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ON WARPED PRODUCT MANIFOLDS —CONFORMAL FLATNESS AND CONSTANT SCALAR CURVATURE PROBLEM

KWANG-WU YANG

Abstract. In this paper, we study some geometric properties on doubly or singly warped-product manifolds. In general, on a fixed topological product manifold, the problem for finding warped-product metrics satisfying certain curvature conditions are finally reduced to find positive solutions of linear or non-linear differential equations. Here, we are mainly interested in the following problems on essentially warped-product manifolds: one is the sufficient and necessary conditions for conformal flatness, and the other is to find warped-product metrics so that their scalar curvatures are contants.

1. Introduction

Since the famous work of B. L. Bishop and B. O'Neill [2], the method of warped-product had been studied, and proved to be important for constructing new metrics explicitly. In general, if (M^m, g) and (N^n, h) are smooth manifolds, Riemannian or pseudo-Riemannian, and $f : M \to R$ and $\phi : N \to R$ are positive smooth functions. The doubly warped-product metric $\tilde{g} \equiv g_\phi \times_f h$ on the (topological) product space $M \times N$ is defined by

$$\tilde{g}_{(p,q)}(X,Y) = \phi(q) g_p(\pi_*(X), \pi_*(Y)) + f(p) h_q(\sigma_*(X), \sigma_*(Y)),$$

where $(p,q) \in M \times N$, $\pi : M \times N \to M$ and $\sigma : M \times N \to N$ being the natural projections. We shall denote this new manifold by $(M \times N, g_\phi \times_f h)$, or simply $M_\phi \times_f N$ if there is no confusions on metrics. If it is the case $\phi \equiv 1$, we obtain a (singly) warped-product manifold $(M \times N, g \times_f h)$ or $M \times_f N$.

It is useful to establish the elementary formulae and equations relating the geometric objects between g, h, and $g_\phi \times_f h$. Standing on this situation, one consider the following problems: Among the warped=product metrics, find one to satisfy certain geometric or topological conditions. For example, find a metric which is locally symmetric, or conformally flat, or whose scalar curvature is a constant, etc. We shall generalize a result of M. Hotlös ([13], Lemma 1) so that in all dimensions, it is necessary that M and N are

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conformally flat if $M_\phi \times_f N$ does (Theorem 4). On the other hand, we shall see, after general arguments, that one cannot obtain metrics satisfying some curvature conditions by using warped=product constructions. For instance, if M and N are 2-dimensional compact Riemannian manifolds without boundaries such that $\chi(M)\chi(N) > 0$, then $M_\phi \times_f N$ cannot be conformally flat, no matter whether ϕ and f are constant functions ro not (Theorem 3).

We shall prove in Theorem 1 that an essentially (i.e., the warpping functions are not constants) doubly warped-product manifold cannot have constant scalar curvature except possibly for zero. On the other hand, the constant scalar curvature problem on singly warped-product cases have very different and more interesting results. In [10], F. Dobarro and E. Lami-Dozo looked for warpping functions such that a singly warped-product manifold has constant scalar curvature and then determined which constants will be attained. Following their works, we obtain a lot of results for the essential case (Corollary 2; Theorem 11).

2. Doubly Warped-Product Manifolds

2.1. Notaion and Formulae

Let $(\tilde{M}, \tilde{g}) = (M^m, g)_\phi \times_f (N^n, h)$ be a doubly warped-product Riemannian manifold, where $\dim M = m$, $\dim N = n$, and f and ϕ are positive smooth functions defined on M and N, respectively. The components of geometric objects on M, N, and $\tilde{M}$ will be labeled as in the following table:

Manifold	M	N	$\tilde{M}$
Local coordinate	(x^i)	(y^α)	(x^i, y^α)
Metric tensor	g_{ij}	$h_{\alpha\beta}$	$\tilde{g}_{AB}$
Levi-Civita connection	D	∇	$\tilde{D}$
Riemann-Christoffel symbols	Γ^i_{jk}	$H^\alpha_{\beta\gamma}$	$\tilde{\Gamma}^A_{BC}$
Riemann curvature tensor	R^i_{jkl}	$K^\alpha_{\beta\gamma\delta}$	$\tilde{R}^A_{BCD}$
Ricci curvature tensor	R_{ij}	$K_{\alpha\beta}$	$\tilde{R}_{AB}$
Scalar curvature	R	K	$\tilde{R}$

where, now and in the sequel, the ranges of indices are $1 \le i, j, k, l, \ldots \le m$; $m+1 \le \alpha, \beta, \gamma, \delta, \ldots \le m+n$; $1 \le A, B, C, D, \ldots \le m+n$. We also use the following abbreviations:

$$\partial_i = \frac{\partial}{\partial x^i}, \quad \partial_\alpha = \frac{\partial}{\partial y^\alpha}, \quad D_i = D_{\partial i}, \quad \nabla_\alpha = \nabla_{\partial_\alpha}, \quad \Delta_M f = g^{ij} D_i f_j,$$

$$f_i = \frac{\partial f}{\partial x^i}, \quad \phi_\alpha = \frac{\partial \phi}{\partial y^\alpha}, \quad f^j = f_i g^{ij}, \quad \phi^\beta = \phi_\alpha h^{\alpha\beta}, \quad \Delta_N \phi = h^{\alpha\beta} \nabla_{\alpha\phi\beta},$$

$F = f^j \partial_j =$ the gradient vector field of f,

$\Phi = \phi^\beta \partial_\beta =$ the gradient vector field of ϕ,

where $(g^{ij}) = (g_{ij})^{-1}, (h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$, and we use the convention for summing the repeated indices.

The quantities defined below and equations from (1) to (7), except for (5), were firstly established by M. Hotlös [13] (One should note here that we adopt the convention of notations of the Riemann curvature tensors as in [14], so the following equations are not all the same as in [13]).

