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ON SOME FIXED POINT THEOREMS FOR MAPPINGS WITH GENERALIZED LIPSCHITZIAN ITERATES

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Abstract. In this paper we prove the following theorem: Let $p > 1$ and let E be a p-uniformly convex Banach space, K a nonempty bounded closed convex subset of E, and $A = [t_{n,k}]_{n,k\geq 1}$ a strongly ergodic matrix. Let $T : K \to K$ be a continuous mapping satisfying: for each x, y in K and $i = 1, 2, \ldots$.

$$\|T^i x - T^i y\| \leq a_i\|x - y\| + b_i(\|x - T^i x\| + \|y - T^i y\|) + c_i(\|x - T^i y\| + \|y - T^i x\|)$$

where a_i, b_i, c_i are nonnegative, $3b_i + 3c_i \leq 1$, and

$$\liminf_{n\to\infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} t_{n,k} \cdot 2^{p-1}\{\alpha^p_{k+m} + \beta^p_{k+m}\} < 1 + c_p$$

where $\alpha_i = \frac{a_i+b_i+c_i}{1-b_i-c_i}$, $\beta_i = \frac{2b_i+2c_i}{1-b_i-c_i}$, and $c_p > 0$ is some constant. Then T has a fixed point in K.

1. Introduction and Preliminaries

Let K be a nonempty subset of a Banach space E. A mapping T: $K \to K$ is said to be Lipschitzian if for each $n \geq 1$ there exists a positive real number k_n such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|$$

for all x, y in K. A Lipschitzian mapping is said to be nonexpansive if $k_n = 1$ for all $n \geq 1$ and asymptotically nonexpansive [5] if $\lim_{n\to\infty} k_n = 1$.

A Lipschitzian mapping is said to be uniformly Lipschitzian if $k_n = k$ for all $n \geq 1$ or in other words, if the Lipschitz constant of T^n,

$$|||T^n||| = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x - y\|} : x \neq y, x, y \in K\right\} < k, n \geq 1$$

holds for some $k > 0$.

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In [8], Lifshitz proved the following result:

Theorem A. *Let K be a nonempty closed convex bounded subset of a Hilbert space. If $T : K \to K$ is a mapping such that*

$$\limsup_{n\to\infty} |||T^n||| < \sqrt{2},$$

then T has a fixed point in K.

Let $p > 1$ and denote by λ a number in [0,1] and by $\omega_p(\lambda)$ the function $\lambda \cdot (1-\lambda)^p + \lambda^p \cdot (1-\lambda)$. The functional $||\cdot||^p$ is said to be uniformly convex (c. f. Zalinescu [16]) on the Banach space E if there exists a positive constant c_p such that for all $\lambda \in [0,1]$ and $x, y \in E$ the following inequality holds:

$$||\lambda_x + (1-\lambda)y||^p \leq \lambda||x||^p + (1-\lambda)||y||^p - c_p \cdot \omega_p(\lambda) \cdot ||x-y||^p \tag{1}$$

Xu [15] proved that the functional $||\cdot||^p$ is uniformly convex on the whole Banach space E if and only if E is p-uniformly convex, i.e., there exists a constant $c_p > 0$ such that the moduli of convexity, $\delta_E(\varepsilon) \geq c_p \cdot \varepsilon^p$ for all $0 \leq \varepsilon \leq 2$.

Górnicki and Krüppel [7] extended Theorem A via an inequality in Banach space and proved the following:

Theorem B. *Let E be a uniformly convex Banach space with diam $E > 2$ for which norm satisfies (1) for some $p \geq 2$ and let K be a nonempty bounded closed convex subset of E. If $T : K \to K$ is a mapping such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |||T^i|||^p < 1 + c_p,$$

then T has a fixed point in K.

Recently, Górnicki [6] generalized Theorem B to the following result via Banach space inequalities and more general summation methods involving

$$\sum_{k=1}^{\infty} t_{n,k} \cdot |||T^k|||^p, \quad n = 1, 2, \ldots$$

where $A = [t_{n,k}]_{n,k\geq 1}$ is strongly ergodic matrix [2]:

(a) $\Lambda_{n,k} t_{n,k} \geq 0$,
(b) $\Lambda_k \lim_{n\to\infty} t_{n,k} = 0$,
(c) $\Lambda_n \sum_{k=1}^{\infty} t_{n,k} = 1$,
(d) $\lim_{n\to\infty} \sum_{k=1}^{\infty} |t_{n,k+1} - t_{n,k}| = 0$

Theorem C. *Let $p > 1$ and let E be a p-uniformly convex Banach space, K a nonempty bounded closed convex subset of E, and $A = [t_{n,k}]_{n,k\geq 1}$ a strongly ergodic matrix. If $T : K \to K$ is a mapping such that*

$$g = \liminf_{n\to\infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} t_{n,k} |||T^{k+m}|||^p < 1 + c_p$$

then T has a fixed point in K.

