Abstract. In this paper we prove the following theorem: Let \( p > 1 \) and let \( E \) be a \( p \)-uniformly convex Banach space, \( K \) a nonempty bounded closed convex subset of \( E \), and \( A = \{ t_{n,k} \}_{n,k \geq 1} \) a strongly ergodic matrix. Let \( T : K \to K \) be a continuous mapping satisfying: for each \( x, y \) in \( K \) and \( i = 1, 2, \ldots \)

\[
\|T^{i}x - T^{i}y\| \leq a_{i}\|x - y\| + b_{i}(\|x - T^{i}x\| + \|y - T^{i}y\|) + c_{i}(\|x - T^{i}y\| + \|y - T^{i}x\|)
\]

where \( a_{i}, b_{i}, c_{i} \) are nonnegative, \( 3b_{i} + 3c_{i} \leq 1 \), and

\[
\liminf_{n \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} t_{n,k} \cdot 2^{P-1}(\alpha_{k+m}^{p} + \beta_{k+m}^{p}) < 1 + c_{p}
\]

where \( \alpha_{i} = \frac{a_{i} + b_{i} + c_{i}}{1 - b_{i} - c_{i}} \), \( \beta_{i} = \frac{2b_{i} + 2c_{i}}{1 - b_{i} - c_{i}} \), and \( c_{p} > 0 \) is some constant. Then \( T \) has a fixed point in \( K \).

1. Introduction and Preliminaries

Let \( K \) be a nonempty subset of a Banach space \( E \). A mapping \( T : K \to K \) is said to be Lipschitzian if for each \( n \geq 1 \) there exists a positive real number \( k_{n} \) such that

\[
\|T^{n}x - T^{n}y\| \leq k_{n}\|x - y\|
\]

for all \( x, y \) in \( K \). A Lipschitzian mapping is said to be nonexpansive if \( k_{n} = 1 \) for all \( n \geq 1 \) and asymptotically nonexpansive \([5]\) if \( \lim_{n \to \infty} k_{n} = 1 \).

A Lipschitzian mapping is said to be uniformly Lipschitzian if \( k_{n} = k \) for all \( n \geq 1 \) or in other words, if the Lipschitz constant of \( T^{n} \),

\[
\|T^{n}\| = \sup \left\{ \frac{\|T^{n}x - T^{n}y\|}{\|x - y\|} : x \neq y, x, y \in K \right\} < k, n \geq 1
\]

holds for some \( k > 0 \).

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In [8], Lifshitz proved the following result:

**Theorem A.** Let \( K \) be a nonempty closed convex bounded subset of a Hilbert space. If \( T : K \to K \) is a mapping such that

\[
\limsup_{n \to \infty} \|T^n\| < \sqrt{2},
\]

then \( T \) has a fixed point in \( K \).

Let \( p > 1 \) and denote by \( \lambda \) a number in \([0,1]\) and by \( \omega_p(\lambda) \) the function \( \lambda \cdot (1 - \lambda)^p + \lambda^p \cdot (1 - \lambda) \). The functional \( \| \cdot \|^p \) is said to be uniformly convex (c. f. Zalinescu [16]) on the Banach space \( E \) if there exists a positive constant \( c_p \) such that for all \( \lambda \in [0,1] \) and \( x, y \in E \) the following inequality holds:

\[
\| \lambda x + (1 - \lambda)y \|^p \leq \lambda \| x \|^p + (1 - \lambda)\| y \|^p - c_p \cdot \omega_p(\lambda) \cdot \| x - y \|^p
\]

(1)

Xu [15] proved that the functional \( \| \cdot \|^p \) is uniformly convex on the whole Banach space \( E \) if and only if \( E \) is \( p \)-uniformly convex, i.e., there exists a constant \( c_p > 0 \) such that the moduli of convexity, \( \delta_E(\varepsilon) \geq c_p \cdot \varepsilon^p \) for all \( 0 \leq \varepsilon \leq 2 \).

