TAMKANG JOURNAL OF MATHEMATICS
Volume 29, Number 3, Aunumn 1998

CERTAIN CLASSES OF ANALYTIC AND MULTIVALENT FUNCTIONS

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Abstract. The main object of the present paper is to investigate the special classes

$$\mathcal{P}_\alpha^*(p, A, B) \text{ and } \mathcal{R}_\alpha^*(p, A, B)$$

$$(0 \leq \alpha < p; -1 \leq B < A \leq 1; p \in \mathbb{N} := \{1, 2, 3, \ldots\})$$

of analytic and p-valent functions in the open unit disk $\mathcal{U}$. In particular, various growth and distortion theorems, and several coefficient estimates, are obtained for these as well as related classes of analytic and p-valent functions in $\mathcal{U}$.

1. Introduction and Definitions

Let $\mathcal{S}(p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by Ω the class of *bounded analytic functions* $w(z)$ in $\mathcal{U}$ satisfying the conditions:

$$w(0) = 0 \text{ and } |w(z)| < 1 \quad (z \in \mathcal{U}).$$

A function $f(z)$ in $\mathcal{S}(p)$ is called p-valent starlike of order α in $\mathcal{U}$ if it satisfies the following conditions:

$$\mathcal{R}\Big\{\frac{zf'(z)}{f(z)}\Big\} > \alpha \text{ and } \int_0^{2\pi} \mathcal{R}\Big\{\frac{zf'(z)}{f(z)}\Big\} d\theta = 2p\pi \quad (z = e^{i\theta})$$
$$(0 \leq \alpha < p;\ p \in \mathbb{N};\ z \in \mathcal{U}). \tag{1.2}$$

Received November 19, 1997.
1991 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Analytic functions, multivalent functions, growth and distortion theorems, coefficient estimates, starlike functions, convex functions.

We denote by $\mathcal{S}^*(p,\alpha)$ the class of all p-valent starlike functions of order α in $\mathcal{U}$. Further, a function $f(z)$ in $\mathcal{S}(p)$ is called p-valent convex of order α in $\mathcal{U}$ if it satisfies the following codnitions:

$$\mathcal{R}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \text{ and } \int_0^{2\pi}\mathcal{R}\left\{1+\frac{zf''(z)}{f'(z)}\right\}d\theta=2p\pi \quad (z=e^{i\theta})$$
$$(0\leq\alpha<p;\ p\in\mathbb{N};z\in\mathcal{U}). \tag{1.3}$$

We donote by $\mathcal{K}(p,\alpha)$ the class of all p-valent convex functions of order α in $\mathcal{U}$. It follows from (1.2) and (1.3) that

$$f(z)\in\mathcal{K}(p,\alpha)\Leftrightarrow\frac{zf'(z)}{p}\in\mathcal{S}^*(p,\alpha). \tag{1.4}$$

The class $\mathcal{S}^*(p,\alpha)$ was studied by Patil and Thakare [9] and the class $\mathcal{K}(p,\alpha)$ was considered by Owa [8].

Let $\mathcal{T}(p)$ denote the subclass of $\mathcal{S}(p)$ consisting of functions $f(z)$ of the form:

$$f(z)=z^p-\sum_{n=1}^{\infty}a_{p+n}z^{p+n} \quad (a_{p+n}\geq 0;\ p\in\mathbb{N}). \tag{1.5}$$

We denote by $\mathcal{T}^*(p,\alpha)$ and $\mathcal{C}(p,\alpha)$ the classes obtained by taking intersections, respectively, of the classes $\mathcal{S}^*(p,\alpha)$ and $\mathcal{K}(p,\alpha)$ with $\mathcal{T}(p)$, that is,

$$\mathcal{T}^*(p,\alpha)=\mathcal{S}^*(p,\alpha)\cap\mathcal{T}(p) \tag{1.6}$$

and

$$\mathcal{C}(p,\alpha)=\mathcal{K}(p,\alpha)\cap\mathcal{T}(p). \tag{1.7}$$

The classes $\mathcal{T}^*(p,\alpha)$ and $\mathcal{C}(p,\alpha)$ were investigated by Owa [8], who proved the following results for these classes:

Lemma 1. *Let the function $f(z)$ be defined by (1.5). Then $f(z)$ is in the class $\mathcal{T}^*(p,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}(p+n-\alpha)a_{p+n}\leq p-\alpha. \tag{1.8}$$

The result is sharp.