$$G_{ijkl} = g_{ik}g_{il} - g_{il}g_{ik}\ ;\ G'_{\alpha\beta\gamma\delta} = h_{\alpha\gamma}h_{\beta\delta} - h_{\alpha\delta}h_{\beta\gamma}$$
$$T_{ij} = \frac{-1}{2f}\Big(D_i f_j - \frac{1}{2f} f_i f_j\Big)\ ;\ T'_{\alpha\beta} = \frac{-1}{2\phi}\Big(\nabla_{\alpha\phi\beta} - \frac{1}{2\phi}\phi_\alpha\phi_\beta\Big)$$
$$tr(T) = g^{ij}T_{ij}\ ;\ tr(T') = h^{\alpha\beta}T'_{\alpha\beta}$$
$$I = \frac{1}{4f^2}g^{ij}f_i f_j = \frac{\|F\|^2}{4f^2}\ ;\ I' = \frac{1}{4\phi^2}h^{\alpha\beta}\phi_\alpha\phi_\beta = \frac{\|\Phi\|^2}{4\phi^2}$$
$$J = f[(n-1)I - tr(T)]\ ;\ J' = \phi[(m-1)I' - tr(T')]$$

where $\|\cdot\|$ is the length of vector fields. Note that, $tr(T) = [-(\Delta_M f)/(2f)] + I$ and $tr(T') = [-(\Delta_N \phi)/(2\phi)] + I'$. Thus, using the formulae $\Gamma^i_{jk} \doteq g^{il}(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})/2$, etc, the Riemann-Christoffel symbols between M, N, and $M_\phi \times_f N$ are related as

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \tilde{\Gamma}^\alpha_{\beta\gamma} = H^\alpha_{\beta\gamma}, \quad \tilde{\Gamma}^i_{j\gamma} = \frac{1}{2\phi}\phi_\gamma\delta^i_j, \tag{1}$$
$$\tilde{\Gamma}^\alpha_{\beta k} = \frac{1}{2f}f_k\delta^\alpha_\beta, \quad \tilde{\Gamma}^i_{\beta\gamma} = \frac{-1}{2\phi}f^i h_{\beta\gamma}, \quad \tilde{\Gamma}^\alpha_{jk} = \frac{-1}{2f}\phi^\alpha g_{jk}.$$

Passing through direct computations, the only non-vanishing components of the Riemann curvature tensor on $\tilde{M}$ are

$$\tilde{R}_{ijkl} = \phi R_{ijkl} - \frac{\|\Phi\|^2}{4f}G_{ijkl}, \quad \tilde{R}_{\alpha\beta\gamma\delta} = fK_{\alpha\beta\gamma\delta} - \frac{\|F\|^2}{4\phi}G'_{\alpha\beta\gamma\delta}, \tag{2}$$
$$\tilde{R}_{i\beta k\delta} = fT_{ik}h_{\beta\delta} + \phi T'_{\beta\delta}g_{ik}, \quad \tilde{R}_{ijk\delta} = \frac{\phi_\delta}{4f}(f_j g_{ik} - f_i g_{jk}),$$
$$\tilde{R}_{\alpha\beta\gamma l} = \frac{f_l}{4\phi}(\phi_\beta h_{\alpha\gamma} - \phi_\alpha h_{\beta\gamma}).$$

It follows from the contracting processes that the only non-vanishing components of the Ricci curvature tensor on $\tilde{M}$ are

$$\tilde{R}_{ij} = R_{ij} + nT_{ij} - \frac{J'}{f}g_{ij}, \quad \tilde{R}_{\alpha\beta} = K_{\alpha\beta} + mT'_{\alpha\beta} - \frac{J}{\phi}h_{\alpha\beta}, \tag{3}$$
$$\tilde{R}_{i\beta} = \frac{m+n-2}{4f\phi}f_i\phi_\beta.$$

Hence the scalar curvatures of M, N, and $\tilde{M}$ are related as

$$\tilde{R} = \frac{1}{\phi}[R - n(n-1)I + 2n(tr(T))] + \frac{1}{f}[K - m(m-1)I' + 2m(tr(T'))]. \tag{4}$$

Now let W, C, and $\tilde{W}$ be the Weyl conformal curvature tensors on M, N, and $\tilde{M}$, respectively. That is

$$W_{ijkl} = R_{ijkl} - \frac{1}{m-2}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{ik}R_{il}) \tag{5}$$
$$+\frac{R}{(m-1)(m-2)}G_{ijkl},$$

and similar expressions for those of $C_{\alpha\beta\gamma\delta}$ and $\tilde{W}_{ABCD}$. Therefore, the only non-vanishing components of $\tilde{W}$ are

$$\tilde{W}_{ijkl} = \phi\Big[R_{ijkl} - \frac{1}{m+n-2}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il})\Big] \tag{6}$$
$$-\frac{n\phi}{m+n-2}(g_{ik}T_{jl} + g_{jl}T_{ik} - g_{il}T_{jk} - g_{jk}T_{il})$$
$$-G_{ijkl}\Big[\frac{\|\Phi\|^2}{4f} - \frac{2\phi J'}{(m+n-2)f} - \frac{\phi^2\tilde{R}}{(m+n-1)(m+n-2)}\Big],$$
$$\tilde{W}_{\alpha\beta\gamma\delta} = f\Big[K_{\alpha\beta\gamma\delta} - \frac{1}{m+n-2}(h_{\alpha\gamma}K_{\beta\delta} + h_{\beta\delta}K_{\alpha\gamma} - h_{\alpha\delta}K_{\beta\gamma} - h_{\beta\gamma}K_{\alpha\delta})\Big]$$
$$-\frac{mf}{m+n-2}(h_{\alpha\gamma}T'_{\beta\delta} + h_{\beta\delta}T'_{\beta\gamma} - h_{\alpha\delta}T'_{\beta+\gamma} + h_{\beta\gamma}T'_{\alpha\delta})$$
$$-G'_{\alpha\beta\gamma\delta}\Big[\frac{\|F\|^2}{4\phi} - \frac{2fJ}{(m+n-2)\phi} - \frac{f^2\tilde{R}}{(m+n-1)(m+n-2)}\Big],$$
$$\tilde{W}_{i\beta k\delta} = \frac{1}{m+n-2}\{[(n-2)T'_{\beta\delta} - K_{\beta\delta}]\phi g_{ik} + [(m-2)T_{ik} - R_{ik}]fh_{\beta\delta}$$
$$+\Big[J + J' + \frac{f\phi\tilde{R}}{m+n-1}\Big]g_{ik}f_{\beta\delta}\}.$$