Görnicki and Krüppel [7] extended Theorem A via an inequality in Banach space and proved the following:

**Theorem B.** Let \( E \) be a uniformly convex Banach space with \( \text{diam } E > 2 \) for which norm satisfies (1) for some \( p \geq 2 \) and let \( K \) be a nonempty bounded closed convex subset of \( E \). If \( T : K \to K \) is a mapping such that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \|T^i\|^p < 1 + c_p,
\]

then \( T \) has a fixed point in \( K \).

Recently, Görnicki [6] generalized Theorem B to the following result via Banach space inequalities and more general summation methods involving

\[
\sum_{k=1}^{\infty} t_{n,k} \cdot \|T^k\|^p, \quad n = 1, 2, \ldots
\]

where \( A = [t_{n,k}]_{n,k \geq 1} \) is strongly ergodic matrix [2]:

(a) \( \Lambda_{n,k} t_{n,k} \geq 0 \),
(b) \( \Lambda_k \lim_{n \to \infty} t_{n,k} = 0 \),
(c) \( \Lambda_n \sum_{k=1}^{\infty} t_{n,k} = 1 \),
(d) \( \lim_{n \to \infty} \sum_{k=1}^{\infty} |t_{n,k+1} - t_{n,k}| = 0 \)

**Theorem C.** Let \( p > 1 \) and let \( E \) be a \( p \)-uniformly convex Banach space, \( K \) a nonempty bounded closed convex subset of \( E \), and \( A = [t_{n,k}]_{n,k \geq 1} \) a strongly ergodic matrix. If \( T : K \to K \) is a mapping such that

\[
g = \liminf_{n \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} t_{n,k} \|T^{k+m}\|^p < 1 + c_p
\]
then $T$ has a fixed point in $K$.

We now consider the following class of mappings which we call generalized Lipschitzian mapping:

A mapping $T : K \to K$ is said to be generalized Lipschitzian if

$$
\|T^n x - T^n y\| \\
\leq a_n \|x - y\| + b_n (\|x - T^n x\| + \|y - T^n y\|) + c_n (\|x - T^n y\| + \|y - T^n x\|)
$$

(2)

for each $x, y$ in $K$ and $n \geq 1$, where $a_n, b_n, c_n$ are nonnegative constants such that $3(b_n + c_n) \leq 1$ for all $n \geq 1$.

This class of mappings are more general than nonexpansive, asymptotically nonexpansive, Lipschitzian and uniformly Lipschitzian mappings and it can be seen by taking $b_n = c_n = 0$.

In the present paper, we extend the result of Górnicki [6] and consequently Górnicki and Krüppel [7] and Lipschitz [8] for generalized Lipschitzian mappings and in $p$-uniformly convex Banach space. Further, we establish for these mappings some fixed point theorems in Hilbert space, $L^p$ spaces, Hardy space $H^p$ or Sobolev spaces $H^{k,p}$ for $1 < p < \infty$ and $k \geq 0$.

2. Main Results

Before presenting our main result, we need the following:

Lemma 1 [6]. Let $p > 1$ and let $E$ be a $p$-uniformly convex Banach space, $K$ a nonempty closed convex subset of $E$, and $\{x_n\} \subset E$ a bounded sequence. Then there exists a unique point $z$ in $K$ such that

$$
\limsup_{n \to \infty} \sum_{k=1}^{\infty} t_{n,k} \cdot \|x_k - z\|^p \\
\leq \limsup_{n \to \infty} \sum_{k=1}^{\infty} t_{n,k} \cdot \|x_k - x\|^p - c_p \cdot \|x - z\|^p
$$

(3)

for every $x$ in $K$, where $c_p$ is the constant given in (1) and $A = [t_{n,k}]_{n,k \geq 1}$ is a strongly ergodic matrix.