We now consider the following class of mappings which we call generalized Lipschitzian mapping:

A mapping $T : K \to K$ is said to be generalized Lipschitzian if

$$\begin{aligned} &||T^n x - T^n y|| \\ &\leq a_n||x-y|| + b_n(||x - T^n x|| + ||y - T^n y||) + c_n(||x - T^n y|| + ||y - T^n x||) \end{aligned} \tag{2}$$

for each x, y in K and $n \geq 1$, where a_n, b_n, c_n are nonnegative constants such that $3(b_n + c_n) \leq 1$ for all $n \geq 1$.

This class of mappings are more general than nonexpansive, asymptotically nonespansive, Lipschitzian and uniformly Lipschitzian mappings and it can be seen by taking $b_n = c_n = 0$.

In the present paper, we extend the result of Górnicki [6] and consequently Górnicki and Krüppel [7] and Lipschitz [8] for generalized Lipschitzian mappings and in p-uniformly convex Banach space.Further, we establish for these mappings some fixed point theorems in Hilbert space, L^p spaces, Hardy space H^p or Sobolev spaces $H^{k,p}$ for $1 < p < \infty$ and $k \geq 0$.

2. Main Results

Before presenting our main result, we need the following:

Lemma 1 [6]. *Let $p > 1$ and let E be a p-uniformly convex Banach space, K a nonempty closed convex subset of E, and $\{x_n\} \subset E$ a bounded sequence. Then there exists a unique point z in K such that*

$$\begin{aligned} &\limsup_{n\to\infty} \sum_{k=1}^{\infty} t_{n,k} \cdot ||x_k - z||^p \\ &\leq \limsup_{n\to\infty} \sum_{k=1}^{\infty} t_{n,k} \cdot ||x_k - x||^p - c_p \cdot ||x - z||^p \end{aligned} \tag{3}$$

for every x in K, where c_p is the constant given in (1) and $A = [t_{n,k}]_{n,k\geq 1}$ is a strongly ergodic matrix.

We are now in position to give our result:

Theorem 1. *Let $p > 1$ and let E be a p-uniformly convex Banach space, K a nonempty bounded closed convex subset of E, and $A = [t_{n,k}]_{n,k\geq 1}$ a strongly ergodic matrix. If $T : K \to K$ is a continuous generalized Lipschitzian mapping such that*

$$h = \liminf_{n\to\infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} t_{n,k} \cdot 2^{p-1}\{\alpha_{k+m}^p + \beta_{k+m}^p\} < 1 + c_p,$$

where

$$\alpha_{k+m} = \frac{a_{k+m} + b_{k+m} + c_{k+m}}{1 - b_{k+m} - c_{k+m}}$$

and

$$\beta_{k+m} = \frac{2b_{k+m} + 2c_{k+m}}{1 - b_{k+m} - c_{k+m}},$$

then T has a fixed point in K.

Proof. Let $\{n_i\}$ and $\{m_i\}$ be sequences of natural numbers such that

$$h = \liminf_{i\to\infty} \inf_{m=0,1,2,\dots} \sum_{k=1}^{\infty} t_{n_i,k} \cdot 2^{p-1}\{\alpha^p_{k+m_i} + \beta^p_{k+m_i}\} < 1 + c_p.$$

For any $z_0 \in K$, we can inductively define a sequence $\{z_j\}$ in the following manner: z_j is the unique asymptotic center in K of the sequence $\{T^n z_{j-1}\}_{n\geq 1}$, i.e., z_j is the unique point in K that minimizes the functional

$$r_{j-1}(x) = \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|x - T^{k+m_i} z_{j-1}\|^p$$

over x in K. Using (2), after a simple calculation, for each $x, y \in K$, $k > N \geq 1$, we have

$$\begin{aligned} &\|T^N x - T^k y\| \\ &\leq \frac{a_N + b_N + c_N}{1 - b_N - c_N} \cdot \|x - T^{k-N} y\| + \frac{2b_N + 2c_N}{1 - b_N - c_N} \cdot \|x - T^k y\| \\ &= \alpha_N \cdot \|x - T^{k-N} y\| + \beta_N \cdot \|x - T^k y\| \end{aligned} \tag{4}$$

