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SOME MÖBIUS-TYPE FUNCTIONS AND INVERSIONS CONSTRUCTED VIA DIFFERENCE OPERATORS*

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Abstract. It is shown that some difference operators and their inverses, defined on the hyper-real field $^*R$, can be used to generate a pair of reciprocal relations that implies both the Möbius inversion formulae and the fundamental theorem of calculus as special consequences. As suggested by the form for the Möbius function of integral order, some explicitly constructive extensions of Möbius-type functions are presented; and accordingly, certain general Möbius-type inversion pairs are obtained in a natrural way.

0. Introduction

This paper consists of four sections. As a preparation for the second section (§2), we shall expound in §1 that certain difference operators and their inverses can be used to show that the classical Möbius inversion pair in number theory just represents a discrete analogue of the Newton-Leibniz fundamental formulae in calculus (cf. [6]). The main point is that the Möbius inversion pair may be expressed as a simple reciprocal pair of difference equations. The second section will deal with a similar but quite general reciprocal pair of non-standard difference equations using non-standard difference operators. Such a general reciprocal pair just provides a non-standard model that implies the most basic inversion formulas in analysis and number theory as articular cases. Both sections §3 and §4 are concerned with certain explicity constructive generalizations of the Möbius function as suggested by the expressions displayed in §1. The well-known convolution formula of Hagen and Rothe will be ultilized in proving a general subindex law which implies a pair of generalized Möbius-type inversions. Some applications of our results will be given briefly as examples.

1. Möbius Inversion Using Difference Operations

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A quite natural and interesting fact is that the Möbius inversion pair in number theory (cf. e.g., [3] and [9])

$$f(n) = \sum_{d|n} g(d) \tag{1.1}$$

$$g(n) = \sum_{d|n} f(d)\mu(n/d) = \sum_{d|n} f(n/d)\mu(d) \tag{1.2}$$

may be precisely viewed as a discrete analogue of the Newton-Leibniz fundamental formulae

$$F(x_1, \ldots, x_s) = \int_{a_1}^{x_1} \cdots \int_{a_s}^{x_s} G(t_1, \ldots, t_s) dt_s \cdots dt_1 \tag{1.3}$$

$$G(x_1, \ldots, x_s) = \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_s} F(x_1, \ldots, x_s) \tag{1.4}$$

where $G(t_1, \ldots, t_s)$ is an integrable function so that $F(x_1, \ldots, x_s) = 0$ whenever there is some $x_i = a_i$ $(1 \le i \le s)$. Clearly it is no real restriction to assume all $a_i = 0$.

For positive integers n and d with $d|n$, write

$$n = p_1^{x_1} \cdots p_s^{x_s}, \; d = p_1^{t_1} \cdots p_s^{t_s},$$

where p_i are primes with $p_1 < p_2 < \cdots, x_i$ and t_i are non-negative integers with $0 \le t_i \le x_i$, $(i = 1, 2, \ldots, s)$, s being an unspecified positive integer. Let us replace $f(n)$ and $g(d)$ of (1.1) by $f((x)) \equiv f(x_1, \ldots, x_s)$ and $g((t)) \equiv g(t_1, \ldots, t_s)$, respectively. Then (1.1)$\Longleftrightarrow$(1.2) may be replaced by the reciprocal pair

$$f((x)) = \sum_{0 \le t_i \le x_i} g((t)), \tag{1.5}$$

$$g((x)) = \sum_{0 \le t_i \le x_i} f((x) - (t))\mu^*((t)), \tag{1.6}$$

where $(x) \equiv (x_1, \ldots, x_s)$, $(t) \equiv (t_1, \ldots, t_s)$ and $(x) - (t) \equiv (x_1 - t_1, \ldots, x_s - t_s)$ and $\mu^*((t))$ is defined by

$$\mu^*((t)) = \begin{cases} (-1)^{t_1 + \cdots + t_s} & \text{if all } t_i \le 1, \\ 0 & \text{if there is a } t_i \ge 2. \end{cases}$$

It is clear that $\mu^*((t)) = \mu(d)$ just represents the classical Möbius function defined on Z_+, the set of natural numbers.