Finally, the non-vanishing components of those of $\tilde{D}_E\tilde{R}_{ABCD}$ are

$$
\begin{aligned}
\tilde{D}_h\tilde{R}_{ijkl} &= \phi D_h R_{ijkl} + \frac{\|\Phi\|^2}{8f^2}(2f_h G_{ijkl} \\
&\quad + f_i G_{hjkl} + f_j G_{ihkl} + f_k G_{ijhl} + f_l G_{ijkh}), \\
\tilde{D}_\varepsilon\tilde{R}_{\alpha\beta\gamma\delta} &= f\nabla_\varepsilon K_{\alpha\beta\gamma\delta} + \frac{\|F\|^2}{8\phi^2}(2\phi_\varepsilon G'_{\alpha\beta\gamma\varepsilon} \\
&\quad + \phi_\alpha G'_{\varepsilon\beta\gamma\delta} + \phi_\beta G'_{\alpha\varepsilon\gamma\delta} + \phi_\gamma G'_{\alpha\beta\varepsilon\delta} + \phi_\delta G'_{\alpha\beta\gamma\varepsilon}), \\
\tilde{D}_\varepsilon\tilde{R}_{ijkl} &= -\Big[\phi_\varepsilon R_{ijkl} + \frac{\phi^2}{f}(\partial_\varepsilon I')G_{ijkl} \\
&\quad + \frac{\phi_\varepsilon}{4f^2}(f_i f_k g_{jl} + f_j f_l g_{ik} - f_i f_l g_{jk} - f_j f_k g_{il})\Big], \\
\tilde{D}_h\tilde{R}_{\alpha\beta\gamma\delta} &= -\Big[f_n K_{\alpha\beta\gamma\delta} + \frac{f^2}{\phi}(\partial_h I)G'_{\alpha\beta\gamma\delta} \\
&\quad + \frac{f_h}{4\phi^2}(\phi_\alpha\phi_\gamma h_{\beta\delta} + \phi_\beta\phi_\delta h_{\alpha\delta} - \phi_\alpha\phi_\delta h_{\beta\gamma} - \phi_\beta\phi_\gamma h_{\alpha\delta}\Big)], \\
\tilde{D}_h\tilde{R}_{ijk\delta} &= \frac{f_h\phi_\delta}{4f^2}(f_i g_{jk} - f_j g_{jk}) - \frac{\phi^2}{2f}(\partial_\delta I')G_{ijkh} \\
&\quad + \frac{\phi_\delta}{2}(g_{jh}T_{ik} - g_{ik}T_{jh} + g_{jk}T_{ih} - g_{ih}T_{jk} - R_{ijkh}), \\
\tilde{D}_\varepsilon\tilde{R}_{\alpha\beta\gamma l} &= \frac{f_l\phi_\varepsilon}{4\phi^2}(\phi_\alpha h_{\beta\gamma} - \phi_\beta h_{\alpha\gamma}) - \frac{f^2}{2\phi}(\partial_l I)G'_{\alpha\beta\gamma\varepsilon} \\
&\quad + \frac{f_l}{2}(h_{\beta\varepsilon}T'_{\alpha\gamma} - h_{\alpha\gamma}T'_{\beta\varepsilon} + h_{\beta\gamma}T'_{\alpha\varepsilon} - h_{\alpha\varepsilon}T'_{\beta\gamma} \dot{-} K_{\alpha\beta\gamma\varepsilon}), \\
\tilde{D}_\varepsilon\tilde{R}_{ijk\delta} &= \frac{\phi T'_{\varepsilon\delta}}{f}(f_i g_{ik} - f_j g_{ik}) + \frac{h_{\varepsilon\delta}}{2}(f_i T_{jk} - f_j T_{ik} + R_{ijkl}f^l) \\
&\quad + \frac{1}{8f\phi}(f_i g_{jk} - f_j g_{jk})(\|\Phi\|^2 h_{\varepsilon\delta} + 2\phi_\varepsilon\phi_\delta), \\
\tilde{D}_h\tilde{R}_{\alpha\beta\gamma l} &= \frac{fT_{hl}}{\phi}(\phi_\alpha h_{\beta\gamma} - \phi_\beta h_{\alpha\gamma}) + \frac{g_{hl}}{2}\Big(\phi_\alpha T'_{\beta\gamma} - \phi_\beta T'_{\alpha\gamma} + K_{\alpha\beta\gamma\delta}\phi^\delta\Big) \\
&\quad + \frac{1}{8f\phi}(\phi_\alpha h_{\beta\gamma} - \phi_\beta h_{\alpha\gamma})(\|F\|^2 g_{hl} + 2f_h f_l), \\
\tilde{D}_h\tilde{R}_{i\beta k\delta} &= f h_{\beta\delta}D_h T_{ik} + \frac{\|\Phi\|^2 h_{\beta\delta}}{8f\phi}(f_i g_{hk} + f_k g_{ih}) \\
&\quad - \frac{f_h g_{ik}}{f}\Big(\phi T'_{\beta\delta} + \frac{\phi_\beta\phi_\delta}{4\phi}\Big), \\
\tilde{D}_\varepsilon\tilde{R}_{i\beta k\delta} &= \phi g_{ik}\nabla_\varepsilon T'_{\beta\delta} + \frac{\|F\|^2 g_{ik}}{8f\phi}(\phi_\beta h_{\varepsilon\delta} + \phi_\delta h_{\beta\varepsilon}) \\
&\quad - \frac{\phi_\varepsilon h_{\beta\delta}}{\phi}\Big(fT_{ik} + \frac{f_i f_k}{4f}\Big).
\end{aligned} \tag{7}
$$

Clearyly, it follows from the last equation in (3) that a doubly warped-product manifold $M_\phi \times_f N$ is never Einstein if both of the warpping functions f and ϕ are not constant. In particular, it's never a space of constant curvature. Actually, we have

Theorem 1. *If $\tilde{M} = M_\phi \times_f N$ has constant scalar curvature $\tilde{R}$, where f and ϕ are not constant functions, then $\tilde{R} = 0$.*

Proof. Suppose that $\tilde{R}$ is a non-zero constant. It follows from (4) that

$$\begin{aligned} 0 &= \phi \partial_k \tilde{R} \\ &= \partial_k[R - n(n-1)I + 2n(tr(T))] - \frac{\phi f_k}{f^2}[K - m(m-1)I' + 2m(tr(T'))]. \end{aligned}$$

Since f_k/f^2 and the function in the first bracket are those defined on M, while the function in the second bracket and ϕ are defined on N, we see that, in virtue of $f_k \not\equiv 0$,

$$\phi[K - m(m-1)I' + 2m(tr(T'))] = \text{constant}.$$

Similarly, by $f\partial_\varepsilon \tilde{R} = 0$ and $\phi_\varepsilon \not\equiv 0$, we have

$$f[R - n(n-1)I + 2n(tr(T))] = \text{constant}.$$

Adding these two constants together then going back to (4), we find that $\phi f \tilde{R}$ is a constant. However, since $\tilde{R} \neq 0$ is itself a constant, this implies that both of f and ϕ are constants, which is a contradiction. Therefore $\tilde{R} = 0$.

Remark. It would be an interesting problem to determine whether there does exist an essentially doubly warped-product manifold with vanishing scalar curvature or not.