Lemma 2. *Let the function $f(z)$ be defined by (1.5). Then $f(z)$ is in the class $C(p,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)(p+n-\alpha)a_{p+n}\leq p-\alpha. \tag{1.9}$$

The result is sharp.

Lemma 3. *Let the function $f(z)$ defined by (1.5) be in the class $\mathcal{T}^*(p,\alpha)$. Then*

$$|z|^p - \frac{p-\alpha}{p+1-\alpha}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{p-\alpha}{p+1-\alpha}|z|^{p+1}, \tag{1.10}$$

and

$$p|z|^{p-1} - \frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{(p+1)(p-\alpha)}{p+1-\alpha}|z|^p. \tag{1.11}$$

Each of these results is sharp.

Lemma 4. *Let the function $f(z)$ defined by (1.5) be in the class $C(p,\alpha)$. Then*

$$|z|^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}|z|^{p+1} \tag{1.12}$$

and

$$p|z|^{p-1} - \frac{p(p-\alpha)}{p+1-\alpha}|z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{p(p-\alpha)}{p+1-\alpha}|z|^p. \tag{1.13}$$

Each of these results is sharp.

Let $\mathcal{P}^*_\alpha(p,A,B)$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the condition:

$$\frac{f(z)}{g(z)} = \frac{1+A\omega(z)}{1+B\omega(z)} \quad (\omega \in \Omega;\ -1 \leq B < A \leq 1;\ z \in \mathcal{U}) \tag{1.14}$$

for a function

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}z^{p+n} \quad (b_{p+n} \geq 0;\ p \in \mathbb{N}) \tag{1.15}$$

in the class $\mathcal{T}^*(p,\alpha)(0 \leq \alpha < p)$. Further, let $\mathcal{R}^*_\alpha(p,A,B)$ denote the class of functions $f(z)$ of the form (1.1) which satisfy the condition (1.14) for a function $g(z)$, defined by (1.17), but belonging to the class $C(p,\alpha)(0 \leq \alpha < p)$.

Clearly, we have

(i) $\mathcal{P}^*_\alpha(1,A,B) = \tilde{P}_\alpha(A,B)$ and $\mathcal{R}^*_\alpha(1,A,B) = \tilde{R}_\alpha(A,B)$ (Owa [5], [7]);

(ii) $\mathcal{P}^*_\alpha(1,\beta,-\lambda\beta) = \tilde{S}_\lambda(\alpha,\beta)$ and $\mathcal{R}^*_\alpha(1,\beta,-\lambda\beta) = \tilde{C}_\lambda(\alpha,\beta)$ $(0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ 0 \leq \lambda \leq 1)$ (Owa [4], [6]);

(iii) $\mathcal{P}^*_\alpha(1,1,-\mu) = S_\mu(0,\alpha)(0 \leq \mu \leq 1)$ (Altintas [1]).

2. Growth and Distortion Theorems

Theorem 1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{P}^*_\alpha(p,A,B)$. Then*

$$\frac{(1-A|z|)\{(p+1-\alpha)-(p-\alpha)|z|\}|z|^p}{(1-B|z|)(p+1-\alpha)} \leq |f(z)|$$

$$\leq \frac{(1+A|z|)\{(p+1-\alpha)+(p-\alpha)|z|\}|z|^p}{(1+B|z|)(p+1-\alpha)} \quad (z \in \mathcal{U}). \tag{2.1}$$

Each of these estimates is sharp.

Proof. We employ the technique used earlier by Goel and Sohi [2], and by Owa ([4] to [7]). Since $f(z) \in \mathcal{P}_\alpha^*(p, A, B)$, we have

$$\frac{f(z)}{g(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \quad (\omega \in \Omega;\ -1 \leq B < A \leq 1; z \in \mathcal{U}; g \in \mathcal{T}^*(p, \alpha)). \tag{2.2}$$

Thus, by using Schwarz's lemma [3], we have $|\omega(z)| \leq |z|$ for $z \in \mathcal{U}$. After a simple computation, we find that

$$\left|\frac{f(z)}{g(z)} - \frac{1 - AB|z|^2}{1 - B^2|z|^2}\right| \leq \frac{(A - B)|z|}{1 - B^2|z|^2} \quad (z \in \mathcal{U}). \tag{2.3}$$

Consequently, we obtain

$$\frac{1 - A|z|}{1 - B|z|} \leq \left|\frac{f(z)}{g(z)}\right| \leq \frac{1 + A|z|}{1 + B|z|} \quad (z \in \mathcal{U}), \tag{2.4}$$

which gives the inequalities in (2.1) with the aid of Lemma 3.