In view of inequality (1), for any fixed N, k, $m_i \in \mathbb{N}$ and $0 \leq \lambda \leq 1$, we have

$$\begin{aligned} &\|\lambda z_j + (1-\lambda) \cdot T^N z_j - T^{k+m_i} z_{j-1}\|^p \\ &= \|\lambda(z_j - T^{k+m_i} z_{j-1}) + (1-\lambda)(T^N z_j - T^{k+m_i} z_{j-1})\|^p \\ &\leq \lambda \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p + (1-\lambda) \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\ &-c_p \cdot \omega_p(\lambda) \cdot \|z_j - T^N z_j\|^p. \end{aligned}$$

Multiplying both sides of this inequality by suitable elements of the matrix A and summing, we have

$$\begin{aligned} &\sum_{k=1}^{\infty} t_{n_i,k} \cdot \|\lambda z_j + (1-\lambda) T^N z_j - T^{k+m_i} z_{j-1}\|^p \\ &\leq \lambda \cdot \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\ &\quad + (1-\lambda) \cdot \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^P \\ &\quad - c_p \cdot \omega_p(\lambda) \cdot \|z_j - T^N z_j\|^p \quad \text{for } i = 1, 2, \dots. \end{aligned}$$

Taking the limit superior on each side as $i \to \infty$, we obtain

$$\begin{aligned}
&\limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \|\lambda z_j + (1-\lambda) T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\le \lambda \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad + (1-\lambda) \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^P \\
&\quad - c_p \cdot \omega_p(\lambda) \cdot \|z_j - T^N z_j\|^p
\end{aligned}$$

and

$$\begin{aligned}
&c_p \cdot \omega_p(\lambda) \cdot \|z_j - T^N z_j\|^p \\
&\le \lambda \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad + (1-\lambda) \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad - \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|\lambda z_j + (1-\lambda) T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\le (1-\lambda) \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad - (1-\lambda) \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p
\end{aligned}$$

and

$$\begin{aligned}
&c_p \cdot [\lambda(1-\lambda)^{p-1} + \lambda^p] \cdot \|z_j - T^N z_j\|^p \\
&\le \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad - \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p.
\end{aligned}$$

Taking $\lambda = 1$, we get

$$\begin{aligned}
c_p \cdot \|z_j - T^N z_j\|^p \le{}& \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&- \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p.
\end{aligned}$$

In view of inequality (4), we have

$$
\begin{aligned}
&c_p \cdot \|z_j - T^N z_j\|^p \\
&\le \limsup_{i\to\infty} \Big[\sum_{k=1}^{N} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad + \sum_{k=N+1}^{\infty} t_{n_i,k} \{\alpha N \cdot \|z_j - T^{k-N+m_i} z_{j-1}\| + \beta_N \cdot \|z_j - T^{k+m_i} z_{j-1}\|\}^p \Big] \\
&\quad - \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
&\le \limsup_{i\to\infty} \Big[\sum_{k=N+1}^{N} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad + 2^{p-1} \Big\{ \alpha_N^p \sum_{k=N+1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k-N+m_i} z_{j-1}\|^p \\
&\quad + \beta_N^p \sum_{k=N+1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \Big\} \Big] \\
&\quad - \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
&\le \limsup_{i\to\infty} \Big[\sum_{k=1}^{N} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad + 2^{p-1} \Big\{ \alpha_N^p \sum_{k=1}^{\infty} t_{n_i,k+N} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad + \beta_N^p \sum_{k=N+1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \Big\} \Big] \\
&\quad - \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
&= \limsup_{i\to\infty} \Big[\sum_{k=1}^{N} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad + 2^{p-1} \Big\{ \alpha_N^p \cdot \Big(\sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \\
&\quad - \sum_{k=1}^{\infty} (t_{n_1,k} - t_{n_i,k+N}) \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \Big) \\
&\quad + \beta_N^p \Big(\sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p - \sum_{k=1}^{N} t_{n_i,k} \|z_j - T^{k+m_i} z_{j-1}\|^p \Big) \Big\} \Big]
\end{aligned}
$$

$$- \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p$$

$$\le [2^{p-1}(\alpha_N^p + \beta_N^p) - 1] \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_{i-1}\|^p$$