2.2. Conformal Flatness

A Riemannian manifold (M, g) is called conformally flat if g is locally conformally equivalent to the Euclidean metric. Thus for dim $M \geq 4$, (M, g) is conformally flat if its Weyl conformal curvature tensor vanishes identically. For dim $M = 3$, we need the following classical result due to H. Weyl. (c.f. [17], p. 21)

Theorem 2. *Let (M, g) be a 3-dimensional Riemannian manifold with Ricci curvature tensor R_{ij} and scalar curvature R. Then (M, g) is conformally flat if* *and only if*

$$D_k R_{ij} - D_j R_{ik} = \frac{1}{4}(g_{ij}\partial_k R - g_{ik}\partial_j R)$$

holds for all i, j, and k.

The following lemma, due to M. Hotlös, will play an important role in the rest of this thesis. For the proof, see [13]. Now, let $\tilde{M} = M_\phi \times_f N$, dim $M = m$, dim $N = n$.

Lemma 1. (i) *If $\tilde{W}_{ijkl} = 0$, then*

$$n(n-1)f[R - m(m-1)I - 2(m-1)tr(T)] \\ +m(m-1)\phi[K - n(n-1)I' - 2(n-1)tr(T')] = 0 \tag{8}$$

$$R_{ik} - (m-2)T_{ik} = \frac{g_{ik}}{m}[R - (m-2)tr(T)]. \tag{9}$$

Furthermore, if $m > 1$, then

$$(m-2)R_{ijkl} = (g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}) - \frac{R}{m-1}G_{ijkl}. \tag{10}$$

(ii) If $\tilde{W}_{\alpha\beta\gamma\delta} = 0$, then (8) still holds and

$$K_{\alpha\beta} - (n-2)T'_{\alpha\beta} = \frac{h_{\alpha\beta}}{n}[K - (n-2)tr(T')]. \tag{11}$$

Similarly, for $n > 1$, we have

$$(n-2)K_{\alpha\beta\gamma\delta} = (h_{\alpha\gamma}K_{\beta\delta} + h_{\beta\delta}K_{\alpha\gamma} - h_{\alpha\delta}K_{\beta\gamma} - h_{\beta\gamma}K_{\alpha\delta}) - \frac{K}{n-1}G'_{\alpha\beta\gamma\delta}. \tag{12}$$

(iii) If $\tilde{W}_{ijkl} = \tilde{W}_{\alpha\beta\gamma\delta} = 0$, then $\tilde{W}_{i\beta k\delta} = 0$. Consequently, $\tilde{M}$ is conformally flat provided $m + n \geq 4$.

For dim$M$ =dim$N = 2$, we have the following topological obstruction result:

Theorem 3. *Let (M, g) and (N, h) be 2-dimensional compact surfaces without boundaries. If $M_\phi \times_f N$ is conformally flat, then $\chi(M)\chi(N) \leq 0$ with equality holds if and only if M and N are homeomorphic to T^2—the 2-torus, where χ is the Eular-Poincaré characteristic number.*

Proof. It follows from (8) that for some $\lambda \in R$

$$f[R - 2I - 2tr(T)] = \lambda, \quad \phi[K - 2I' - 2tr(T')] = -\lambda.$$

which , in virtue of definitions of I, I', $tr(T)$, $tr(T')$, are equivalent to

$$R - \Delta_M(\ln f) = \frac{\lambda}{f}, \quad K - \Delta_N(\ln \phi) = \frac{-\lambda}{\phi}.$$

Integrating these equations on M and N, respectively, then applying the Stoke's Theorem and Gauss-Bonnet formula, we find

$$4\pi\chi(M) = \lambda \int_M \frac{1}{f}, \quad 4\pi\chi(N) = -\lambda \int_N \frac{1}{\phi}.$$

Therefore, we have

$$\chi(M)\chi(N) = \frac{-\lambda^2}{16\pi^2}\Big(\int_M \frac{1}{f}\Big)\Big(\int_N \frac{1}{\phi}\Big) \leq 0.$$

Finally, $\chi(M)\chi(N) = 0$ if and only if $\lambda = 0$, which is equivalent to $\chi(M) = \chi(N) = 0$, thus M and N are topological 2-torus.

Theorem 4. *Let $M_\phi \times_f N$ be conformally flat. Then M and N are also conformally flat.*

Proof. It is trivial if dim$M$ =dim$N = 1$. For dim$M$ =dim$N = 2$, it follows from the existence of isothermal coordinates. Thus it suffices to show for dim $M \geq 3$ and dim$N \geq 3$. Firstly, It follows from (10) and (12) that M and N are conformally flat if dim$M \geq 4$ and dim$N \geq 4$.

Next, for dim$M = m = 3$, it follows from (9) that

$$\begin{aligned} D_kR_{ij} - D_jR_{ik} = &\frac{1}{3}(g_{ij}\partial_k R - g_{ik}\partial_j R) + (D_kT_{ij} - D_jT_{ik}) \\ &-\frac{1}{3}[g_{ij}\partial_k tr(T) - g_{ik}\partial_j tr(T)]. \end{aligned} \tag{13}$$

Applying the Ricci identity and (9), (10), we get

$$\begin{aligned} D_kT_{ij} - D_jT_{ik} = &-\frac{1}{2f}(g_{ik}R_{jl}f^l - g_{ij}R_{kl}f^l) \\ &+\frac{1}{12f}(f_jg_{ik} - f_kg_{ij})[R + 2tr(T)]. \end{aligned} \tag{14}$$

On the other hand, in virtue of (8), we have

$$\begin{aligned} \frac{1}{3}[g_{ij}\partial_k tr(T) - g_{ik}\partial_j tr(T)] = &\frac{1}{12}(g_{ij}\partial_k R - g_{ik}\partial_j R) \\ &-\frac{1}{2}(g_{ij}\partial_k I - g_{ik}\partial_j I) + \frac{1}{12f}(f_kg_{ij} - f_jg_{ik}). \end{aligned} \tag{15}$$

Therefore, substituting (14) and (15) into (13), then by Theorem 2, we see that M is conformally flat. A similar argument shows that N is also conformally flat if $\dim N = 3$.

Theorem 5. *Let L be an interval in the real line R with the standard Euclidean metric. Then $L_\phi \times_f N^n$ $(n \geq 3)$ is conformally flat if and only if N is conformally flat and ϕ satisfies the equation* (11).

Proof. It suffices to show that if N is conformally flat and (11) holds, then $L_\phi \times_f N$ is conformally flat.

Note that, (12) holds whenever (i) $n = 3$, or (ii) $n \geq 4$ and N is conformally flat. Therefore, substituting (11) and (12) into the right hand side of the second equation in (6), we see that $\tilde{W}_{\alpha\beta\gamma\delta} = 0$. Moreover, since $\dim L = 1$, $\tilde{W}_{ijkl} \equiv 0$. Hence by (iii) of Lemma 1, $L_\phi \times_f N$ is conformally flat.

