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PARTIAL SUMS OF A CLASS OF UNIVALENT FUNCTIONS

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Abstract. We investigate the sequence of partial sums f_n of univalent functions f for which $\mathrm{Re} f' > 0$ in the unit disk. Radii of univalence for f_n are tracked as a function of n.

1. Introduction

Denote by S the family of functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ that are analytic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$ and by R the subfamily for which $\mathrm{Re} f'(z) > 0$, $z \in \Delta$. In [2], MacGregor investigated the sequence of partial sums $f_n(z) = z+\sum_{k=2}^{n} a_k z^k$ for $f \in R$. He showed that Re $f_n'(z) > 0$, $|z| < 1/2$, and that the radius of univalence for f_n is $1/2$. The result is sharp only when $n = 2$.

In this note, we track the radius of univalence of f_n, $f \in R$, as a function of n. Sharp results are found when n is even. The radius of univalence for f_n, $f \in R$, is shown to be an increasing function of n, $n \geq 4$.

Characterizing the extreme points of the closed convex hull of various subfamilies of S has enabled us to apply the Krein-Milman Theorem to solve many linear extremal problems. In [1] it is shown that $f \in R$ if and only if $f(z) = -z - 2\int_X \overline{x} \log(1 - xz) d\mu(x)$, where $|x| = 1$ and μ is a probability measure defined on the unit circle X. Consequently, the extreme points of R are $f_x(z) = -z - 2\overline{x}\log(1 - xz)$, $|x| = 1$. Thus, to minimize Re $f_n'(z)$, $|z| = r$, we need only consider the sequence of partial sums of

$$f(z) = -z - 2\log(1 - z) = z + 2\sum_{k=2}^{\infty} \frac{z^k}{k}. \tag{1}$$

2. Main Results

Theorem 1. *If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in R$, then the sequence of partial sums $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ is univalent in the disk $|z| < r_n$, where r_n is the smallest positive root of the polynomial equation $1 - r - 2r^n = 0$. The result is sharp for n even.*

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Proof. In view of the extremal function (1), it suffices to show that $\mathrm{Re} f_n'(z) = \mathrm{Re}\left(1 + 2\sum_{k=2}^{n} z^{k-1}\right) > 0$ for $|z| < r_n$. We have

$$f_n'(z) = 1 + \frac{2(z - z^n)}{1 - z} = \frac{1 + z - 2z^n}{1 - z}$$

and

$$\mathrm{Re} f_n'(z) = \frac{\mathrm{Re}(1 + z - 2z^n)(1 - \bar{z})}{|1 - z|^2} > 0$$

if

$$g(r, n, \theta) = 1 - r^2 - 2r^n(\cos n\theta - r\cos(n - 1)\theta) \tag{2}$$

is positive. But

$$g(r, n, \theta) \geq 1 - r^2 - 2r^n(1 + r) = (1 + r)(1 - r - 2r^n),$$

which is positive when $|z| < r_n$.

Note that $g(r_{2n}, 2n, \pi) = 0$. Hence if n is even we have $f_n'(-r_n) = 0$, so that the radius of univalence cannot be extended.

It is of interest to obtain a "reasonable" approximation of r_n in Theorem 1 that does not involve extracting roots from an nth degree polynomial. To this end, we give the following corollaries.

Corollary 1. *If $f \in R$, then f_n is univalent for $|z| < (\frac{1}{2n})^{1/n}$, $n \geq 2$.*

Proof. It suffices to show that $1 - r - 2r^n \geq 0$ for $r = (1/2n)^{1/n}$. We have

$$1 - \left(\frac{1}{2n}\right)^{1/n} - 2\left(\frac{1}{2n}\right) \geq 0 \text{ if } n\left(1 - \frac{1}{n}\right)^n \geq 1/2.$$

Since an induction argument shows that $n(1 - \frac{1}{n})^n$ is an increasing function of n for $n \geq 2$, the proof is complete.

A similar argument leads to

Corollary 2. *If $f \in R$, then*
(i) *f_n is univalent for $|z| < (1/n)^{1/n}$, $n \geq 10$, and*
(ii) *For any $A > 0$, there exists an $N_0(A)$ such that f_n is univalent for $|z| < (A/n)^{1/n}$, $n \geq N_0(A)$.*

Corollary 3. *If $f \in R$, then f_n is univalent for $|z| < 1 - \frac{\log n}{n}$, $n \geq 5$.*

Proof. A direct substitution shows that $1 - r - 2r^n \geq 0$ when $r = 1 - \frac{\log n}{n}$ for n =5, 6, 7, 8, 9. In view of Corollary 2, for $n \geq 10$ it suffices to show that $1 - \log n/n < (1/n)^{1/n}$, or equivalently $\log n + n \log\left(1 - \frac{\log n}{n}\right) < 0$. Since $\log\left(1 - \frac{\log n}{n}\right) < -\frac{\log n}{n}$, the result follows.

The value r_n in Theorem 1 is sharp only when n is even. For g defined by (2) and n odd, we have $\min_\theta g(r,n,\theta) > (1+r)(1-r-2r^n)$, which shows that the radius of univalence can be extended beyond r_n. In general, it appears to be a computationally difficult problem to find the smallest r for which $\min_\theta g(r,n,\theta) = 0$ when n is odd. We do this in the special case of $n = 3$.