since
(i) $\sum_{k=1}^{N} t_{n_i,k} \cdot \|T^N z_j - T^{k+m_i} z_{j-1}\|^p \to 0$ as $i \to +\infty$,
(ii) $\sum_{k=1}^{\infty} (t_{n_i,k} - t_{n_i,k+N}) \cdot \|z_j - T^{k+m_i} z_{j-1}\|^p \to 0$ as $i \to +\infty$,
(iii) $r_{j-1}(z_j) \le r_{j-1}(z_{j-1})$.
For any fixed $N \in \mathbb{N}$, we have

$$c_p \cdot \|z_j - T^N z_j\|^p \le [2^{p-1}(\alpha_N^p + \beta_N^p) - 1] \cdot r_{j-1}(z_{j-1}).$$

We multiply this inequality for $N = k + m_i$ by suitable elements $t_{n_i,k}$ for $k = 1, 2, \ldots$. Summing up these inequalities and taking the limit superior on each side as $i \to +\infty$, we obtain

$$c_p \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_j - T^{k+m_i} z_j\|^p$$

$$\le \lim_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \{2^{p-1}(\alpha_{k+m_i}^p + \beta_{k+m_i}^p) - 1\} \cdot r_{j-1}(z_{j-1})$$

and

$$r_j(z_j) \le B \cdot r_{j-1}(z_{j-1}),$$

where

$$B = \frac{1}{c_p}\Big[\lim_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \{2^{p-1}(\alpha_{k+m_i}^p + \beta_{k+m_i}^p) - 1\}\Big] < 1.$$

In a similar way, we obtain

$$r_j(z_j) \le B^j \cdot r_0(z_0), \quad j = 1, 2, \ldots.$$

Next, we show the convergence of the sequence $\{z_j\}$. For a fixed $N \in \mathbb{N}$, we have

$$\|z_{j+1} - z_j\|^p \le 2^{p-1}(\|z_{j+1} - T^N z_j\|^p + \|T^N z_j - z_j\|^p)$$

We multiply this inequality for $N = k + m_i$ by suitable elements $t_{n_i,k}$ for $k = 1, 2, \ldots$. Summing up these inequalities and taking the limit superior on each side as $i \to +\infty$, we obtain

$$\|z_{j+1} - z_j\|^p \le 2^{p-1}[\limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z_{j+1} - T^N z_j\|^p$$

$$+ \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|T^N z_j - z_j\|^p]$$

$$= 2^{p-1} \cdot (r_j(z_{j+1}) + r_j(z_j))$$

$$\le 2^p \cdot r_j(z_j)$$

$$\le 2^p \cdot B^j \cdot r_0(z_0)$$

and

$$\|z_{j+1} - z_j\| \leq 2[B^j \cdot r_0(z_0)]^{\frac{1}{p}} \to 0 \text{ as } j \to +\infty,$$

which show that $\{z_j\}$ is a Cauchy sequence. Let $z = \lim_{j\to\infty} z_j$. For fixed $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\|z - T^N z\| \\
&\leq \|z - z_j\| + \|z_j - T^N z_j\| + \|T^N z_j - T^N z\| \\
&\leq \frac{1 + a_N + 2c_N}{1 - b_N - c_N} \cdot \|z - z_j\| + \frac{1 + b_N - c_N}{1 - b_N - c_N} \cdot \|z_j - T^N z_j\|
\end{aligned}$$

or

$$\begin{aligned}
&\|z - T^N z\|^p \\
&\leq 2^{p-1} \cdot \Big[\Big(\frac{1 + a_N + 2c_N}{1 - b_N - c_N}\Big)^p \cdot \|z - z_j\|^p + \Big(\frac{1 + b_N + c_N}{1 - b_N - c_n}\Big)^p \cdot \|z_j - T^N z_j\|^p\Big] \\
&\leq 2^{p-1} \cdot \Big[\Big(\frac{1 + a_N + 2c_N}{1 - b_N - c_N}\Big)^p \cdot \|z - z_j\|^p + 2^p \cdot \|z_j - T^N z_j\|^p\Big].
\end{aligned}$$

We multiply this inequality for $N = k + m_i$ by suitable elements $t_{n_i,k}$ for $k = 1, 2, \ldots$. Summing up these inequalities, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z - T^{k+m_i} z\|^p \\
&\leq 2^{p-1} \Big[\|z - z_j\|^p \cdot \sum_{k=1}^{\infty} t_{n_i,k} \cdot \Big(\frac{1 + a_{k+m_i} + 2c_{k+m_i}}{1 - b_{k+m_i} - c_{k+m_i}}\Big)^p \\
&\quad + 2^p \cdot \sum_{k=1}^{\infty} t_{n_i,k} \|z_j - T^{k+m_i} z_j\|^p\Big].
\end{aligned}$$