Let us now introduce the backwaard difference operator $\underset{x}{\Delta}$ and its inverse $\underset{x}{\Delta}^{-1}$ (summation operator) by the following

$$\underset{x}{\Delta} f(x) = f(x) - f(x-1), \quad \underset{x}{\Delta}^{-1} g(x) = \sum_{0 \le t \le x} g(t),$$

so that $\underset{x}{\Delta}\underset{x}{\Delta}^{-1}g(x) = g(x)$, $\underset{x}{\Delta}^{-1}\underset{x}{\Delta}f(x) = f(x)$ and we may denote $\underset{x}{\Delta}\underset{x}{\Delta}^{-1} = \underset{x}{\Delta}^{-1}\underset{x}{\Delta} = I$ with If $(x) \equiv f(x)$, where we assume that $f(x) = g(x) = 0$ for $x < 0$.

Henceforth we always assume that every $f((x))$ and $g((x))$ have the property that $f((x)) = g((x)) = 0$ whenever there is some $x_i < 0$ $(1 \le i \le s)$.

Evidently (1.5) and (1.6) may be rewritten using difference operators

$$f((x)) = \underset{x_1}{\Delta}^{-1} \cdots \underset{x_s}{\Delta}^{-1} g((x)) \tag{1.7}$$

$$g((x)) = \underset{x_1}{\Delta} \cdots \underset{x_s}{\Delta} f((x)), \tag{1.8}$$

where the fact that (1.8) is equivalent to (1.6) may be verified at once using the definition of $\mu^*((t))$. Note that (1.7) and (1.8) are precisely the discrete analogue of (1.3) and (1.4) respectively. Hence it follows that (1.1) $\Longleftrightarrow$ (1.2) is just a discrete analogue of (1.3) $\Longleftrightarrow$ (1.4), as claimed.

Inductively, we may define higher order operators, viz.

$$\underset{x}{\Delta}^r = \underset{x}{\Delta}\underset{x}{\Delta}^{r-1}, \ \underset{x}{\Delta}^{-r} = \underset{x}{\Delta}^{-1}\underset{x}{\Delta}^{-(r-1)}, \ (r \ge 2), \Delta^0 = I.$$

As may be verified, for any $r \in Z_+$ we have $\underset{x}{\Delta}^r\underset{x}{\Delta}^{-r} = \underset{x}{\Delta}^{-r}\underset{x}{\Delta}^r = I$, and consequently we can obtain the following reciprocal pair

$$f((x)) = \Big(\prod_{i=1}^{s} \underset{x_i}{\Delta}^{-r_i}\Big) g((x)), \tag{1.9}$$

$$g((x)) = \Big(\prod_{i=1}^{s} \underset{x_i}{\Delta}^{r_i}\Big) f((x)), \tag{1.10}$$

where $(r) \equiv (r_1, \dots, r_s) \in Z_+ \times \cdots \times Z_+ = Z_+^s$.

It is not difficult to find explicit expressions for the right-hand sides of (1.9) and (1.10) in terms of binomial coefficients. Indeed, some easy computations with idfferences show that (1.9) and (1.10) may be rewritten in the forms (cf. [5],[6])

$$f((x)) = \sum_{(0)\le(t)\le(x)} \mu_{(-r)}((t)) g((x) - (t)) \tag{1.11}$$

$$g((x)) = \sum_{(0)\le(t)\le(x)} \mu_{(r)}((t)) f((x) - (t)) \tag{1.12}$$

where $(0) \le (t) \le (x)$ stands for the summation condition $0 \le t_i \le x_i$, $(i = 1, \dots, s)$, and $\mu_{(-r)}((t))$ and $\mu_{(r)}((t))$ are defined by the following

$$\mu_{(r)}((t)) = \prod_{i=1}^{s} \binom{r_i}{t_i} (-1)^{t_i}, \mu_{(-r)}((t)) = \prod_{i=1}^{s} \binom{-r_i}{t_i} (-1)^{t_i}. \tag{1.13}$$