2.3. Conformally Symmetric Doubly Warped-Product Manifolds

A Riemannian manifold (M, g), with $\dim M \geq 4$, is said to be conformally symmetric if its Weyl conformal curvature tensor W is parallel, that is, $D_h W_{ijkl} = 0$ for all i, j, k, l, and h. This was firstly studied by M. C. Chaki and B. Gupta [4]. clearly, conformally flat or locally symmetric manifolds are conformally symmetric. A conformally symmetric manifold is said to be essential if it's neither conformally flat nor locally symmetric. This class of manifolds had been deeply studied by A. Derdzinski and W. Roter [7], [8], [16]. Using their method, M. Hotlös [13] firstly proved the result described in Theorem 6. We shall give a simple and direct proof by applying some observations to $\tilde{D}_E\tilde{W}_{ABCD}$.

Lemma 2. *On $M_\phi \times_f N$, the followings hold:*

$$\phi\tilde{D}_\varepsilon\tilde{W}_{ijkl} = -\phi_\varepsilon\tilde{W}_{ijkl} + \frac{\phi^2 G_{ijkl}\partial_\varepsilon\{\phi[K - n(n-1)I' - 2(n-1)tr(T')]\}}{(m+n-1)(m+n-2)f}. \quad (16)$$

$$f\tilde{D}_h\tilde{W}_{\alpha\beta\gamma\delta} = -f_h\tilde{W}_{\alpha\beta\gamma\delta} + \frac{f^2 G'_{\alpha\beta\gamma\delta}\partial_h\{f[R - m(m-1)I - 2(m-1)tr(T)]\}}{(m+n-1)(m+n-2)\phi}. \quad (17)$$

Theorem 6. *Let $M_\phi \times_f N$ be conformally symmetric. Then it is conformally flat.*

Proof. By assumption, $\tilde{D}_E\tilde{W}_{ABCD} \equiv 0$. It follows from direct calculations that

$$0 = \frac{1}{\phi}\tilde{D}_h\tilde{W}_{ijkl}$$
$$= \partial_h\bar{f} - \frac{f_h G_{ijkl}}{f^2}\phi[K - n(n-1)I' - 2(n-1)tr(T')],$$

where $\bar{f}$ is a function defined on M. Thus we find

$$\phi[K - n(n-1)I' - 2(n-1)tr(T')] = \text{constant}. \quad (18)$$

Similarly, in virtue of $(\tilde{D}_\varepsilon\tilde{W}_{\alpha\beta\gamma\delta})/f = 0$, we have

$$f[R - m(m-1)I - 2(m-1)tr(T)] = \text{constant}. \quad (19)$$

Finally, we substitute (18) and (19) into (16) and (17), respectively. Then, by our assumption, we get $\phi_\varepsilon\tilde{W}_{ijkl} = 0$ and $f_h\tilde{W}_{\alpha\beta\gamma\delta} = 0$. Since f and ϕ are not constant functions, we see that $\tilde{W}_{ijkl} = \tilde{W}_{\alpha\beta\gamma\delta} = 0$, which, by (iii) of Lemma 1, implies that $M_\phi \times_f N$ is conformally flat.

3. Singly Warped-Product Manifolds

In this section, we study some problems on singly warped-product manifolds. More precisely, we consider problems on $M \times_f N$. It seems to be a special (hence simpler) case of that on $M_\phi \times_f N$, however, we shall see that at the same time, we lose some equations under considerations. Throughout this section, $(\tilde{M}, \tilde{g})$ will be the warped-product manifold $(M^m \times N^n, g \times_f h)$.

3.1. Elementary Formulae

Although all of the formulae and equations for geometric objects on $(\tilde{M}, \tilde{g})$ can be obtained by taking $\phi = 1$ in Subsection 2. 1, we list some of them below for the sake of convenience. The possible non-vanishing components of the Riemann curvature tensor on $\tilde{M}$ are

$$\tilde{R}_{ijkl} = R_{ijkl},\ \tilde{R}_{\alpha\beta\gamma\delta} = fK_{\alpha\beta\gamma\delta} - \frac{\|F\|^2}{4}G'_{\alpha\beta\gamma\delta},\ \tilde{R}_{i\beta k\delta} = fT_{ik}h_{\beta\delta}, \quad (20)$$

which implies that the non-vanishing components of the Ricci curvature tensor on $\tilde{M}$ are

$$\tilde{R}_{ij} = R_{ij} + nT_{ij}, \quad \tilde{R}_{\alpha\beta} = K_{\alpha\beta} - Jh_{\alpha\beta}. \quad (21)$$

The scalar curvature of $\tilde{M}$ are given by

$$\tilde{R} = \frac{K}{f} + R - n[(n-1)I - 2tr(T)]. \tag{22}$$

Moreover, the Weyl conformal curvature tensor $\tilde{W}$ on $\tilde{M}$ has non-vanishing components of the same kinds as on doubly warped-product case, say

$$\begin{aligned}
\tilde{W}_{ijkl} &= R_{ijkl} - \frac{1}{m+n-2}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{ik}R_{il}) \\
&\quad - \frac{n}{m+n-2}(g_{ik}T_{jl} + g_{jl}T_{jk} - g_{il}T_{jk} - g_{jk}T_{il}) \\
&\quad + \frac{\tilde{R}G_{ijkl}}{(m+n-1)(m+n-2)}, \\
\tilde{W}_{\alpha\beta\gamma\delta} &= f\Big[K_{\alpha\beta\gamma\delta} - \frac{1}{m+n-2}(h_{\alpha\gamma}K_{\beta\delta} + h_{\beta\delta}K_{\alpha\gamma} - h_{\alpha\delta}K_{\beta\gamma} - h_{\beta\gamma}K_{\alpha\delta})\Big] \\
&\quad - G'_{\alpha\beta\gamma\delta}\Big[\frac{\|F\|^2}{4} - \frac{2fJ}{m+n-2} - \frac{f^2\tilde{R}}{(m+n-1)(m+n-2)}\Big], \\
\tilde{W}_{i\beta k\delta} &= \frac{1}{m+n-2}\{[(m-2)T_{ik} - R_{ik}]fh_{\beta\delta} - g_{ik}K_{\beta\delta} \\
&\quad + \Big[J + \frac{f\tilde{R}}{m+n-1}\Big]g_{ik}h_{\beta\delta}\Big\}.
\end{aligned} \tag{23}$$