By taking

$$\frac{f(z)}{g(z)} = \frac{1 + Az}{1 + Bz} \tag{2.5}$$

and

$$g(z) = z^p - \frac{p - \alpha}{p + 1 - \alpha} z^{p+1}, \tag{2.6}$$

we can see that the estimates in (3.1) are sharp. This completes the proof of Theorem 1.

Theorem 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{R}_\alpha^*(p, A, B)$. Then*

$$\frac{(1 - A(z))\{(p+1)(p+1-\alpha) - p(p-\alpha)|z|\}|z|^p}{(1 - B|z|)(p+1)(p+1-\alpha)} \leq |f(z)|$$
$$\leq \frac{(1 + A(z))\{(p+1)(p+1-\alpha) + p(p-\alpha)|z|\}|z|^p}{(1 + B|z|)(p+1)(p+1-\alpha)} \quad (z \in \mathcal{U}). \tag{2.7}$$

Each of these estimates is sharp for

$$\frac{f(z)}{g(z)} = \frac{1 + Az}{1 + Bz} \tag{2.8}$$

and

$$g(z) = z^p - \frac{p(p - \alpha)}{(p+1)(p+1-\alpha)} z^{p+1}. \tag{2.9}$$

The proof of Theorem 2 is completed by using the same technique as in the proof of Theorem 1 with the aid of Lemma 4.

Theorem 3. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{P}_\alpha^*(p, A, B)$. *Then*

$$|f'(z)| \leq \frac{(1+A|z|)\{p(p+1-\alpha)+(p+1)(p-\alpha)|z|\}|z|^{p-1}}{(1+B|z|)(p+1-\alpha)} + \frac{(A-B)\{(p+1-\alpha)+(p-\alpha)|z|\}|z|^p}{(1+B|z|)^2(1-|z|^2)(p+1-\alpha)} \quad (z \in \mathcal{U}). \tag{2.10}$$

Proof. Since $f(z) \in \mathcal{P}_\alpha^*(p, A, B)$, by using (2.2), we obtain

$$f'(z) = \frac{1+A\omega(z)}{1+B\omega(z)}g'(z) - \frac{(A-B)\omega'(z)}{\{1+B\omega(z)\}^2}g(z) \quad (\omega \in \Omega). \tag{2.11}$$

We also have

$$|\omega'(z)| \leq \frac{1}{1-|z|^2} \quad (z \in \mathcal{U}),$$

by means of Carathéodory's theorem [3]. Hence we obtain Theorem 3 with the aid of Lemma 3.

Remark 1. We have not been able to obtain a sharp estimate for $|f'(z)|$ for

$$f(z) \in \mathcal{P}_\alpha^*(p, A, B).$$

Theorem 4. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{R}_\alpha^*(p, A, B)$. *Then*

$$|f'(z)| \leq \frac{p(1+A|z|)\{(p+1-\alpha)+(p-\alpha)|z|\}|z|^{p-1}}{(1+B|z|)(p+1-\alpha)} + \frac{(A-B)\{(p+1)(p+1-\alpha)+p(p-\alpha)|z|\}|z|^p}{(p+1)(1+B|z|)^2(1-|z|^2)(p+1-\alpha)} \quad (z \in \mathcal{U}). \tag{2.12}$$

The proof of Theorem 4 would make use of Lemma 4 in the same manner as we applied Lemma 3 in the proof of Theorem 3.