Notice that for the case $(r) \equiv (1, \dots, 1)$ the function $\mu_{(r)}((t))$ becomes the ordinary Möbius function $\mu(d) \equiv \mu(p_1^{t_1} \cdots p_s^{t_s})$ and $\mu_{(-r)}((t)) \equiv 1$, so that (1.11) $\Longleftrightarrow$ (1.12) reduces to the classical inversion formulae of Möbius. Moreover, it is clear that (1.11) $\Longleftrightarrow$ (1.12) may be expressed as "convolutions"

$$f((x)) = (\mu_{(-r)} * g)((x)), \quad g((x)) = (\mu_{(r)} * f)((x)), \tag{1.14}$$

or, in brief, $f = \mu_{(-r)} * g, g = \mu_{(r)} * f$.

Remark 1. There is also a dual form of the inversion formulae (1.11) $\Longleftrightarrow$ (1.12), of which the special case with $(r) \equiv (1, \ldots, 1)$ has a nice application to the Combinatorial Probability Theory of Arbitrary Events (cf. [1]). Indeed, the dual form of (1.11) $\Longleftrightarrow$ (1.12) can be used to obtain a kind of generalized Poincaré formula for the calculus of probabilities.

Remark 2. (1.11) $\Longleftrightarrow$ (1.12) with μ's being defined by (1.13) can also be verified by direct substitutions (making use of a simple combinatorial identity). In such a way of verification one may find that $r_i (i = 1, \ldots, s)$ can even be replaced by any positive real numbers. This implies that the orders r_i of Δ^{r_i} (and $-r_i$ of Δ^{-r_i}) may be extended to positive (negative) real numbers via (1.13). Actually, some extensions of the Möbius function considred in §3-§4 are just suggested by the expressions appearing in (1.13).

2. Möbuis Inversion Using Non-standard Difference Operators

In this section we will make use of some basic concepts and operations occurred in non-standard analysis. (cf. [10] and [11]). Denote by $^*R$ and $^*Z$ the hyper-real field and the set of integers (including infinite integers) contained in $^*R$. All functions of a single variable considered in what follows are defined on some set $D \subset^* R$ and taking values in $^*R$, namely, they belong to Map $(D,^* R)$. For any real number $\delta \in^* R$ we denote its standard part by $(\delta)^\circ$ or $\mathrm{st}(\delta)$.

As may be observed, the analogy between (1.7) $\Longleftrightarrow$ (1.8) and (1.3) $\Longleftrightarrow$ (1.4) suggests that it should be possible to build up a non-standard model implying both (1.7) $\Longleftrightarrow$ (1.8) and (1.3) $\Longleftrightarrow$ (1.4), if a kind of non-standard difference operator, say,

$$\overset{\delta}{\underset{x}{\Delta}} f(x) = f(x) - f(x - \delta), \quad (\delta > 0, \delta \in^* R)$$

is utilized instead of the ordinary difference $\underset{x}{\Delta}$ as employed in §1. The reason is clear: If δ is a positive infinitesimal with $(\delta)^\circ = 0$ and if $f(x)$ is differentiable, we have

$$(\delta^{-1} \overset{\delta}{\underset{x}{\Delta}} f(x))^\circ = \frac{d}{dx} f(x) = f'(x).$$

Correspondingly, the inverse operator $\delta \overset{\delta}{\underset{x}{\Delta}}{}^{-1}$ will represent an integral operator such that

$$(\delta \overset{\delta}{\underset{x}{\Delta}}{}^{-1} g(x))^\circ = \int_a^x g(t) dx, (a < x, a \in^* R).$$

Let us give some details as follows.