Taking the limit superior on each side as $i \to +\infty$, we get

$$\begin{aligned}
&\limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \|z - T^{k+m_i} z\|^p \\
&\leq 2^{p-1} \cdot \Big[\|z - z_j\|^p \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \Big(\frac{1 + a_{k+m_i} + 2c_{k+m_i}}{1 - b_{k+m_i} - c_{k+m_i}}\Big)^p \\
&\quad + 2^p \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \|z_j - T^{k+m_i} z_j\|^p\Big] \\
&\leq 2^{p-1} \cdot \Big[\|z - z_j\|^p \cdot \limsup_{i\to\infty} \sum_{k=1}^{\infty} t_{n_i,k} \cdot \Big(\frac{1 + a_{k+m_i} + 2c_{k+m_i}}{1 - b_{k+m_i} - c_{k+m_i}}\Big)^p + 2^p \cdot B^j r_0(z_0)\Big] \\
&\to 0 \text{ as } j \to +\infty
\end{aligned}$$

Therefore,

$$\limsup_{i\to\infty}\sum_{k=1}^{\infty} t_{n_i,k}\cdot\|z - T^{k+m_i}z\|^p = 0.$$

This implies that $T_z = z$. Indeed, for any $\varepsilon > 0$ there exists natural numbers n, $n+1$ such that

$$\|z - T^n z\| < \varepsilon \text{ and } \|z - T^{n+1}z\| < \varepsilon.$$

Otherwise, we have for any n and m.

$$\sum_{k=1}^{\infty} t_{n,k}\cdot\|z - T^{k+m}z\|^p \geq \frac{1}{2}\varepsilon^p$$

and hence

$$\limsup_{i\to\infty}\sum_{k=1}^{\infty} t_{n_i,k}\cdot\|z - T^{k+m_i}z\|^p \geq \limsup_{i\to\infty}\sum_{k=1}^{\infty} t_{n_i,k}\cdot\|z - T^{k+m_i}z\|^p \geq \frac{1}{2}\varepsilon^p.$$

Thus for every natural number l there exists a natural number n_1 such that

$$\|z - T^{n_l}z\| < \frac{1}{l} \text{ and } \|z - T^{n_l+1}z\| < \frac{1}{l}.$$

It follows that

$$T^{n_l}z \to z \text{ and } T^{n_l+1} \to z \text{ as } l \to \infty.$$

Since T is continuous, we have

$$T_z = T(\lim_{l\to\infty} T^{n_l}z) = \lim_{l\to\infty} T^{n_l+1}z = z.$$

This completes the proof.

3. Some Applications

In a Hilbert space H, the following identity holds:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{5}$$

for all x, y in H and $\lambda \in [0,1]$.

By (5), we immediately obtain from Theorem 1 the following:

Corollary 1. *Let K be a nonempty bounded closed convex subset of a Hilbert space H and $A = [t_{n,k}]_{n,k\geq 1}$ a strongly ergodic matrix. If $T : K \to K$ is a continuous generalized Lipschitzian mapping such that*

$$\liminf_{i\to\infty}\inf_{m=0,1,2,\ldots}\sum_{k=1}^{\infty} t_{n,k}\cdot(\alpha_{n+m}^2 + \beta_{k+m}^2) < 1,$$

then T has a fixed point in K.

If $1 < p \leq 2$, then we have for all x, y in L^P and $\lambda \in [0,1]$

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)\cdot(p-1)||x-y||^2 \tag{6}$$

(The inequality (6) is contained in Lim, Xu and Xu [10] and Smarzewski [14].)

Assume that $2 < p < +\infty$ and h_p is the unique zero of the function $g(x) = -x^{p-1} + (p-1)x + p - 2$ in the interval $(1,\infty)$. Let

$$c_p = (p-1)\cdot(1+h_p)^{2-p} = \frac{1+h_p^{p-1}}{(1+h_p)^{p-1}}.$$

Then we have the following inequality:

$$||\lambda x + (1-\lambda)y||^p \leq \lambda||x||^p + (1-\lambda)||y||^p = \omega_p(\lambda)\cdot c_p \cdot ||x-y||^p \tag{7}$$

for all x, y in L^p and $\lambda \in [0,1]$ (The inequality (7) is essentially due to Lim [9]).