We are now in position to give our result:

Theorem 1. Let $p > 1$ and let $E$ be a $p$-uniformly convex Banach space, $K$ a nonempty bounded closed convex subset of $E$, and $A = [t_{n,k}]_{n,k \geq 1}$ a strongly ergodic matrix. If $T : K \to K$ is a continuous generalized Lipschitzian mapping such that

$$
h = \liminf_{n \to \infty} \inf_{m = 0, 1, 2, \ldots} \sum_{k=1}^{\infty} t_{n,k} \cdot 2^{p-1} \{\alpha_{k+m}^p + \beta_{k+m}^p\} < 1 + c_p,
$$

where
where
\[ \alpha_{k+m} = \frac{a_{k+m} + b_{k+m} + c_{k+m}}{1 - b_{k+m} - c_{k+m}} \]
and
\[ \beta_{k+m} = \frac{2b_{k+m} + 2c_{k+m}}{1 - b_{k+m} - c_{k+m}} , \]
then \( T \) has a fixed point in \( K \).

Proof. Let \( \{n_i\} \) and \( \{m_i\} \) be sequences of natural numbers such that
\[ h = \liminf_{i \to \infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} t_{n_i,k} \cdot 2^{p-1} \{\alpha_{k+m, i} + \beta_{k+m, i}\} < 1 + c_p . \]
For any \( z_0 \in K \), we can inductively define a sequence \( \{z_j\} \) in the following manner: \( z_j \)
the unique asymptotic center in \( K \) of the sequence \( \{T^n z_{j-1}\}_{n \geq 1} \), i.e., \( z_j \) is the unique point in \( K \) that minimizes the functional
\[ r_{j-1}(x) = \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|x - T^{k+m} z_{j-1}\|^p \]
over \( x \) in \( K \). Using (2), after a simple calculation, for each \( x, y \in K \), \( k > N \geq 1 \), we have
\[ \|T^N x - T^k y\| \leq a_N + b_N + c_N \cdot \|x - T^{k-N} y\| + 2b_N + 2c_N \cdot \|x - T^k y\| \quad (4) \]
\[ = \alpha_N \cdot \|x - T^{k-N} y\| + \beta_N \cdot \|x - T^k y\| \]
In view of inequality (1), for any fixed \( N, k, m_i \in \mathbb{N} \) and \( 0 \leq \lambda \leq 1 \), we have
\[ \|\lambda z_j + (1 - \lambda) T^N z_j - T^{k+m_i} z_{j-1}\|^p \]
\[ = \|\lambda (z_j - T^{k+m_i} z_{j-1}) + (1 - \lambda)(T^N z_j - T^{k+m_i} z_{j-1})\|^p \]
\[ \leq \lambda \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p + (1 - \lambda) \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \]
\[ -c_p \cdot \omega_p(\lambda) \cdot \|z_j - T^N z_j\|^p . \]
Multiplying both sides of this inequality by suitable elements of the matrix \( A \) and summing, we have
\[ \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|\lambda z_j + (1 - \lambda) T^N z_j - T^{k+m_i} z_{j-1}\|^p \]
\[ \leq \lambda \cdot \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \]
\[ + (1 - \lambda) \cdot \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \]
\[ -c_p \cdot \omega_p(\lambda) \cdot \|z_j - T^N z_j\|^p \quad \text{for } i = 1, 2, \ldots . \]
Taking the limit superior on each side as $i \to \infty$, we obtain

\[ \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \| \lambda z_j + (1 - \lambda) T^N z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ \leq \lambda \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ + (1 - \lambda) \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| T^N z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ - c_p \cdot \omega_p(\lambda) \cdot \| z_j - T^N z_j \|^p \]

and

\[ c_p \cdot \omega_p(\lambda) \cdot \| z_j - T^N z_j \|^p \]

\[ \leq \lambda \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ + (1 - \lambda) \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| T^N z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ - \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| \lambda z_j + (1 - \lambda) T^N z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ \leq (1 - \lambda) \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| T^N z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ - (1 - \lambda) \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| z_j - T^{k+m_i} z_{j-1} \|^p \]

and

\[ c_p \cdot [\lambda(1 - \lambda)^{p-1} + \lambda^p] \cdot \| z_j - T^N z_j \|^p \]

\[ \leq \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| T^N z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ - \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| z_j - T^{k+m_i} z_{j-1} \|^p. \]