For a given $\delta \in^* R$ with $\delta > 0$, denote by $[1/\delta]$ the integral part of $1/\delta$, so that it may be an infinite integer, if $(\delta)^\circ = 0$. Let ω be a positive infinite integer, viz. $\omega \in^* Z\backslash Z_+$, and let Z_ω be a set of integers defined by

$$Z_\omega := \{m | 0 \leq m \leq [1/\delta] \cdot \omega\}.$$

Denote $W := \{x = m\delta | m \in Z_\omega\}$. Clearly W just gives fine partition of an interval containing $(0, (1-\delta)\omega)$ whenever $(\delta)^\circ = 0$. We shall consider functions of $\mathrm{Map}(W,^* R)$, denote by f, g, etc. For the sake of simplicity we consider all operators acting on functions of x within $[0, \infty)$, so that we always assume that

$$f(x) = g(x) = 0 \text{ for } x < 0. \tag{2.1}$$

Let us now introduce some difference operators and their inverses with step-length $\delta > 0$ for the function $f(x) \in \mathrm{Map}(W,^* R)$, namely

(i) Divided difference:

$$\overset{\delta}{\underset{x}{\Lambda}} f(x) := \delta^{-1} \overset{\delta}{\underset{x}{\Delta}} f(x) = \frac{1}{\delta}(f(x) - f(x-\delta))$$

(ii) Higher differences (defined by induction):

$$\overset{\delta}{\underset{x}{\Lambda}}{}^1 = \overset{\delta}{\underset{x}{\Lambda}}, \qquad \overset{\delta}{\underset{x}{\Lambda}}{}^n = \overset{\delta}{\underset{x}{\Lambda}}{}^1 \overset{\delta}{\underset{x}{\Lambda}}{}^{n-1}, \quad (n \geq 2)$$
$$\overset{\delta}{\underset{x}{\Lambda}}{}^0 = I, \quad \mathrm{I}f(x) \equiv f(x).$$

(iii) Inverse difference:

$$\overset{\delta}{\underset{x}{\Delta}}{}^{-1} g(x) = \sum_{j=0}^{m} g(j\delta) = \sum_{0 \leq j\delta \leq x} g(j\delta), \quad (x = m\delta),$$
$$\overset{\delta}{\underset{x}{\Lambda}}{}^{-1} g(x) = \delta \overset{\delta}{\underset{x}{\Delta}}{}^{-1} g(x) = \sum_{0 \leq j\delta \leq x} g(j\delta)\delta.$$

(iv) Higher inverse differences:

$$\overset{\delta}{\underset{x}{\Lambda}}{}^{-n} = \overset{\delta}{\underset{x}{\Lambda}}{}^{-1} \overset{\delta}{\underset{x}{\Lambda}}{}^{-(n-1)}, \quad (n \geq 2)$$

Briefly one may write $\Lambda \equiv \delta^{-1}\Delta$ and $\Lambda^{-1} \equiv \delta\Delta^{-1}$. As may be verified immediately, we have with the condition (2.1)

$$\Lambda^{-r}\Lambda^r = \Lambda^r\Lambda^{-r} = I, \quad (r \in Z_+). \tag{2.2}$$

Suppose that $f(x)$ is differentiable for $x > 0$ and $g(x)$ is integrable. Then for $\delta \in^* R$ with $\delta > 0$ and its standard part $(\delta)^\circ$ =st$(\delta) = 0$, and for $x = m\delta \in W$, we have by taking standard part

$$\Big(\overset{\delta}{\underset{x}{\Lambda}} f(x)\Big)^\circ = \frac{d}{dx} f(x) = f'(x),$$
$$\Big(\overset{\delta}{\underset{x}{\Lambda}}{}^{-1} g(x)\Big)^\circ = \Big(\sum_{j=0}^{m} g(j\delta)\delta\Big)^\circ = \int_0^x g(t)dt.$$

More generally, by induction one may show that for $r \geq 2$

$$\Big(\overset{\delta}{\underset{x}{\Lambda}}{}^{-r} g(x)\Big)^\circ = \Big(\Big(\delta \overset{\delta}{\underset{x}{\Delta}}{}^{-1}\Big)^r g(x)\Big)^\circ = \int_0^x \frac{(x-t)^{r-1}}{\Gamma(r)} g(t)dt.$$

Here the order r of integration may even by replaced by any positive real number in accordance with the idea of Riemann and Liouville (cf. [12], Chap.2 §8).