Remark 2. We have not been able to obtain a sharp estimate for $|f'(z)|$ for

$$f(z) \in \mathcal{R}_\alpha^*(p, A, B).$$

3. A Set of Coefficient Estimates

Theorem 5. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\mathcal{P}_\alpha^*(p, A, B)$. *Then*

$$|a_{p+1}| \leq A - B + \frac{p-\alpha}{p+1-\alpha}. \tag{3.1}$$

The estimate is sharp. Furthermore, for $B \geq 0$,

$$|a_{p+2}| \leq (A-B)B + (A-B)\frac{p-\alpha}{p+1-\alpha} + \frac{p-\alpha}{p+2-\alpha}. \tag{3.2}$$

Proof. Let

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n \in \Omega. \tag{3.3}$$

Then we obtain (*cf.* [3])

$$|c_1| \leq 1 \text{ and } |c_2| \leq 1 - |c_1|^2. \tag{3.4}$$

Since $f(z) \in \mathcal{P}_\alpha^*(p, A, B)$, by using (2.2), we have

$$f(z)[1 + B\omega(z)] = g(z)[1 + A\omega(z)] \quad (\omega \in \Omega). \tag{3.5}$$

Then, on substituting into (3.5) the power series (1.1), (1.15), and (3.3) for the functions $f(z)$, $g(z)$, and $\omega(z)$, respectively, we find that

$$\begin{aligned}&\left(z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}\right)\left(1 + B\sum_{n=1}^{\infty} c_n z^n\right) \\ &= \left(z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}\right)\left(1 + A\sum_{n=1}^{\infty} c_n z^n\right).\end{aligned} \tag{3.6}$$

Equating coefficients of z^{p+1} and z^{p+2} on both sides of (3.6), we obtain

$$a_{p+1} = (A - B)c_1 - b_{p+1} \tag{3.7}$$

and

$$\begin{aligned}a_{p+2} &= (A - B)c_2 - Ab_{p+1}c_1 - Ba_{p+1}c_1 - b_{p+2} \\ &= (A - B)c_2 - (A - B)b_{p+1}c_1 - B(A - B)c_1^2 - b_{p+2}.\end{aligned} \tag{3.8}$$

Since $g(z) \in \mathcal{T}^*(p, \alpha)$, by using Lemma 1, we have

$$b_{p+1} \leq \frac{p - \alpha}{p + 1 - \alpha} \text{ and } b_{p+2} \leq \frac{p - \alpha}{p + 2 - \alpha}, \tag{3.9}$$

which, in conjunction with (3.7) and (3.8), would lead us to the inequalities (3.1) and (3.2), respectively. The estimate for $|a_{p+1}|$ in (3.1) is sharp for

$$\frac{f(z)}{g(z)} = \frac{1 - Az}{1 - Bz} \tag{3.10}$$

and

$$g(z) = z^p - \frac{p - \alpha}{p + 1 - \alpha} z^{p+1}. \tag{3.11}$$

Similarly, with the aid of Lemma 2, we can prove

Theorem 6. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{R}_\alpha^*(p, A, B)$. Then*

$$|a_{p+1}| \leq A - B + \frac{p(p - \alpha)}{(p + 1)(p + 1 - \alpha)}. \tag{3.12}$$

Furthermore, for $B \geq 0$,

$$|a_{p+2}| \leq (A-B)B + (A-B)\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} + \frac{p(p-\alpha)}{(p+2)(p+2-\alpha)}. \tag{3.13}$$

The estimate for $|a_{p+1}|$ in (3.12) is sharp for

$$\frac{f(z)}{g(z)} = \frac{1-A_z}{1-B_z} \tag{3.14}$$

and

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}z^{p+1}. \tag{3.15}$$

Theorem 7. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{P}_\alpha^*(p, A, B)$. Then, for $B \geq 0$,*

$$\begin{aligned} |a_{p+3} &\leq (A-B)(1+B+B^2) + \{A+B+(A-B)B\}\frac{p-\alpha}{p+1-\alpha} \\ &+(A+B)\frac{p-\alpha}{p+2-\alpha} + \frac{p-\alpha}{p+3-\alpha}. \end{aligned} \tag{3.16}$$

Proof. Equating the coefficients of z^{p+3} on both sides of (3.6), we have

$$\begin{aligned} a_{p+3} &= -b_{p+3} + (A-B)c_3 - (Ab_{p+1} + Ba_{p+1})c_2 \\ &-(Ab_{p+2} - Ba_{p+2})c_1. \end{aligned} \tag{3.17}$$