Taking $\lambda = 1$, we get

\[ c_p \cdot \| z_j - T^N z_j \|^p \leq \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| T^N z_j - T^{k+m_i} z_{j-1} \|^p \]

\[ - \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \| z_j - T^{k+m_i} z_{j-1} \|^p. \]
In view of inequality (4), we have

\[ c_p \cdot \|z_j - T^N z_j\|^p \leq \limsup_{t \to \infty} \left[ \sum_{k=1}^{N} t_{n_i, k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \right. \\
+ \sum_{k=N+1}^{\infty} t_{n_i, k} \left\{ \alpha_N \cdot \|z_j - T^{k-N+m_i} z_{j-1}\| + \beta_N \cdot \|z_j - T^{k+m_i} z_{j-1}\| \right\}^p \right] \\
- \limsup_{t \to \infty} \sum_{k=1}^{\infty} t_{n_i, k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
\leq \limsup_{t \to \infty} \left[ \sum_{k=N+1}^{\infty} t_{n_i, k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \right. \\
+ 2^{p-1} \left\{ \alpha_N^p \sum_{k=N+1}^{\infty} t_{n_i, k} \cdot \|z_j - T^{k-N+m_i} z_{j-1}\|^p \right. \\
+ \beta_N^p \sum_{k=N+1}^{\infty} t_{n_i, k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \left\} \right] \\
- \limsup_{t \to \infty} \sum_{k=1}^{\infty} t_{n_i, k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
\leq \limsup_{t \to \infty} \left[ \sum_{k=1}^{N} t_{n_i, k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \right. \\
+ 2^{p-1} \left\{ \alpha_N^p \sum_{k=1}^{\infty} t_{n_i, k+N} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \right. \\
+ \beta_N^p \sum_{k=N+1}^{\infty} t_{n_i, k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \left\} \right] \\
- \limsup_{t \to \infty} \sum_{k=1}^{\infty} t_{n_i, k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
= \limsup_{t \to \infty} \left[ \sum_{k=1}^{N} t_{n_i, k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \right. \\
+ 2^{p-1} \left\{ \alpha_N^p \cdot \left( \sum_{k=1}^{\infty} t_{n_i, k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \right) \right. \\
- \sum_{k=1}^{\infty} (t_{n_i, k} - t_{n_i, k+N}) \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \right\} \\
+ \beta_N^p \left[ \sum_{k=1}^{\infty} t_{n_i, k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \right] \left\{ \sum_{k=1}^{N} t_{n_i, k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \right\} \right] \]
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\[- \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i}z_{j-1}\|^p \]

\[\leq \left[2^{p-1}(\alpha_N^p + \beta_N^p) - 1\right] \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i}z_{i-1}\|^p\]

Since

(i) \(\sum_{k=1}^{N} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i}z_{j-1}\|^p \to 0\) as \(i \to +\infty\),

(ii) \(\sum_{k=1}^{\infty} (t_{n_i,k} - t_{n_i,k+N}) \cdot \|z_j - T^{k+m_i}z_{j-1}\|^p \to 0\) as \(i \to +\infty\),

(iii) \(r_{j-1}(z_j) \leq r_{j-1}(z_{j-1})\).

For any fixed \(N \in \mathbb{N}\), we have

\[c_p \cdot \|z_j - T^N z_{j}\|^p \leq \left[2^{p-1}(\alpha_N^p + \beta_N^p) - 1\right] \cdot r_{j-1}(z_{j-1}).\]

We multiply this inequality for \(N = k + m_i\) by suitable elements \(t_{n_i,k}\) for \(k = 1, 2, \ldots\).