Let us consider multivarite functions of Map$(W^s,{}^* R^s)$, denoted by $f((x)) \equiv f(x_1, \ldots, x_s)$ and $g((x)) \equiv g(x_1, \ldots, x_s)$, etc. with $(x) \in W^s$. It is always assumed that $f((x)) = g((x)) = 0$ whenever there is some $x_i < 0$ $(1 \leq i \leq s)$. Then the method of non-standard analysis permits us to obtain an extended form of (1.9) $\Longleftrightarrow$ (1.10) as follows.

Theorem 1. *(operational inversion formulae) For every s-tuple* $(r_1, \ldots, r_s) \in Z_+^s$ *we have the inversion formulae*

$$f((x)) = \Big(\prod_{i=1}^{s} \overset{\delta}{\underset{x_i}{\Lambda}}{}^{-r_i}\Big) g((x)), \tag{2.3}$$

$$g((x)) = \Big(\prod_{i=1}^{s} \overset{\delta}{\underset{x_i}{\Lambda}}{}^{r_i}\Big) f((x)), \tag{2.4}$$

where $(x) \in W^s$, $f((x)) \in Map(W^s,{}^* R^s)$ *and* $g((x)) \in Map(W^s,{}^* R^s)$.

In fact (2.3) $\Longleftrightarrow$ (2.4) can be verified by repeated applications of the basic relation (2.2).

Example 1. For $\delta = 1$ and $r_i = 1$ $(1 \leq i \leq s)$ we see that the relation (2.3) $\Longleftrightarrow$ (2.4) implies the Möbius inversion (1.1) $\Longleftrightarrow$ (1.2) via its equivalent form (1.7) $\Longleftrightarrow$ (1.8).

Example 2. For $r_i = 1$ $(1 \leq i \leq s)$ and $\delta \in {}^*R$, $\delta > 0$ with st$(\delta) = 0$, it is clear that (2.3) $\Longleftrightarrow$ (2.4) with $f = F$ and $g = G$ implies the the Newton-Leibniz reciprocal formulae (1.3) $\Longleftrightarrow$ (1.4) by taking standard parts of the both sides of (2.3) and (2.4).

Example 3. The case $\delta = 1$ of Theorem 1 clearly reduces to the generalized Möbius inversion (1.11) $\Longleftrightarrow$ (1.12) with $\mu_{(r)}$ and $\mu_{(-r)}$ being defined by (1.13).

Remark 3. Compairing (2.3)-(2.4) with (1.9)-(1-10) and recalling that (1.9)-(1.10) have explicit expressions (1.11)-(1.12) with (1.13), in which r_i may take any positive real numbers, one may imagine that (2.3) $\Longleftrightarrow$ (2.4) could also be extended to the case where r_i are any positive real numbers. In particular, for the case $(\delta)^\circ = 0$ and taking standard parts of (2.3) and (2.4), one may get certain reciprocal relations (under some conditions for functions) between iterated integrals and partial derviatives of fractional orders (positive real orders). A detailed investigation of the above matter appears to be of some interest and may be left to the interested reader.

3. A Kind of Möbius-type Function with Number-Theoretic Applications

The forms given by (1.13) suggest the following definition:
For any real or complex number $\alpha \neq 0$, a Möbius function of order α is defined by

$$\mu_\alpha(n) = \prod_{p|n} \binom{\alpha}{\partial_p(n)} (-1)^{\partial_p(n)} \tag{3.1}$$

where p runs through all the prime divisors of n, $(n \in Z_+)$, and $\partial_p(n) = ord_p(n)$ denotes the highest index k of p that p^k divides n.

