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MAJORIZING SEQUENCES FOR NEWTON'S METHOD

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Abstract. Majorizing sequences for Newton's method are analysed from a new standpoint. As a consequence, we give convergence results under assumptions different from the classical Kantorovich conditions.

One of the most known techniques for studying a sequence $\{x_n\}$ defined in Banach spaces is majorizing theory. In brief, consider an equation $x = G(x)$, where G is an operator defined on the ball $\|x - x_0\| < R$ in some Banach space X. Take also the real equation $t = g(t)$, where g is defined in the interval $[a_0, a_1]$, $(a_1 - a_0 < R)$. We shall say that equation $t = g(t)$ (or function g) majorizes equation $x = G(x)$ (or operator G) if

$$\|G(x_0) - x_0\| \leq g(t_0) - t_0, \quad t_0 \in [a_0, a_1] \tag{1}$$

$$\|G'(x)\| \leq g'(x), \quad \text{whenever } \|x - x_0\| \leq t - t_0. \tag{2}$$

In this situation, the convergence of the iterative process $x_{n+1} = G(x_n)$ in X can be deduced from the one of the iteration $t_{n+1} = g(t_n)$ on the real line [1], [2].

For solving a nonlinear operator equation

$$F(x) = 0 \tag{3}$$

we can consider Newton's method:

$$x_{n+1} = G(x_n), \quad G(x) = x - F'(x)^{-1}F(x). \tag{4}$$

Here F is an operator defined from a Banach space X into another Banach space Y. Following Kantorovich [1], we shall assume that F is defined and has a continuous second derivative in a closed ball $\Omega_0 = \{x \in X; \|x - x_0\| \leq R_0\}$. Assume, in addition, that

(i) $\Gamma_0 = F'(x_0)^{-1}$ exists and is a continuous linear operator,
(ii) $\|\Gamma_0 F(x_0)\| \leq a$,
(iii) $\|\Gamma_0 F''(x)\| \leq b$, $x \in \Omega_0$.

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Then if $h = ab \leq 1/2$, and $(1-\sqrt{1-2h})/b \leq R_0$, the equation (3) has a solution x^* and the sequence $\{x_n\}$ defined in (4) converges to x^*. Additional information on uniqueness of solution and error bounds can be seen in [1], [3].

The convergence of the sequence $\{x_n\}$ follows from the convergence of the real sequence

$$t_{n+1} = t_n - \frac{p(t_n)}{p'(t_n)}, \quad t_0 = 0,$$

where

$$p(t) = \frac{b}{2}t^2 - t + a. \tag{5}$$

On the other hand, in [4] it was shown that the derivative of G defined in (4) is the linear operator $L_F(x) : X \to X$ given by

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x).$$

Now we are going to study Newton sequence in terms of this linear operator $L_F(x)$ and its correspondent real expression.

Notice that condition (1) tells us $||x_1 - x_0|| = ||\Gamma_0 F(x_0)|| \leq a = t_1 - t_0$. Besides, if $x \in [x_0, x_1]$, i.e., $x = x_0 + s(x_1 - x_0)$, $0 \leq s \leq 1$, then we have $||L_F(x)|| \leq L_p(t)$ where $t \in [t_0, t_1]$. Indeed,

$$\begin{aligned}\Gamma_0 F(x) &= \Gamma_0 F(x_0) + (x - x_0) + \int_{x_0}^{x} \Gamma_0 F''(z)(x-z)dz \\ &= \Gamma_0 F(x_0)(1-s) + \int_{x_0}^{x} \Gamma_0 F''(z)(x-z)dz.\end{aligned}$$

Then

$$||\Gamma_0 F(x)|| \leq (1-s)a + \frac{b}{2}s^2a^2 = a - t + \frac{b}{2}t^2,$$

where $t = sa \in [0, t_1]$.

Following a standard technique and using Banach lemma on invertion of operators [3], we have

$$||[\Gamma_0 F'(x)]^{-1}|| \leq \frac{1}{1 - b||x - x_0||} \leq \frac{1}{1-bt}.$$

Then

$$||L_F(x)|| \leq \frac{b(a - t + (b/2)t^2)}{(1-bt)^2} = L_p(t),$$

and therefore

$$||x_2 - x_1|| = ||G(x_1) - G(x_0)|| = ||\int_{x_0}^{x_1} L_F(x)dx|| \leq \int_{t_0}^{t_1} L_p(t)dt = t_2 - t_1.$$

In general, we can replace x_0 by x_1, x_1 by x_2 and so on, obtaining $||L_F(x)|| \leq L_p(t)$ for $x \in [x_n, x_{n+1}]$ and $t \in [t_n, t_{n+1}]$. What we have just proved is that, under Kantorovich assumptions, conditions (1) and (2) are fulfilled.