Corollary 2. *Let K be a nonempty bounded closed convex subset of $L^p(1 < p < +\infty)$ and $A = [t_{n,k}]_{n,k\geq 1}$ a strongly ergodic matrix. If $T : K \to K$ is a continuous generalized Lipschitzian mapping such that*

$$\liminf_{n\to\infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} t_{n,k}\cdot 2(\alpha_{k+m}^2 + \beta_{k+m}^2) < p, \text{ for } 1 < p \leq 2$$

and

$$\liminf_{n\to\infty} \inf_{m=0,1,2,\ldots} \sum_{k=1}^{\infty} t_{n,k}\cdot 2^{p-1}(\alpha_{k+m}^p + \beta_{k+m}^p) < 1 + c_p, \quad \text{for } p > 2,$$

Then T has a fixed point in K.

Let H^p, $1 < p < +\infty$, denote the Hardy space [4] of all functions x analytic in unit disc $|z| < 1$ of the complex plane and such that

$$||x|| = \lim_{r\to 1^-}\left(\frac{1}{2\pi}\int_0^{2\pi}|x(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} < \infty$$

Now, let Ω be an open subset of $\mathbb{R}^n$. Denote by $H^{k,p}(\Omega), k \geq 0, 1 < p < +\infty$, the Sobolev space [1, p. 149] of distributions x such that $D^\alpha x \in L^p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq k$ equipped with the norm.

$$||x|| = \Big(\sum_{|\alpha|\leq k}\int_\Omega |D^\alpha x(\omega)|^p d\omega\Big)^{\frac{1}{p}}.$$

Let $(\Omega_\alpha, \sum_\alpha, \mu_\alpha), \alpha \in \Lambda$, be a sequence of positive measure spaces, where index set Λ is finite or countable. Given a sequence of linear subspaces X_α in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$, we denote by $L_{q,p}, 1 < p < +\infty$ and $q = \max\{2,p\}$ [11], the linear space of all sequence $x\{x_\alpha \in X_\alpha : \alpha \in \Lambda\}$ equipped with the norm

$$||x|| = \Big(\sum_{\alpha\in\Lambda}(||x_\alpha||_{p,\alpha})^q\Big)^{\frac{1}{\varepsilon}}.$$

where $||\cdot||_{p,\alpha}$ denotes the norm in $L^p(\Omega_\alpha, \sum_\alpha, \mu_\alpha)$.

Finally, let $L_p = L^p(S_1, \sum_1, \mu_1)$ and $L_q = L^q(S_2, \sum_2, \mu_2)$, where $1 < p < +\infty$, $q = \max\{2,p\}$ and $(S_i, \sum_i, \mu_i)$ are positive measure spaces. Denote by $L_q(L_p)$ the Banach spaces [3, III. 2.10] of all measureble L_p−value function x on S_2 such that

$$||x|| = \Big(\int_{S_2}(||x(s)||_p)^q \mu_2(ds)\Big)^{\frac{1}{\varepsilon}}.$$

These spaces are q-uniformly convex with $q = \max\{2,p\}$ [12, 13] and the norm in these spaces satisfies

$$||\lambda x + (1-\lambda)y||^q \leq \lambda||x||^q + (1-\lambda)||y||^q - d\cdot\omega_q(\lambda)\cdot||x-y||^q$$

with a constant

$$d = d_p = \begin{cases} \frac{p-1}{8} & \text{for } 1 < p \leq 2 \\ \frac{1}{p\cdot 2^p} & \text{for } 2 < p < +\infty \end{cases}$$

Here, from Theorem 1, we have the following result:

Corollary 3. *Let K be a nonempty bounded closed convex subset of the space E, where $E = H^p$, or $E = H^{k,p}(\Omega)$, or $E = L_{q,p}$, or $E = L_q(L_p)$, and $1 < p < +\infty$, $q = \max\{2,p\}$, $k \geq 0$ and $A = [t_{n,k}]_{n,k\geq 1}$ is a strongly ergodic matrix. If $T : K \to K$ is a continuous generalized Lipschitzian mapping such that*

$$\liminf_{n\to\infty}\ \inf_{m=0,1,2,\dots}\sum_{k=1}^{\infty} t_{n,k}\cdot 2^{q-1}\{\alpha_{k+m}^q + \beta_{k+m}^q\} < 1 + d_p,$$

then T has a fixed point in K.

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