Note that $\partial_p(1) = 0$. Thus we have $\mu_0(n) = [1/n] = e_0(n), (n \in Z_+)$. Recall that the Dirichlet convolution of two arithmetic functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d), \tag{3.2}$$

where the summation is taken over all the divisors of n. Evidently we have

$$(f * \mu_0)(n) = (\mu_0 * f)(n) = f(n), \tag{3.3}$$

so that μ_0 is just the identity element of the Dirichlet convolution algebra.

Using some combinatorial computations we easily verify

$$(\mu_\alpha * \mu_\beta)(n) = \mu_{\alpha+\beta}(n), \tag{3.4}$$

where α and β are any two real or complex numbers. (A much more general result will be proved in the next section.) In particular, $\mu_\alpha * \mu_{-\alpha} = \mu_0$, ie., μ_α and $\mu_{-\alpha}$ are reciprocal elements of the convolution algebra. Consequently, there hold the Möbius-type inversion formulae

$$f(n) = \sum_{d|n} \mu_\alpha(n/d)g(d) \Longleftrightarrow g(n) = \sum_{d|n} \mu_{-\alpha}(n/d)f(d). \tag{3.5}$$

A few examples of some interest may be briefly mentioned as follows.

Example 4. It is easily seen that the ordinary Möbuius function and the zeta-function of the convolution algebra are given by $\mu = \mu_1$ and $\zeta = \mu_{-1}$ respectively. Moreover, it is also easy to verify that $\mu_{-2}(n)$ just represents the divisor function $\tau(n)$, namely

$$\mu_{-2}(n) = (\mu_{-1} * \mu_{-1})(n) = \sum_{d|n} 1 = \tau(n).$$

Example 5. As consequences of (3.4) we easily find

$$\sum_{d|n} \mu(d)\mu(n/d) = (\mu_1 * \mu_1)(n) = \mu_2(n) = \prod_{p|n} \binom{2}{\partial_p(n)} (-1)^{\partial_p(n)},$$

$$\sum_{d|n} \tau(d)\tau(n/d) = (\mu_{-2} * \mu_{-2})(n) = \mu_{-4}(n) = \prod_{p|n} \binom{-4}{\partial_p(n)} (-1)^{\partial_p(n)},$$

$$\sum_{d|n} \mu_\alpha(d) = \sum_{d|n} \mu_\alpha(d)\zeta(n/d) = (\mu_\alpha * \mu_{-1})(n) = \mu_{\alpha-1}(n).$$

Exmaple 6. As a remarkable application of the Möbius function $\mu_r(n)(n \in Z_+)$ to the number-theory, we introduce an arithmetical function which has the number-theoretic meaning familiar to Euler's totient.

Let n be a fixed positive integer. An integer a is said to be r-th degree prime to n, written as $(a, n)_r = 1$, if for each prime divisor p of n, there are $a_0, a_1, \ldots, a_{r-1}$ with $0 < a_i < p$ such that

$$a \equiv a_0 + a_1 p + \cdots + a_{r-1} p^{r-1} \pmod{p^r}. \tag{3.6}$$

Then, the function

$$\phi_{(r)}(n) = \sum_{d|n} d\mu_r(n/d) = n \sum_{d|n} \mu_r(d)/d \tag{3.7}$$

just counts the number of a's such that $1 \le a \le n$ and $(a, n)_r = 1$, whenever n is r-powerful, viz. $\partial_p(n) \ge r$ for every prime divisor p of n. Moreover, $\phi_{(r)}$ has a product formula if n is r-powerful, viz.

$$\phi_{(r)}(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)^r. \tag{3.8}$$

In fact, let $N_r(n)$ denote the number of integers a such that $1 \le a \le n$ and $(a, n)_r = 1$. By Chinese Remainder Theorem (cf. [7], p.34, Theorem 1), it is easily shown that $N_r(n)$ is multiplicative in n. When $n = p^e$, $e \ge r$, we can see from (3.6) that $N_r(p^e) = p^{e-r}(p-1)^r = \phi_{(r)}(p^e)$. So, $N_r(n) = \phi_{(r)}(n)$ holds for every r-powerful numbers n.