Here, since

$$c_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{\omega(z)}{z^{n+1}}dz \quad (0 < r < 1; n \in \mathbb{N}) \tag{3.18}$$

for the coefficients c_n of an analytic function $\omega(z)$ in the open unit disk $\mathcal{U}$, we obtain

$$\begin{aligned} a_{p+3} = -b_{p+3} &+ \frac{A-B}{2\pi i}\int_{|z|=r} \frac{\omega(z)}{z^4}dz \\ &- \frac{Ab_{p+1} + Ba_{p+1}}{2\pi i}\int_{|z|=r} \frac{\omega(z)}{z^3}dz - \frac{Ab_{p+2} - Ba_{p+2}}{2\pi i}\int_{|z|=r} \frac{\omega(z)}{z^2}dz. \end{aligned} \tag{3.19}$$

Now, by using Schwarz's lemma [3], $|\omega(z)| \leq |z|$ for $z \in \mathcal{U}$. Therefore, we get

$$|a_{p+3}| \leq |b_{p+3}| + \frac{(A-B)}{r^3} + \frac{A|b_{p+1}| + B|a_{p+1}|}{r^2} + \frac{A|b_{p+2}| + B|a_{p+2}|}{r}. \tag{3.20}$$

Consequently, we have Theorem 7 with the aid of Lemma 1 and Theorem 5.

Similarly, by applying Lemma 2 and Theorem 6, we have

Theorem 8. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{R}_\alpha^*(p,A,B)$. Then, for $B \geq 0$,*

$$|a_{p+3} \leq (A-B)(1+B+B^2) + \{A+B+(A-B)B\}\frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} + (A+B)\frac{p(p-\alpha)}{(p+2)(p+2-\alpha)} + \frac{p(p-\alpha)}{(p+3)(p+3-\alpha)}. \tag{3.21}$$

Remark 3. We have not been able to obtain sharp estimates for

$$|a_{p+n}| \quad (p \in \mathbb{N}; n \in \mathbb{N}\backslash\{1\})$$

for the function $f(z)$ belonging to the classes $\mathcal{P}_\alpha^*(p,A,B)$ and $\mathcal{R}_\alpha^*(p,A,B)$.

Theorem 9. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{P}_\alpha^*(p,A,0)$. Then*

$$|a_{p+n}| \leq A\left(\frac{2p+1-2\alpha}{p+1-\alpha}\right) + \frac{p-\alpha}{p+n-\alpha} \quad (p \in \mathbb{N}; n \in \mathbb{N}). \tag{3.22}$$

Proof. Since $f(z)$ belongs to the class $\mathcal{P}_\alpha^*(p,A,0)$, from (3.6) we have

$$\sum_{n=1}^{\infty}(a_{p+n}+b_{p+n})z^{p+n} = A\left(z^p - \sum_{n=1}^{\infty} b_{p+n}z^{p+n}\right)\left(\sum_{n=1}^{\infty} c_n z^n\right). \tag{3.23}$$

Equating the coefficients of z^{p+n} on both sides of (3.23), we have

$$a_{p+n} + b_{p+n} = A\left(c_n - \sum_{m=1}^{n-1} b_{p+m}c_{n-m}\right), \tag{3.24}$$

which, in view of (3.18), yields

$$a_{p+n} + b_{p+n} = \frac{A}{2\pi i}\int_{|z|=r}\frac{\omega(z)}{z^{n+1}}\left(1 - \sum_{m=1}^{n-1} b_{p+m}z^m\right)dz. \tag{3.25}$$

Consequently, by using Schwarz's lemma [3] once again, we find that

$$\begin{aligned}
|a_{p+n} + b_{p+n}| &\leq \frac{A}{2\pi}\int_0^{2\pi}\frac{1}{r^n}\left|1 - \sum_{m=1}^{n-1} b_{p+m}r^m c^{im\theta}\right|d\theta \\
&\leq \frac{A}{2\pi r^n}\int_0^{2\pi}\left(1 + \sum_{m=1}^{n-1} b_{p+m}r^m\right)d\theta \\
&= \frac{A}{r^n}\left(1 + \sum_{m=1}^{n-1} b_{p+m}r^m\right) \\
&\leq \frac{A}{r^n}\left(1 + \sum_{m=1}^{n-1} b_{p+m}\right).
\end{aligned} \tag{3.26}$$