Summing up these inequalities and taking the limit superior on each side as \(i \to +\infty\), we obtain

\[c_p \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i}z_{j}\|^p \]

\[\leq \lim_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \left\{2^{p-1}(\alpha_{k+m_i}^p + \beta_{k+m_i}^p) - 1\right\} \cdot r_{j-1}(z_{j-1})\]

and

\[r_j(z_j) \leq B \cdot r_{j-1}(z_{j-1}),\]

where

\[B = \frac{1}{c_p} \left[ \lim_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \left\{2^{p-1}(\alpha_{k+m_i}^p + \beta_{k+m_i}^p) - 1\right\} \right] < 1.\]

In a similar way, we obtain

\[r_j(z_j) \leq B^j \cdot r_0(z_0), \quad j = 1, 2, \ldots.\]

Next, we show the convergence of the sequence \(\{z_j\}\). For a fixed \(N \in \mathbb{N}\), we have

\[\|z_{j+1} - z_j\|^p \leq \left[2^{p-1}(\|z_{j+1} - T^N z_j\|^p + \|T^N z_j - z_j\|^p)\right] \]

We multiply this inequality for \(N = k + m_i\) by suitable elements \(t_{n_i,k}\) for \(k = 1, 2, \ldots\).

Summing up these inequalities and taking the limit superior on each side as \(i \to +\infty\), we obtain

\[\|z_{j+1} - z_j\|^p \leq 2^{p-1} \left[ \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_{j+1} - T^N z_j\|^p \right]

\[+ \lim_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - z_j\|^p\]

\[= 2^{p-1} \cdot (r_j(z_{j+1}) + r_j(z_j)) \]

\[\leq 2^{p} \cdot r_j(z_j) \]

\[\leq 2^{p} \cdot B^j \cdot r_0(z_0)\]
and
\[ \|z_{j+1} - z_j\| \leq 2[B^j \cdot r_0(z_0)]^\frac{1}{p} \to 0 \text{ as } j \to +\infty, \]
which show that \( \{z_j\} \) is a Cauchy sequence. Let \( z = \lim_{j \to \infty} z_j \). For fixed \( N \in \mathbb{N} \), we have
\[
\begin{align*}
\|z - T^N z\| &\leq \|z - z_j\| + \|z_j - T^N z_j\| + \|T^N z_j - T^N z\| \\
&\leq \frac{1 + a_N + 2c_N}{1 - b_N - c_N} \cdot \|z - z_j\| + \frac{1 + b_N - c_N}{1 - b_N - c_N} \cdot \|z_j - T^N z_j\| \\
&\quad \text{or} \\
\|z - T^N z\|^p &\leq 2^{p-1} \cdot \left[ \left( \frac{1 + a_N + 2c_N}{1 - b_N - c_N} \right)^p \cdot \|z - z_j\|^p + \left( \frac{1 + b_N - c_N}{1 - b_N - c_N} \right)^p \cdot \|z_j - T^N z_j\|^p \right] \\
&\leq 2^{p-1} \cdot \left[ \left( \frac{1 + a_N + 2c_N}{1 - b_N - c_N} \right)^p \cdot \|z - z_j\|^p + 2^p \cdot \|z_j - T^N z_j\|^p \right].
\end{align*}
\]
We multiply this inequality for \( N = k + m_i \) by suitable elements \( t_{n_i,k} \) for \( k = 1, 2, \ldots \). Summing up these inequalities, we obtain
\[
\begin{align*}
\sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z - T^{k+m_i} z\|^p &\leq 2^{p-1} \cdot \left[ \|z - z_j\|^p \cdot \sum_{k=1}^{\infty} t_{n_i,k} \cdot \left( \frac{1 + a_{k+m_i} + 2c_{k+m_i}}{1 - b_{k+m_i} - c_{k+m_i}} \right)^p \\
&\quad + 2^p \cdot \sum_{k=1}^{\infty} t_{n_i,k} \|z_j - T^{k+m_i} z_j\|^p \right].
\end{align*}
\]
Taking the limit superior on each side as \( i \to +\infty \), we get
\[
\begin{align*}
\limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z - T^{k+m_i} z\|^p &\leq 2^{p-1} \cdot \left[ \|z - z_j\|^p \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \left( \frac{1 + a_{k+m_i} + 2c_{k+m_i}}{1 - b_{k+m_i} - c_{k+m_i}} \right)^p \\
&\quad + 2^p \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \|z_j - T^{k+m_i} z_j\|^p \right] \\
&\leq 2^{p-1} \cdot \left[ \|z - z_j\|^p \cdot \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \left( \frac{1 + a_{k+m_i} + 2c_{k+m_i}}{1 - b_{k+m_i} - c_{k+m_i}} \right)^p + 2^p \cdot B^j r_0(z_0) \right] \\
&\to 0 \text{ as } j \to +\infty.
\end{align*}
\]
Therefore,