4. Further Extended Möbius-type Functions and Inversions

Here we will introduce two further extensions of the Möbuius function that have explicit representations.

(i) It is easy to observe that the constant α contained in (3.1) may be replaced by any arithmetic function α $\in$Map(Z_+, C), where C denotes the set of complex numbers. More precisely, one may define

$$\mu_{(\alpha)}(n) = \prod_{p|n} \binom{\alpha(p)}{\partial_p(n)} (-1)^{\partial_p(n)}. \tag{4.1}$$

Correspondigly, the subindex law (3.4) can be extended to the form

$$(\mu_{(\alpha)} * \mu_{(\beta)})(n) = \mu_{(\alpha+\beta)}(n), \tag{4.2}$$

where both α and β are arithmetic functions. The proof follows from the similar lines as that of (4.4) which will be given immediately.

Certainly the Möbius-type inversion (3.5) still holds when μ_α and $\mu_{-\alpha}$ are replaced by $\mu_{(\alpha)}$ and $\mu_{(-\alpha)}$, respectively.

(ii) Let $\alpha \in \mathrm{Map}(Z_+, C)$ and $z \in C$. Then an extensive generalization of the Möbius function may be introduced as follows

$$\mu_{(\alpha)}(n,z) = \prod_{p|n} \frac{\alpha(p)}{\alpha(p) + z\partial(n)} \binom{\alpha(p) + z\partial_p(n)}{\partial_p(n)} (-1)^{\partial_p(n)}. \tag{4.3}$$

This will reduce to (4.1) when $z = 0$. For the particular case $\alpha \equiv 0$ we define $\mu_0(n,z) = [1/n] = e_0(n)(n \in Z_+)$. Also we see that $\mu_1(n,0) = \mu_1(n) = \mu(n)$.

Theorem 2. *(general subindex law) For any fixed parameter $z \in C$ we have*

$$(\mu_{(\alpha)} * \mu_{(\beta)})(n,z) = \mu_{(\alpha+\beta)}(n,z), \tag{4.4}$$

where both α and β are arithmetic functions.

Proof. Notice that $\mu_{(\alpha)}(n,z)$ and $\mu_{(\beta)}(n,z)$ are multiplicative functions of n, $(n \in Z_+)$, so that the convolution $(\mu_{(\alpha)} * \mu_{(\beta)})(n,z)$ is also multiplicative with respect to n (For basic properties of multiplicative functions, see, e.g., [9], Chap.6). Thus, it suffices to verity (4.4) for the case $n = p^e$ with p being a prime and $e \in Z_+$.

Let us recall Hagen-Rothe's convolution formula (cf. [2])

$$\sum_{k=0}^{n} \frac{x}{x+kz}\binom{x+kz}{k}\frac{y}{y+(n-k)z}\binom{y+(n-k)z}{n-k} = \frac{x+y}{x+y+nz}\binom{x+y+nz}{n} \tag{4.5}$$

Thus, making use of (4.5) we find

$$\begin{aligned}
(\mu_{(\alpha)} * \mu_{(\beta)})(p^e,z) &= \sum_{d|p^e} \mu_{(\alpha)}(d,z)\mu_{(\beta)}(p^e/d,z) \\
&= \sum_{i=0}^{e} \mu_{(\alpha)}(p^i,z)\mu_{(\beta)}(p^{e-i},z) \\
&= \sum_{i=0}^{e} \frac{\alpha(p)}{\alpha(p)+zi}\binom{\alpha(p)+zi}{i}\frac{\beta(p)}{\beta(p)+z(e-i)}\binom{\beta(p)+z(e-i)}{e-i}(-1)^e \\
&= (-1)^e \frac{\alpha(p)+\beta(p)}{\alpha(p)+\beta(p)+ze}\binom{\alpha(p)+\beta(p)+ze}{e} \\
&= \mu_{(\alpha+\beta)}(p^e,z).
\end{aligned}$$

Thus is what we expected.