Since (3.26) holds true for any $r(0 < r < 1)$ and since

$$\sum_{m=1}^{n-1} b_{p+m} \leq \frac{p-\alpha}{p+1-\alpha} \quad (p \in \mathbb{N}; n \in \mathbb{N}), \tag{3.27}$$

by Lemma 1, we have

$$|a_{p+n} + b_{p+n}| \leq A\Big(\frac{2p+1-2\alpha}{p+1-\alpha}\Big). \tag{3.28}$$

Hence we obtain

$$\begin{aligned} |a_{p+n}| &\leq |a_{p+n} + b_{p+n}| + |b_{p+n}| \\ &\leq A\Big(\frac{2p+1-2\alpha}{p+1-\alpha}\Big) + \frac{p-\alpha}{p+n-\alpha} \end{aligned} \tag{3.29}$$

because

$$b_{p+n} \leq \frac{p-\alpha}{p+n-\alpha} \quad (p \in \mathbb{N}; n \in \mathbb{N}), \tag{3.30}$$

again by Lemma 1.

Similarly, with the aid of Lemma 2, we can prove

Theorem 10. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{R}_\alpha^*(p, A, 0)$. Then*

$$|a_{p+n}| \leq A\Big(\frac{2p^2+2p+1-(2p+1)\alpha}{(p+1)(p+1-\alpha)}\Big) + \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)} \quad (p \in \mathbb{N}; n \in \mathbb{N}). \tag{3.31}$$

Theorem 11. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{P}_\alpha^*(p, A, B)$ for*

$$-1 \leq B < 0, \ 0 < A \leq 1, \ \text{and } \Re(a_{p+n}) \geq 0 \ (n = 1, \ldots, k-p-1).$$

Then

$$|a_{p+n}| \leq A - B + \frac{p-\alpha}{p+n-\alpha} (p \in \mathbb{N}; c \in \mathbb{N}). \tag{3.32}$$

Proof. Since $f(z) \in \mathcal{P}_\alpha^*(p, A, B)$, from (3.6) we have

$$\sum_{n=1}^{\infty} (a_{p+n} + b_{p+n}) z^{p+n} = \Big[(A-B)z^p - \sum_{n=1}^{\infty} (Ba_{p+n} + Ab_{p+n}) z^{p+n}\Big]\omega(z). \tag{3.33}$$

We thus find that

$$\sum_{n=0}^{\infty} (a_{p+n} + b_{p+n}) z^n = -\sum_{n=0}^{\infty} [(Ba_{p+n} + Ab_{p+n}) z^n]\omega(z), \tag{3.34}$$

where

$$a_p = 1, \ b_p = -1, \ \text{and } \omega(z) = \sum_{m=0}^{\infty} c_{m+1} z^{m+1}.$$

Equating the coefficients of z^m on both sides of (3.34), we obtain

$$a_{p+m} + b_{p+m} = -\sum_{n=0}^{m-1} (Ba_{p+n} + Ab_{p+n})c_{m-n}. \tag{3.35}$$

Hence, from (3.34) and (3.35), we find that

$$\sum_{n=0}^{m} (a_{p+n} + b_{p+n})z^n + \sum_{n=m+1}^{\infty} E_n z^n = -\sum_{n=0}^{m-1} [(Ba_{p+n} + Ab_{p+n})z^n]\omega(z) \quad (m \in \mathbb{N}) \tag{3.36}$$

for some coefficients E_n.

Using $|\omega(z)| < 1$ for $z \in \mathcal{U}$ and Parseval's identity [3] on both sides of (3.36), we obtain

$$\sum_{n=0}^{m} |a_{p+n} + b_{p+n}|^2 r^{2n} \leq \sum_{n=0}^{m} |a_{p+n} + b_{p+n}|^2 r^{2n} + \sum_{n=m+1}^{\infty} |E_n|^2 r^{2n} \leq \sum_{n=0}^{m-1} |Ba_{p+n} + Ab_{p+n}|^2 r^{2n}. \tag{3.37}$$

