 & (\text{mod}4) \\ m_1+m_2 \equiv 0 & (\text{mod}4). \end{cases}$$

In this case, $(m_1, m_2, m_3, m_4; n+1)$ can be presented as follows.

m_1	m_2	m_3	m_4	$n+1$	(n_2, n_1, n)
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$(\bar{3}, 1, \bar{3})$
$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$(\bar{3}, 1, \bar{1})$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$(\bar{3}, 1, \bar{1})$
$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$([\bar{3}]_7, 1, \bar{3})$.

For the other case $(m, s) = (1, 1)$, we have

m_1	m_2	m_3	m_4	$n+2$	(n_2, n_1, n)
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$(\bar{3}, 1, \bar{2})$
$\bar{0}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$(\bar{3}, 1, \bar{0})$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{0}$	$\bar{2}$	$(\bar{3}, 1, \bar{0})$
$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{0}$	$([\bar{3}]_7, 1, \bar{2})$.

Hence, for $\sum_{i=1}^{4} m_i \geq 3$, spin real projective bundle $\mathbb{RP}(\beta_I)$ can be constructed if and only if $(n_2, n_1, n) = (\bar{3}, \bar{3}, \text{all})$, $(\bar{3}, \bar{1}, \text{all})$ except for $(3, \bar{3}, \bar{2})$, $(3, \bar{3}, \bar{0})$. This proves the half part of the Proposition C.

2.2.3. $\mathbb{RP}(\beta_I)$, $\sum_{i=1}^{4} m_i = 2$

For the case $\sum_{i=1}^{4} m_i = 2$, we have the relation $(y_0 + y_1 + y_2)^{m_1}(y_1 + y_2)^{m_2}(y_0 + y_2)^{m_3}(y_2)^{m_4}$ coming from $w_2(\text{Hom}(L_2, p^*\beta_I)) = 0$. It follows from

$$\begin{aligned} w(\mathbb{RP}(\beta_I)) &= (1+y_0)^{n+1}(1+y_0+y_1)^m(1+y_1)^s \\ &\quad (1+y_0+y_1+y_2)^{m_1}(1+y_1+y_2)^{m_2}(1+y_0+y_2)^{m_3}(1+y_2)^{m_4} \\ &= (1+y_0)^{n+1}(1+y_0+y_1)^m(1+y_1)^s(1+(m_1+m_3)y_0+(m_1+m_2)y_1) \\ &= \begin{cases} (1+y_0)^{n+1}(1+y_0+y_1)^m(1+y_1)^s & \text{oif } m_i = 2 \text{ for some } i = 1,2,3,4 \\ (1+y_0)^{n+2}(1+y_0+y_1)^m(1+y_1)^s & \text{if } m_1 = m_2 = 1, \text{ or } m_3 = m_4 = 1 \\ (1+y_0)^{n+1}(1+y_0+y_1)^{m+1}m(1+y_1)^s & \text{if } m_1 = m_3 = 1, \text{ or } m_2 = m_4 = 1 \\ (1+y_0)^{n+1}(1+y_0+y_1)^m(1+y_1)^{s+1} & \text{if } m_1 = m_4 = 1, \text{ or } m_2 = m_3 = 1. \end{cases} \end{aligned}$$

that spin real projective bundle $\mathbb{RP}(\beta_I)$ can be constructed if and only if

$$(n_2, n_1, n) = \begin{cases} (1,1,\bar{3}), \ (1,1,\bar{2}), \ (1,\bar{3},\bar{1}), \ (1,\bar{3},\bar{3}) \\ (1,1,\bar{2}), \ (1,1,\bar{1}), \ (1,\bar{3},\bar{0}), \ (1,\bar{3},\bar{2}) \\ (1,\bar{2},\bar{1}), \ (1,\bar{2},\bar{3}), \end{cases}$$

or equivalently,

$$(n_2, n_1, n) = (1,1,\bar{1}),\ (1,1,\bar{2}),\ (1,1,\bar{3}),\ (1,\bar{2},\bar{1}),\ (1,\bar{2},\bar{3}),\ (1,\bar{3},\text{all}).$$

This completes the proof of Proposition C.

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