Remark 4. Since the proof is essentially concerned in the exponent e of p^e in which the meaning of p is unimportant, we see that the related results may be also extended to some arithemtical semigroups. For arithmeticl functions on semigroups, see, e.g., [4].

Evidently, from the relation $(\mu_{(\alpha)} * \mu_{(-\alpha)})(n,z) = \mu_0(n,z) = e_0(n)$ we may infer the following

Theorem 3. *(general Möbius-type inversion) For any $\alpha \in Map(Z_+, C)$ and $z \in C$ we have the reciprocal formulae*

$$f(n) = \sum_{d|n} \mu_{(\alpha)}(n/d, z)g(d) \Longleftrightarrow g(n) = \sum_{d|n} \mu_{(-\alpha)}(n/d, z)f(d). \tag{4.6}$$

Note that the classical Möbius inversion is a particular case of (4.6) with $\alpha \equiv 1$ and $z = 0$.

If we introduce the mapping $n = p_1^{x_1} \cdots p_s^{x_s} \longmapsto (x_1, \ldots, x_s) \equiv (x)$, $d = p_1^{t_1} \cdots p_s^{t_s} \longmapsto (t_1, \ldots, t_s) \equiv (t)$ and denote $(\alpha) = (\alpha_1, \ldots, \alpha_s) \in C^s$, it is clear that (4.6) implies the following reciprocal pair as a generalization of (1.11) $\Longleftrightarrow$ (1.12)

$$f((x)) = \sum_{(0) \le (t) \le (x)} \mu_{(\alpha)}((t), z) g((x) - (t)) \tag{4.7}$$

$$g((x)) = \sum_{(p) \le (t) \le (x)} \mu_{(-\alpha)}((t, z) f((x) - (t)) \tag{4.8}$$

where $f((x))$ and $g((x))$ are functions of $\mathrm{Map}(Z_+^s, C)$ and

$$\mu_{(\alpha)}((t), z) = \prod_{i=1}^{s} \frac{\alpha_i}{\alpha_i + zt_i} \binom{\alpha_i + zt_i}{t_i} (-1)^{t_i}, \tag{4.9}$$

and $\mu_{(-\alpha)}((t), z)$ just follows from (4.9) with (α) being replaced by $(-\alpha) \equiv (-\alpha_1, \ldots, -\alpha_s)$.

Finally, let us give an example showing that $\mu_{(\alpha)}((t), z)$ may attain some combinatorial meaning when $z = -1$ and (α) and (t) are suitably specified.

Example 7. *A set of elemnets arranged in a circle is called a cyclic ordered set. Given s cyclic ordered sets $A_1, \ldots, A_s$ with $|A_i| = a_i$ (the number of elements of A_i). Suppose that $(t) \equiv (t_1, \ldots, t_s) \in Z_+^s$ with $t_i \le [\frac{1}{2}a_i]$, $(i = 1, 2, \ldots, s)$. Then the number of ways that t_i elements are selected from A_i such taht no two are neighboring elements is given by Kaplansky's number [8]*

$$f^*(a_i, t_i) = \frac{a_i}{a_i - t_i} \binom{a_i - t_i}{t_i}.$$

Consequently, the total number of different ways of selecting t_i elements of $A_i (i = 1, 2, \ldots, s)$ is equal to the number

$$(-1)^{\Sigma t_i} \mu_{(\alpha)}((t), -1) = \prod_{i=1}^{s} \frac{a_i}{a_i - t_i} \binom{a_i - t_i}{t_i}. \tag{4.10}$$

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