Letting $r \to 1$ in (3.37), we obtain

$$|a_{p+m} + b_{p+m}|^2 \leq \sum_{n=0}^{m-1} [|Ba_{p+n} + Ab_{p+n}|^2 - |a_{p+n} + b_{p+n}|^2]. \tag{3.38}$$

Setting $m = k - p$ in (3.38), we are led finally to the inequality:

$$|a_k + b_k|^2 \leq (A - B)^2 - (1 - B)^2 \sum_{n=1}^{k-p-1} |a_{p+n}|^2 - (1 - A^2) \sum_{n=1}^{k-p-1} |b_{p+n}|^2 -2(1 - AB) \sum_{n=1}^{k-p-1} \Re(a_{p+n})b_{p+n} \quad (k = p + 1, p + 2, p + 3, \ldots). \tag{3.39}$$

Since $b_{p+n} \geq 0 (p \in \mathbb{N}; n \in \mathbb{N})$, making use of the inequalities assumed to hold true in Theorem 11, we find that

$$|a_k + b_k| \leq A - B \quad (k = p + 1, p + 2, p + 3, \ldots ; p \in \mathbb{N}), \tag{3.40}$$

that is, that

$$|a_{p+n} + b_{p+n}| \leq A - B \quad (p \in \mathbb{N}; n \in \mathbb{N}). \tag{3.41}$$

Hence, by using (3.30) and (3.41), we obtain

$$\begin{aligned}|a_{p+n}| &\leq |a_{p+n} + b_{p+n}| + |b_{p+n}| \\ &\leq A - B + \frac{p-\alpha}{p+n-\alpha} \quad (p \in \mathbb{N}, n \in \mathbb{N}).\end{aligned} \tag{3.42}$$

This completes the proof of Theorem 11.

Finally, the result of Theorem 11 is sharp for

$$\frac{f(z)}{g(z)} = \frac{1 - Az^n}{1 - Bz^n} \tag{3.43}$$

and

$$g(z) = z - \frac{p-\alpha}{p+n-\alpha} z^{p+n}. \tag{3.44}$$

Corollary 1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{P}_0^*(p, A, B)$ for*

$$-1 \leq B < 0, \ 0 < A \leq 1, \ \textit{and}\ \Re(a_{p+n}) \geq 0 \quad (n = 1, \ldots, k-p-1).$$

Then

$$|a_{p+n}| \leq A - B + \frac{p}{p+n} \quad (p \in \mathbb{N}; n \in \mathbb{N}). \tag{3.45}$$

The result is sharp.

Remark 4. Since

$$\frac{p-\alpha}{p+n-\alpha}$$

is decreasing in n, Theorem 11 gives us the inequality:

$$|a_{p+n}| \leq A - B + \frac{p-\alpha}{p+1-\alpha} \quad (p \in \mathbb{N}; n \in \mathbb{N}) \tag{3.46}$$

for $f(z) \in \mathcal{P}_\alpha^*(p, A, B)(-1 \leq B < 0; 0 < A \leq 1)$ satisfying the condition:

$$\Re(a_{p+n}) \geq 0 \quad (n = 1, \ldots, k-p-1).$$

With the aid of Lemma 2, we also have

Theorem 12. *Let the function $f(z)$ defined by (1.1) be in the class $\mathfrak{R}_\alpha^*(p, A, B)$ for*

$$-1 \leq B < 0, \ 0 < A \leq 1, \ \textit{and}\ \Re(a_{p+n}) \geq 0 \quad (n = 1, \ldots, k-p-1).$$

Then

$$|a_{p+n}| \leq A - B + \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)} \quad (p \in \mathbb{N}; n \in \mathbb{N}). \tag{3.47}$$

The result is sharp for

$$\frac{f(z)}{g(z)} = \frac{1 - Az^n}{1 - Bz^n} \tag{3.48}$$

and

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)} z^{p+n}. \tag{3.49}$$

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{R}_0^*(p, A, B)$, for*

$$-1 \leq B < 0, \ 0 < A \leq 1, \ \textit{and} \ \Re(a_{p+n}) \geq 0 \quad (n = 1, \ldots, k-p-1).$$

Then

$$|a_{p+n}| \leq A - B + \frac{p^2}{(p+n)^2} \quad (p \in \mathbb{N}; n \in \mathbb{N}). \tag{3.50}$$

The result is sharp.

Acknowledgement

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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