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A RESULT ON BEST APPROXIMATION

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Abstract. Using a common fixed point theorem for noncommuting mappings of Pant [4], we improve and extend a result of Sahab, Khan and Sessa [5] on best approximation.

Throughout this paper, E denotes a normed space. A subset C of E is said to be starshaped with respect to a point $p \in C$ if, for each $x \in C$, the segment joining x to p is contained in C (that is ., $\lambda x + (1-\lambda)\, p \in C$ for each $x \in C$ and real λ with $0 \leq \lambda \leq 1$). $C \subset E$ is said to be starshaped if it is starshaped with respect to one of its elemnts. A convex set obviously starshaped. A mapping $T : E \to E$ is nonexpansive on E (resp. on a subset C of E) if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in E$ (resp. for all $x, y \in C$). The set of fixed points of T in E is denoted by $F(T)$. Suppose $\hat{x} \in E$. An element $y \in C$ is called an element of best approximation of $\hat{x}$ if we have $\|\hat{x} - y\| = \inf_{z \in C} \|\hat{x} - z\|$. We will denote by $P_c(\hat{x})$ the set of all such elements y. We denote the boundary of C by ∂C.

In 1969, Brosowski [1] obtained the following result which generalizes a theorem of Meinardus [3]:

Theorem 1. *Let $T : E \to E$ be a linear and nonexpansive operator on E. Let C be a T-invariant subset of E and let $\hat{x} \in F(T)$. If $P_c(\hat{x})$ is nonempty, compact, and convex then $P_c(\hat{x}) \cap F(T) \neq \phi$.*

In 1979, Singh [6] observed that the linearity of the operator T and the convexity of $P_c(\hat{x})$ in Theorem 1 can be relaxed and gave the following extension of it.

Theorem 2. *Let $T : E \to E$ be a nonexpansive operator on E. Let C be a T-invariant subset of E and let $\hat{x} \in F(T)$. If $P_c(\hat{x})$ is nonempty, compact, and starshaped, then $P_c(\hat{x}) \cap F(T) \neq \phi$.*

Singh [7] showed that if $D' = P_c(\hat{x}) \cup \{\hat{x}\}$, then Theorem 2 remains valid for T satisfying the condition of nonexpansiveness only on D'. Recently, Sahab, Khan and Sessa [5] generalized Theorem 2 with the following result.

Theorem 3. *Let T, $I : E \to E$ be operators, C be a subset of E such that $T : \partial C \to C$, and $\hat{x} \in F(T) \cap F(I)$. Further T and I satisfy*

$$\|Tx - Ty\| \leq \|Ix - Iy\| \tag{1}$$

for all $x, y \in D' = P_c(\hat{x}) \cup \{\hat{x}\}$ and let I be linear, continuous on $P_c(\hat{x})$, and $ITx = TIx$ for all $x \in P_c(\hat{x})$. If $P_c(\hat{x})$ is nonempty, compact and starshaped with respect to a point $p \in F(I)$ and if $I(P_c(\hat{x})) = P_c(\hat{x})$, then $P_c(\hat{x}) \cap F(T) \cap F(I) \neq \phi$.

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Remark 4. In Theorem 3, the hypothesis that I is continuous implies that T is continuous. In [5] Sahab, Khan and Sessa used the continuity of both T and I.

We need the following preliminary definitions and results.

Let (X, d) be a metric space and let T and I be self-mappings of X. The mappings T and I will be called R-weakly commuting on X, provided there exists some positive real number R such that

$$d(TIx, ITx) \leq Rd(Tx, Ix)$$

for each $x \in X$. T and I will be called R-weakly commuting and a point x if $d(TIx, ITx) \leq R\, d(Tx, Ix)$ for some $R > 0$. Obviously, weak commutativity implies R-weak commutativity. However, R-weak commutativity implies weak commutativity only when $R \leq 1$. For details of the above see Pant [4].

The following is a consequence of Theorem 1 of Pant [4].

Theorem 5. *Let (X, d) be a complete metric space and let $T,\ I : X \to X$ be R-weakly commuting mappings such that $T(X) \subseteq I(X)$, and $d(Tx, Ty) < d(Ix, Iy)$ whenever $Ix \neq Iy$. If either T or I is continuous, then $F(T) \cap F(I)$ is singleton.*

Let us continue this paper by observing that even if in Theorem 3 the conditions of continuity and commutativity of operators are somewhat relaxed, the assertion of Theorem 3 remains valid. Thus we get the following interesting result, which is new in the sense that, unlike other authors (see Remark 4), we do not require both T and I to be continuous.