$$\limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n_i, k} \cdot \|z - T^{k+m_i}z\|^p = 0.$$  

This implies that $T_z = z$. Indeed, for any $\varepsilon > 0$ there exists natural numbers $n, n + 1$ such that

$$\|z - T^n z\| < \varepsilon \text{ and } \|z - T^{n+1} z\| < \varepsilon.$$  

Otherwise, we have for any $n$ and $m$.

$$\sum_{k=1}^{\infty} t_{n, k} \cdot \|z - T^{k+m}z\|^p \geq \frac{1}{2} \varepsilon^p$$  

and hence

$$\limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n, k} \cdot \|z - T^{k+m_i}z\|^p \geq \limsup_{i \to \infty} \sum_{k=1}^{\infty} t_{n, k} \cdot \|z - T^{k+m_i}z\|^p \geq \frac{1}{2} \varepsilon^p.$$  

Thus for every natural number $l$ there exists a natural number $n_1$ such that

$$\|z - T^{n_1} z\| < \frac{1}{l} \text{ and } \|z - T^{n_1+1} z\| < \frac{1}{l}.$$  

It follows that

$$T^{n_1}z \to z \text{ and } T^{n_1+1} \to z \text{ as } l \to \infty.$$  

Since $T$ is continuous, we have

$$T_z = T(\lim_{l \to \infty} T^{n_1} z) = \lim_{l \to \infty} T^{n_1+1} z = z.$$  

This completes the proof.

3. Some Applications

In a Hilbert space $H$, the following identity holds:

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda (1 - \lambda)\|x - y\|^2$$  

for all $x, y$ in $H$ and $\lambda \in [0, 1]$.

By (5), we immediately obtain from Theorem 1 the following:

**Corollary 1.** Let $K$ be a nonempty bounded closed convex subset of a Hilbert space $H$ and $A = [t_{n, k}]_{n,k \geq 1}$ a strongly ergodic matrix. If $T : K \to K$ is a continuous generalized Lipschitzian mapping such that

$$\liminf_{i \to \infty} \inf_{m=0,1,2,...} \sum_{k=1}^{\infty} t_{n, k} \cdot (\alpha_{n+m}^2 + \beta_{k+m}^2) < 1,$$
then $T$ has a fixed point in $K$.

If $1 < p \leq 2$, then we have for all $x, y$ in $L^p$ and $\lambda \in [0, 1]$

$$
||\lambda x + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda) \cdot (p - 1)||x - y||^2
$$

(6)

(The inequality (6) is contained in Lim, Xu and Xu [10] and Smarzewski [14].)

Assume that $2 < p < +\infty$ and $h_p$ is the unique zero of the function $g(x) = -x^{p-1} + (p - 1)x + p - 2$ in the interval $(1, \infty)$. Let

$$
c_p = (p - 1) \cdot (1 + h_p)^{-2-p} = \frac{1 + h_p^{-p-1}}{(1 + h_p)^{-p-1}}.
$$

Then we have the following inequality:

$$
||\lambda x + (1 - \lambda)y||^p \leq \lambda||x||^p + (1 - \lambda)||y||^p = \omega_p(\lambda) \cdot c_p \cdot ||x - y||^p
$$

(7)

for all $x, y$ in $L^p$ and $\lambda \in [0, 1]$ (The inequality (7) is essentially due to Lim [9]).