3.2. Conformal Flatness

As the similar arguments as Lemma 1 in Subsection 2.2, we have the following useful criterion for a warped-product manifold $M \times_f N = \tilde{M}$ to be conformally flat:

Lemma 3. (i) *If $\tilde{W}_{ijkl} = 0$, then*

$$m(m-1)K + n(n-1)f[R - 2(m-1)tr(T) - m(m-1)I] = 0, \tag{24}$$

and

$$R_{ij} - (m-2)T_{ij} = \frac{1}{m}[R - (m-2)tr(T)]g_{ij}. \tag{25}$$

Moreover, if $m > 1$, then

$$(m-2)R_{ijkl} = (g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}) - \frac{RG_{ijkl}}{m-1}. \tag{26}$$

(ii) If $\tilde{W}_{\alpha\beta\gamma\delta} = 0$ then (24) still holds and

$$K_{\alpha\beta} = \frac{K}{n}h_{\alpha\beta}. \tag{27}$$

Consequently, (N^n, h) *is Einstein provided* $m > 1$ *or* $n \geq 3$.

(iii) If $\tilde{W}_{ijkl} = \tilde{W}_{\alpha\beta\gamma\delta} = 0$, *then* $\tilde{W}_{i\beta k\delta} = 0$. *Consequently,* $\tilde{M}$ *is conformally flat provided* $m + n \geq 4$.

For the case $\dim M = 1$, we choose $t = x^1$ to be the arc-length parameter on the real line R. We have

Theorem 7. *Let* L *be an interval in the real line* R *with the standard Euclidean metric, and let* f *be any positive function defined on* L. *Then* $L \times_f N^n$ $(n \geq 2)$ *is conformally flat if and only if* N *is a space of constant sectional curvature.*

Proof. Firstly, we consider the case $n = 2$. It follows from Theorem 2 in Subsection 2. 2 that $L \times_f N$ is conformally flat if and only if

$$\tilde{D}_C\tilde{R}_{AB} - \tilde{D}_B\tilde{R}_{AC} = \frac{1}{4}[(\partial_C\tilde{R})\tilde{g}_{AB} - (\partial_B\tilde{R})\tilde{g}_{AC}], \quad \forall A, B, C. \tag{28}$$

A direct computation shows that for any f

$$\tilde{D}_\beta\tilde{R}_{\alpha 1} - \tilde{D}_1\tilde{R}_{\alpha\beta} = \frac{1}{4}[(\partial_\beta\tilde{R})\tilde{g}_{\alpha 1} - (\partial_1\tilde{R})\tilde{g}_{\alpha\beta}]$$

always holds for $m = 1$ and $n = 2$. Moreover, in virtue of $\tilde{D}_\gamma\tilde{R}_{\alpha\beta} = \nabla_\gamma K_{\alpha\beta}$ and $K_{\alpha\beta} = \frac{K}{2}h_{\alpha\beta}$ $(n = 2)$, we find

$$\tilde{D}_\gamma\tilde{R}_{\alpha\beta} - \tilde{D}_\beta\tilde{R}_{\alpha\gamma} = \frac{1}{2}[(\partial_\gamma K)h_{\alpha\beta} - (\partial_\beta K)h_{\alpha\gamma}]. \tag{29}$$

Since $\tilde{D}_\gamma\tilde{R}_{ij} = \tilde{D}_j\tilde{R}_{i\gamma} = 0$ in any case, while for $m = 1$ and $n = 2$,

$$\frac{1}{4}[(\partial_\gamma\tilde{R})\tilde{g}_{11} - (\partial_1\tilde{R})\tilde{g}_{1\gamma} = \frac{\partial_\gamma K}{4f}. \tag{30}$$

Therefore, if N^2 is a space of constant curvature, (28) holds for an f. On the other hand, if $L \times_f N^2$ is conformally flat, then, in view of (28), (29), and (30), we see that K is a constant.

Next, we consider the general case $n \geq 3$ and show that $\tilde{W}_{ABCD} = 0$ if and only if N has constant sectional curvature. The sufficiency being easily follows from direct calculation. Now we suppose that $\tilde{W}_{ABCD} = 0$ for all A, B, C, and D. By (ii) of Lemma 3, (N^n, h) is Einsteinian: $K_{\alpha\beta} = Kh_{\alpha\beta}/n$, K being a constant.

Furthermore, since $m = 1$, the second equation of (23) takes a simpler form

$$\tilde{W}_{\alpha\beta\gamma\delta} = f\Big[K_{\alpha\beta\gamma\delta} + \frac{K}{n(n-1)}G'_{\alpha\beta\gamma\delta} - \frac{1}{n-1}(h_{\alpha\gamma}K_{\beta\delta} + h_{\beta\delta}K_{\alpha\gamma} - h_{\alpha\delta}K_{\beta\gamma} - h_{\beta\gamma}K_{\alpha\delta})\Big].$$

Thus, in virtue of $\tilde{W}_{\alpha\beta\gamma\delta} = 0$ and $K_{\alpha\beta} = Kh_{\alpha\beta}/n$, we find

$$K_{\alpha\beta\gamma\delta} = \frac{K}{n(n-1)}G'_{\alpha\beta\gamma\delta}. \quad (n \geq 3)$$

Hence (N^n, h) is a space of constant sectional curvature. This complete the proof.

For dim$M = m \geq 2$, we have

Theorem 8. *If $M \times_f N$ is conformally flat, where dim$M \geq 2$ and dim$N \geq 2$. Then M is conformally flat and N is a space of constant sectional curvature.*

Proof. Firstly, the conformal flatness of M follows from the same arguments as in the proof of Theorem 4.

Secondly, by Lemma 3, N is Einstein with (constant) scalar curvature K satisfying

$$\frac{K}{f} = \frac{-n(n-1)}{m(m-1)}[R - 2(m-1)tr(T) - m(m-1)I].$$

Substituting this into the equation $\tilde{W}_{\alpha\beta\gamma\delta} = 0$, we find $K_{\alpha\beta\gamma\delta} = \frac{K}{n(n-1)}G'_{\alpha\beta\gamma\delta}$.

Now, we suppose that dim$M \geq 2$, dim$N \geq 2$, and that N has constant sectional curvature $K/n(n-1)$. A direct calculation shows that

$$\tilde{W}_{\alpha\beta\gamma\delta} = \frac{fG'_{\alpha\beta\gamma\delta}}{(m+n-1)(m+n-2)} \times \Big\{\frac{m(m-1)K}{n(n-1)} + f[R - 2(m-1)tr(T) - m(m-1)I]\Big\}.$$

This proves the first part of the following theorem.