Now we are going to follow the inverse reasoning, that is, assume there exists a function $\omega(t)$ satisfying $||L_f(x)|| \leq \omega(t)$. If a function f is such that $L_f(t) = \omega(t)$, we can use it to prove the convergence of (4). Under Kantorovich assumptions we have just seen $\omega(t) = L_p(t)$, where $p(t)$ is the polynomial given by (5). We wonder now, if we have another function $\omega(t)$, can we obtain convergence conditions different from the ones of Kantorovich? The answer is affirmative, as we see next.

Let us consider, for instance, the situation $||L_F(x)|| \leq M < 1$, $x \in \overline{B(x_0, R_0)}$. Functions f with a positive root r_1 and satisfying $L_f(t) = M$ have the form

$$f(t) = (r_1 - t)^{1/(1-M)}. \tag{6}$$

Notice that this kind of functions are decreasing and convex in the interval $[0, r_1]$. Then, Newton's method for solving $f(t) = 0$ is convergent starting at any point of $[0, r_1]$ (see [5]).

If $t_0 = 0$ and $||\Gamma_0 F(x_0)|| \leq a$, then condition (1) holds if and only if $a \leq -f(t_0)/f'(t_0) = r_1(1 - M)$, that is, $r_1 \geq a/(1 - M)$.

Finally, in order to assure that Newton sequence lies in the ball $\overline{B(x_0, R_0)}$ we have to require $r_1 \leq R_0$. So we have just proved the following result.

Theorem 1. *Let $F(x) = 0$ be an equation, where F is an operator defined between two Banach spaces X and Y. Assume that F satisfies $||L_F(x)|| \leq M < 1$, $x \in \overline{B(x_0, R_0)}$. If $||\Gamma_0 F(x_0)|| \leq a$ and $a/(1 - M) \leq R_0$, then the Newton sequence*

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}, \quad t_0 = 0,$$

where f is given by (6), *majorizes the sequence $\{x_n\}$ given by* (4). *Besides, as $\{t_n\}$ converges to r_1, $\{x_n\}$ is also convergent, and the limit x^* is a solution of* (3).

Notice that the above result is more general than Kantorovich theorem. Just taking the scalar function $f(t) = (1 - t)^4$ we have $L_f(t) = 3/4$ for all $t \in \mathbb{R}$ (define $L_f(1) = \lim_{t \to 1} L_f(t)$). However, Kantorovich conditions are not fulfilled in the interval $[0, 1]$. Indeed,

$$a = -\frac{f(0)}{f'(0)} = \frac{1}{4}, \quad b = -\sup_{t \in [0,1]} \frac{f''(t)}{f'(0)} = 3 \quad \text{and } h = ab = \frac{3}{4} > \frac{1}{2}.$$

On the other hand, $a/(1 - M) = 1 = r_1$ and the previous result guarantees the convergence of Newton's method starting at $t_0 = 0$.

By using Theorem 1 we have got a wider domain of starting points for Newton's method. However error estimates are, in general, worse than the ones obtained from polynomial (5). Notice that function (6) has a multiple root and in this situation it is well known [6] that the convergence of Newton's method is linear.

Instead of condition $||L_F(x)|| \leq M < 1$ we can consider other bounds for $||L_F(x)||$, not necessarily constants. For instance, if we have

$$||L_F(x)|| \leq \alpha(r - t) \text{ for } ||x - x_0|| \leq t - t_0,$$

where $\alpha r < 2$ and $||\Gamma_0 F(x_0)|| \leq a \leq r(2 - \alpha r)/2$, the majorizing function is

$$f(t) = \frac{r-t}{\beta+t}, \quad \beta = \frac{2}{\alpha} - r.$$

This kind of conditions have a theorical interest, but in practice it is difficult to find convenient bounds for $||L_F(x)||$. Nevertheless, when this situation happens, we have developed a new method to find majorizing sequences. This method allow us to establish convergence under assumptions different from the classical Kantorovich conditions.

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