**Corollary 2.** Let $K$ be a nonempty bounded closed convex subset of $L^p(1 < p < +\infty)$ and $A = [tn,k]_{n,k \geq 1}$ a strongly ergodic matrix. If $T : K \to K$ is a continuous generalized Lipschitzian mapping such that

$$
\liminf_{n \to \infty} \inf_{m=0,1,2,...} \sum_{k=1}^{\infty} t_{n,k} \cdot 2(\alpha_{k+m}^2 + \beta_{k+m}^2) < p, \text{ for } 1 < p \leq 2
$$

and

$$
\liminf_{n \to \infty} \inf_{m=0,1,2,...} \sum_{k=1}^{\infty} t_{n,k} \cdot 2(p-1)(\alpha_{k+m}^p + \beta_{k+m}^p) < 1 + c_p, \text{ for } p > 2,
$$

Then $T$ has a fixed point in $K$.

Let $H^p$, $1 < p < +\infty$, denote the Hardy space [4] of all functions $x$ analytic in unit disc $|z| < 1$ of the complex plane and such that

$$
||x|| = \lim_{r \to 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty
$$

Now, let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega), k \geq 0, 1 < p < +\infty$, the Sobolev space [1, p. 149] of distributions $x$ such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ equipped with the norm.

$$
||x|| = \left( \sum_{|\alpha| \leq k} \int_\Omega |D^\alpha x(\omega)|^p d\omega \right)^{\frac{1}{p}}.
$$

Let $(\Omega_\alpha, \sum_\alpha, \mu_\alpha), \alpha \in \Lambda$, be a sequence of positive measure spaces, where index set $\Lambda$ is finite or countable. Given a sequence of linear subspaces $X_\alpha$ in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, we
denote by $L_{q,p}, 1 < p < +\infty$ and $q = \max\{2, p\}$ [11], the linear space of all sequence $x\{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$||x|| = \left( \sum_{\alpha \in \Lambda} (||x_\alpha||_{p, \alpha})^q \right)^{\frac{1}{q}}.$$

where $|| \cdot ||_{p, \alpha}$ denotes the norm in $L^p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \Sigma_1, \mu_1)$ and $L_q = L^q(S_2, \Sigma_2, \mu_2)$, where $1 < p < +\infty$, $q = \max\{2, p\}$ and $(S_i, \Sigma_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [3, III. 2.10] of all measurable $L_p$-value function $x$ on $S_2$ such that

$$||x|| = \left( \int_{S_2} (||x(s)||_p)^q \mu_2(ds) \right)^{\frac{1}{q}}.$$

These spaces are $q$-uniformly convex with $q = \max\{2, p\}$ [12, 13] and the norm in these spaces satisfies

$$||\lambda x + (1 - \lambda)y||^q \leq \lambda||x||^q + (1 - \lambda)||y||^q - d \cdot \omega_q(\lambda) \cdot ||x - y||^q$$

with a constant

$$d = d_p = \begin{cases} \frac{E - 1}{8} & \text{for } 1 < p \leq 2 \\ \frac{1}{p - 2} & \text{for } 2 < p < +\infty \end{cases}$$

Here, from Theorem 1, we have the following result:

**Corollary 3.** Let $K$ be a nonempty bounded closed convex subset of the space $E$, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_{q,p}$, or $E = L_q(L_p)$, and $1 < p < +\infty$, $q = \max\{2, p\}$, $k \geq 0$ and $A = [t_{n,k}]_{n,k \geq 1}$ is a strongly ergodic matrix. If $T : K \rightarrow K$ is a continuous generalized Lipschitzian mapping such that

$$\liminf_{n \rightarrow \infty} \inf_{m=0,1,2,...} \sum_{k=1}^{\infty} t_{n,k} \cdot 2^{q-1}\{\alpha_{k+m}^q + \beta_{k+m}^q\} < 1 + d_p,$$

then $T$ has a fixed point in $K$.

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**References**


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