Theorem 9. *Suppose that dim$M \geq 2$, dim$N \geq 2$, M is conformally flat and N has constant sectional curvature. Then $M \times_f N$ is conformally flat if and only if $\tilde{W}_{ijkl} = 0$, or equivalently, the equations (24) and (25) hold.*

In particular, for dimM = 2, $M \times_f N$ is conformally flat if and only if f satisfies

$$\frac{\tilde{R}}{n(n+1)} + \frac{(n-2)R}{2n} - tr(T) = 0. \tag{31}$$

Proof. We show the second part, say dim$M = m = 2$. It follows from a technique in linear algebra (c.f. [9] also) that

$$g_{ik}A_{jl} + g_{jl}A_{ik} - g_{jk}A_{il} - g_{il}A_{jk} = tr(A)G_{ijkl} \tag{32}$$

for any symmetric 2-tensor A or (M, g), where $tr(A)$ being taken with respect to g. Therefore, in view of $R_{ijkl} = RG_{ijkl}/2$ for dime$M = 2$, we find

$$\tilde{W}_{ijkl} = \Big[\frac{\tilde{R}}{n(n+1)} + \frac{(n-2)R}{2n} - tr(T)\Big]G_{ijkl}.$$

Hence, by the first part of the theorem, $M \times_f N$ is conformally flat if and only if (31) holds.

Corollary 1. *Let M be an orientable, compact 2-dimensional surface without boundary, and let $N^n(c)$ be an n-dimensional manifold with constant sectional curvature c. If there is a positive function f on M such that $M \times_f N^n(c)$ is conformally flat, then*

$$2\pi\chi(M) + c\int_M \frac{1}{f} = 0. \tag{33}$$

Proof. By Theorem 9, f satisfies the equation (31), which is equivalent to say that

$$\frac{1}{2}R + \frac{c}{f} - [tr(T) + I] = 0 \quad \text{on } M,$$

where $R/2$ being the Gaussian curvature of M. Hence

$$\frac{1}{2}R + \frac{c}{f} + \frac{1}{2}\Delta_M(\ln f) = 0 \quad \text{on } M.$$

Integrating this equation then using the Stoke's Theorem, we obtain

$$\int_M \Big(\frac{1}{2}R + \frac{c}{f}\Big) = 0.$$

Our assertion now follows from the Gauss-Bonnet formula.

Example 1. Let $S^m(1)$ be the m-sphere with constant sectional curvature 1 and T^2 be the flat 2-trous.

(i) $S^2(1) \times_f N^n(c)$ is never conformally flat if $c \geq 0$ (c.f. Theorem 3, also [15]);

(ii) $T^2 \times_f T^2$ is conformally flat if and only if f is a constant. Therefore, $T^2 \times_f T^2$ is conformally flat only when the product metric is the ordinary Riemannian product.

(iii) In general, $T^2 \times_f N^n(c)$ is conformally flat if and only if $c = 0$ and f is a positive constant function on T^2.

3.3. Warped-Product Manifolds with Constant Scalar Curvature

As we had asserted in Theorem 1 in Subsection 2.1, an essentially doubly warped-product manifold can only take 0 as its constant scalar curvature. In this subsection, we study the corresponding results in the case for singly warped-product manifolds. We start here from the following

Lemma 4. *If $M \times_f N$ has constant scalar curvature $\tilde{R}$, then N has constant scalar curvature K such that*

$$\Delta_M f + \frac{n-3}{4f}\|F\|^2 = \frac{1}{n}[(R - \tilde{R})f + K] \tag{34}$$

holds everywhere on M.

Proof. It follows from the formula (22) that

$$\Delta_M f + \frac{n-3}{4f}\|F\|^2 + \frac{\tilde{R} - R}{n} f = \frac{K}{n}$$

holds on $M \times N$, K being a function defined on N. Therefore, if $\tilde{R}$ is a constant, then K is also a constant and (34) holds everywhere on M.

Example 2. Let M be a compact Riemannian manifold without boundary whose scalar curvature is constant, and let N be a 3-dimensional flat manifold. By Lemma 4, if $M \times_f N$ has constant scalar curvature $\tilde{R}$, then $\Delta_M f = \frac{1}{3}(R - \tilde{R})f$ holds everywhere on M. Integrating this on M and using the Stoke's Theorem, we see that $R = \tilde{R}$. Hence f is a positive harmonic function on M, which implies that f is a constant function. Therefore, the warped-product $M \times_f N$ has constant scalar curvature only when it is an ordinary Riemannian product.

Example 3. Let L be an interval of the real line R with the standard Euclidean metric, and let t be the arc-length parameter on R. If N is a 3-dimensional

Riemannian manifold with constant scalar curvature K. Then $L \times_f N$ has constant scalar curvature $\tilde{R}$ with f given by

$$
\begin{aligned}
\tilde{R} = 0 : f(t) &= \frac{K}{6} t^2 + at + b, \\
&\quad \text{with } K > 0, b > 0, \text{ and } 3a^2 < 2bK; \\
\tilde{R} > 0 : f(t) &= \frac{K}{\tilde{R}} + a \cos\Big(\sqrt{\frac{\tilde{R}}{3}} t\Big) + b \sin\Big(\sqrt{\frac{\tilde{R}}{3}} t\Big) \\
&\quad \text{with } K > 0, \text{ and } \tilde{R}\sqrt{a^2 + b^2} < K; \\
\tilde{R} < 0 : f(t) &= \frac{K}{\tilde{R}} + a \exp\Big(\sqrt{\frac{-\tilde{R}}{3}} t\Big) + b \exp\Big(-\sqrt{\frac{-\tilde{R}}{3}} t\Big) \\
&\quad \text{with } a, b > 0, \text{ and } 4ab\tilde{R}^2 > K^2.
\end{aligned}
$$

Actually, under the assumptions, the equation (34) takes the following simpler forms, say

$$
\begin{aligned}
\tilde{R} = 0 &: \frac{d^2}{dt^2} f = \frac{K}{3}, \\
\tilde{R} \neq 0 &: \frac{d^2}{dt^2}\Big(f - \frac{K}{\tilde{R}}\Big) = -\frac{\tilde{R}}{3}\Big(f - \frac{K}{\tilde{R}}\Big).
\end{aligned}
$$

The result now follows from the elementary solutions to second order ordinary differential equations.