Theorem 6. *Let $T,\ I : E \to E$ be operators, C be a subset of E such that $T : \partial C \to C$, and $\hat{x} \in F(T) \cap F(I)$. Further T and I satisfy (1) on $D' = P_c(\hat{x}) \cup \{\hat{x}\}$ and let I be linear on $P_c(\hat{x})$ and T, I be R-weakly commuting on $P_c(\hat{x})$. If $P_c(\hat{x})$ is nonempty, compact and starshaped with respect ot $p \in F(I)$, if $I(P_c(\hat{x})) = P_c(\hat{x})$, and if either T or I is continuous, then $P_c(\hat{x}) \cap F(T) \cap F(I) \neq \phi$.*

Proof. First, we show that $T : P_c(\hat{x}) \to P_c(\hat{x})$. Let $y \in P_c(\hat{x})$ and hence $Iy \in P_c(\hat{x})$ since $I(P_c(\hat{x})) = P_c(\hat{x})$. Then $y \in \partial C$ (see Hicks and Humphries [2]) implying that $Ty \in C$, since $T : \partial C \to C$. It follows from (1) that

$$||Ty - \hat{x}|| = ||Ty - T\hat{x}|| \leq ||Iy - I\hat{x}|| = ||Iy - \hat{x}||$$

and therefore $Ty \in P_c(\hat{x})$.

Let us define a sequence of maps T_n:

$$T_n x = (1 - k_n)p + k_n Tx,$$

where k_n is a fixed sequence of positive numbers less than 1 and converging to 1. Each T_n maps $P_c(\hat{x})$ into itself because $T : P_c(\hat{x}) \to P_c(\hat{x})$ and $P_c(\hat{x})$ is starshaped with respect to p. Since I is linear and R-weakly commutes with T on $P_c(\hat{x})$, we have

$$T_n Ix = (1 - k_n)Ip + k_n TIx,$$
$$IT_n x = (1 - k_n)Ip + k_n ITx$$

and

$$\begin{aligned}||T_nIx - IT_nx|| &= k_n||TIx - ITx||\\ &\leq k_nR||Tx - Ix||\\ &< R||Tx - Ix||\end{aligned}$$

for all $x \in P_c(\hat{x})$. Thus T_n and I are R-weakly commuting on $P_c(\hat{x})$ for each n and $T_n(P_c(\hat{x})) \subseteq I(P_c(\hat{x}))$. Also, we have

$$||T_nx - T_ny|| = k_n||Tx - Ty|| \leq k_n||Ix - Iy|| < ||Ix - Iy||$$

whenever $Ix \neq Iy$.

Since either T or I is continuous, according to Theorem 5 $F(T_n) \cap F(I) = \{x_n\}$ for each n. Since $P_c(\hat{x})$ is compact, $\{x_n\}$ has a subsequence $\{X_{n_i}\} \to z$ (say) in $P_c(\hat{x})$.

Let us suppose that the mapping T is continuous. Since

$$x_{n_i} = T_{n_i}x_{n_i} = (1 - k_{n_i})p + k_{n_i}Tx_{n_i},$$

we have, as $i \to \infty$, that $z = Tz$, that is $z \in P_c(\hat{x}) \cap F(T)$. Since T_n and I are R-weakly commuting, we have that

$$||T_{n_i}Ix_{n_i} - IT_{n_i}x_{n_i}|| \leq R||T_{n_i}x_{n_i} - Ix_{n_i}||.$$

On letting $i \to \infty$, the above inequality yields $IT_{n_i}x_{n_i} \to Tz = z$ since $T_{n_i}x_{n_i} = Ix_{n_i} = x_{n_i}$. Since $T(P_c(\hat{x})) \subseteq P_c(\hat{x}) = I(P_c(\hat{x}))$ it follows from $z = Tz$ that there exists $z_1 \in P_c(\hat{x})$ such that $z = Tz = Iz_1$. Now

$$||TT_{n_i}x_{n_i} - Tz_1|| \leq ||IT_{n_i}x_{n_i} - Iz_1||.$$

This inequality on letting $i \to \infty$ implies that $Tz = Tz_1$ since $IT_{n_i}x_{n_i} \to Tz$ and $Tz = Tz_1$.

Thus $z = Tz = Tz_1 = Iz_1$. This in turn implies that

$$||Tz - Iz|| = ||TIz_1 - ITz_1|| \leq R||Tz_1 - Iz_1|| = 0,$$

that is, $z = Tz = Iz$ and hence

$$P_c(\hat{x}) \cap F(T) \cap F(I) \neq \phi.$$

The same conclusion is found when I is assumed to be continuous since continuity of I implies continuity of T.

We have improved and extended Theorem 3([5]) for noncommuting maps.

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