Theorem 2. *$f \in R$, the radius of univalence of $f_3(z)$ is $\sqrt{2}/2$. The result is sharp.*

Proof. It suffices to look at $f_3(z) = z + 2\sum_{k=2}^{3} \frac{z^k}{k}$. We have for $|z| = r$,

$$\begin{aligned} \mathrm{Re} f_3'(z) &= \mathrm{Re}(1 + 2z + 2z^2) = 1 + 2r\cos\theta + 2r^2\cos 2\theta \\ &= (1 - 2r^2) + 2r\cos\theta + 4r^2\cos^2\theta. \end{aligned}$$

This last expression takes its minimum when $\cos\theta = -1/2r$, $r \geq 1/2$, which shows that $\mathrm{Re} f_e'(z) > 0$, $|z| < \sqrt{2}/2$. Since $f_3'(z) = 0$ for $z = (-1 \pm i)/2$, f_3 is not univalent in a larger disk.

From Theorems 1 and 2 we see that the radius of univalence of f_2 is $1/2$, of f_3 is $\sqrt{2}/2$, and of f_4 is ≈ 0.648. Thus the radius of univalence of f_n, $f \in R$, is not generally an increasing function of n. On the other hand, f_3 is the only exception. We will show that the radius of univalence of f_n is an increasing function of n for $n \geq 4$. But first we need the following lemma.

Lemma. *Let f_n be the sequence of partial sums of $f \in R$. If Re $f_n'(z) = 0$ for some z, $|z| = r$, then there exists an $h \in R$ such that $h_n'(r) = 0$.*

Proof. If Re $f_n'(z_0) = 0$, $z_0 = re^{i\alpha}$, set $g(z) = e^{-i\alpha} f(e^{i\alpha} z)$ and $h(z) = \frac{1}{2}\left(g(z) + \overline{g(\overline{z})}\right)$. Both g, $h \in R$, with $g_n'(r) = f_n'(z_0)$ and $h_n'(r) = 0$.

Theorem 3. *Let t_n be the radius of univalence of f_n, $f \in R$. Then t_n is an increasing function of n for $n \geq 4$.*

Proof. In view of (2) and the Lemma, for n fixed we see that t_n is the smallest positive r for which

$$\min_\theta (1 - r^2 - 2r^n(\cos\theta - r\cos(n-1)\theta)) = 0. \tag{3}$$

Note that $t_n \geq r_n$, the least positive root of $1 - r - 2r^n = 0$. Since $g(r_{2n}, 2n, \pi) = 0$, g defined by (2), we have $t_{2n} = r_{2n}$.

Now r_n increases with n, so that $t_{2n+1} \geq r_{2n+1} > r_{2n} = t_{2n}$. It thus suffices to show that $t_{2n} = r_{2n} > t_{2n-1}$, $n \geq 3$. By choosing a particular θ in (2), it is clear from (3) that t_{2n-1} is $\leq$ the smallest positive root of

$$\begin{aligned} g\left(r, 2n-1, \pi + \frac{\pi}{2n-1}\right) &= 1 - r^2 - 2r^{2n-1}\left(1 - r\cos\left(\frac{2n-2}{2n-1}\pi\right)\right) \\ &= 1 - r^2 - 2r^{2n-1}\left(1 + r\cos\left(\frac{\pi}{2n-1}\right)\right). \end{aligned} \tag{4}$$

Since $g(r,2n,\theta)\geq 0$ for $r\leq r_{2n}$ and all θ, it will follow that $r_{2n}>t_{2n-1}$ if we can show that $g\left(r_{2n},2n-1,\pi+\frac{\pi}{2n-1}\right)<0$.

Noting that $1-r-2r^{2n}=0$ when $r=r_{2n}$, we get from (4) that $g\left(r,2n-1,\pi+\frac{\pi}{2n-1}\right)=1-r^2-2r^{2n-1}(r+2r^{2n}+r\cos\left(\frac{\pi}{2n-1}\right))=1-r^2-(1-r)(1+2r^{2n-1}+\cos\left(\frac{\pi}{2n-1}\right))=(1-r)(r-\cos\left(\frac{\pi}{2n-1}\right)-2r^{2n-1})$. To see that this last expression is negative at $r=r_{2n}$, it suffices to show that $r<\cos(\frac{\pi}{2n-1})$.

Since $1-r_{2n}-2r_{2n}^{2n}=0$, our result will follow upon demonstrating that

$$1-\cos(\pi/2n-1)-2(\cos(\pi/2n-1))^{2n}<0,\quad n\geq 3. \tag{5}$$

A direct computation shows that (5) holds for $n=3$. Since $\cos(\pi/n)\geq 1-\pi^2/2n^2$, we have

$$1-\cos(\pi/2n-1)-2(\cos(\pi/2n-1))^{2n}\leq\frac{\pi^2}{2(2n-1)^2}-2\Big(1-\frac{\pi^2}{2(2n-1)^2}\Big)^{2n}$$
$$\leq\frac{\pi^2}{2(2n-1)^2}-2\Big(1-\frac{2n\pi^2}{2(2n-1)^2}\Big)=\frac{(4n+1)\pi^2}{2(2n-1)^2}-2<0\quad\text{for } n\geq 4.$$

This completes the proof.

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