Remark. A conformally symmetric manifold with constant scalar curvature has harmonic curvature (c.f. [4]), in particular, a conformally flat manifold with constant scalar curvature has harmonic curvature (c.f. [12]). We see that, in virtue of Theorem 7, all of the 4-dimensional Riemannian manifolds listed in Example 3 have harmonic curvatures, provided N is replaced by 3-manifold with constant sectional curvature. Moreover, the Ricci curvature tensor of them have the properties that they are not parallel and have at each point exactly two distinct eigenvalues. Manifolds with harmonic curvature and these properties had been studied and classified by A. Derdzindki [5], [6]. These are the earliest explicit examples answering negatively to the well-known Bourguignon's conjecture [3], which asserts that a compact Riemannian manifold with harmonic Riemann curvature tensor must have parallel Ricci curvature tensor.

Lemma 5. *$M^m \times_f N^n (m, n, \geq 1)$ has scalar curvature $\tilde{R}$ satisfying*

$$\frac{4n}{n+1}\Delta_M u - Ru - Ku^{\frac{n-3}{n+1}} + \tilde{R}u = 0, \tag{35}$$

where $u = f^{\frac{n+1}{4}}$. In particular, if $K = 0$ and M is a compact Riemannian manifold without boundary whose scalar curvature R is constant, then u (hence f) is a positive constant and $\tilde{R} = R$. (c.f. Example 2)

For more details about the scalar curvature of a warped-product manifold, see [10], where the authors looked for warpping functions such that the warped-product manifolds have constant scalar curvature and then determined which constants will be attained. The main results are sketched as followings.

Let (M^m, g) be a compact and connected Riemannian manifold and (N^n, h) be a Riemannian manifold with constant scalar curvature K. Firstly one consider the elliptic linear operator L on M given by

$$Lu = -\frac{4n}{n+1}\Delta_M u + Ru, \quad u \in C^\infty(M)$$

where R is the scalar curvature of M. It is well known that the linear eigenvalue problem $Lu = \lambda u$ on M has a positive solution, called principal eigenfunction, u_1 with principal eigenvalue λ_1 given by

$$\lambda_1 = \inf\Big\{ \int_M \Big(\frac{4n}{n+1}\|Dv\|^2 + Rv^2\Big) dV_g | v \in H^1(M), \int_M v^2 dV_g = 1\Big\},$$

where $H^1(M) = \{v \in L^2(M) | \|Dv\|^2 \in L^1(M)\}$ is the Sobolev space (c.f. [1]). u_1 is uniquely determined up to a positive multiplicative constant. We shall call (u_1, λ_1) the principal eigenpair of L when u_1 is chosen so that it has maximum 1.

Theorem 10. [10] *Let M be compact and connected with scalar curvature R and N has constant scalar curvature K and $dimN = n \geq 3$.*

(i) If $K = 0$, then there is positive function $f \in C^\infty(M)$ such that $M \times_f N$ has constant scalar curvature $\tilde{R} = \lambda_1$. f is unique up to a positive multiplicative constant.

(ii) If $K < 0$, then for each $\lambda < \lambda_1$, there is a unique positive function $f \in C^\infty(M)$ such that $M \times_f N$ has constant scalar curvature $\tilde{R} = \lambda$. No constant $\geq \lambda_1$ may be the scalar curvature of $M \times_f N$ for any f.

(iii) If $K > 0$, then there is a $\delta > 0$ such that for each $\lambda \in (\lambda_1, \lambda_1+\delta)$, there is a positive function $f \in C^\infty(M)$ such that $M \times_f N$ has constant scalar curvature λ. No constant $\leq \lambda_1$ may be the scalar curvature of $M \times_f N$ for any f.

It follows from the uniqueness results in Theorem 10 that we have

Corollary 2. *Let M, N be as in Theorem 10 with $K \leq 0$, and let $f = u^{\frac{4}{n+1}}$ be such that $M \times_f N$ has constant scalar curvature. Then $M \times_f N$ is essential if and only if R is not a constant.*

Going back to our Example 3, we see that, when $K > 0$, there are essentially warped-product manifolds $M \times_f N$ having constant scalar curvature even if the base manifolds have constant scalar curvature R. Actually, in [10], the authors had shown that for $n = 3$, $K > 0$, and λ_1 as before, each constant $\lambda \in (\lambda_1, +\infty)$ is the constant scalar curvature of $M \times_f N$ for some f if and only if the scalar curvature R on M is constant. For the special case of $\dim M = 1$, for example $M = S^1$ with the standard metric, given an eigenvalue $\mu_k (k \geq 2)$ of $(-3\Delta_M)$, the authors obtained uncountable positive solutions f such that $M \times_f N$ has μ_k as its scalar curvature. This generalized a result obtained by N. Ejiri with $M = S^1$, $N = S^3$, and $k = 3$. (c.f. [11]). Here we are interesting in the essential case. We have

Theorem 11. *Let (M^m, g) be compact and connected with constant scalar curvature R, and let (N, h) be 3-dimensional with constant scalar curvature $K > 0$. If the warped-product manifold $M \times_f N$ has constant scalar curvature $\tilde{R}$, then $M \times_f N$ is essential if and only if $\tilde{R} = R + 3\mu_k (k \geq 2)$, where $0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$ are the eigenvalues of $-\Delta_M$ (that is, the kernel of $(-\Delta_M + \mu_k I)$ is not trivial).*

Proof. Since R is a constant, the principal eigenvalue of $L = -3\Delta_M + RI$ is $\lambda_1 = R$. It follows from Theorem 3.5 in [10] that each constant $\lambda > R$ can be the scalar curvature of $M \times_f N$, for some positive function $f \in C^\infty(M)$. Indeed, f can be chosen to be the constant function $f_c = K/(\lambda - R)$.

Now, for the same λ, if there is a non-constant positive function $f \in C^\infty(M)$ such that $M \times_f N$ has constant scalar curvature λ, then

$$Lf_c + K = \lambda f_c, \quad \text{and} \quad Lf + K = \lambda f,$$

which implies $-3\Delta_M(f - f_c) = (\lambda - R)(f - f_c)$. Therefore $\lambda = R + 3\mu_k$ for some $k \geq 2$, since $\lambda > R$.

Conversely, let $\phi_k \in C^\infty(M)$ be a (non-constant) eigenfunction of $-\Delta_M$ corresponding to $\mu_k (k \geq 2)$. Then for $t > 0$ small enough, the non-constant function $f = t\phi_k + (3\mu_k)^{-1}K \in C^\infty(M)$ is positive and, by a direct computation, it solves $Lf + K = (R + 3\mu_k)f$. Hence $M \times_f N$ has constant scalar curvature $R + 3\mu_k$.

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Yuan-Ze Institute of Technology, Energy Technology Center, 135 Yuan-Tung Road., Ne-Li, Taoyuan Shian, Taiwan 32026, R.O.C.