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ON THE EULER CHARACTERISTIC OF THE SPACE SATISFYING CONDITION (T^{**})

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Abstract. In this paper, we prove some results on the Euler characteristic number of a locally nilpotent space and a space satisfying condition (T^{**}).

1. Introduction

The study of the nilpotent space was begun by A. K. Bousfield, P. Hilton, G. Mislin and others [7,8]. Especially, the Euler characteristic of the nilpotent space was studied by R. H. Lewis [10].

In this paper, we define the condition (T^*) and (T^{**}) and the locally nilpotent spaces as the extensive concept of the nilpotent space. Euler characteristic number of the spaces with relation to the conditions (T^*) and (T^{**}) will be studied.

Furthermore, we study the homotopy equivalent conditions of the locally nilpotent spaces and spaces satisfying condition (T^{**}).

We work in the category of the topological spaces having the homotoppy type of connected CW-complexes with base point and denote as the T.

2. Some Properties of the Condition (T^{**})

In this section, we define the locally nilpotent space and condition (T^*) and condition (T^{**}) and study their properties respectively.

We recall that locally nilpotent group is the group whose finitely generated subgroups are nilpotent groups [12].

And we denote the category of nilpotent spaces and continuous maps as T_N.

Now we extend the concept of the nilpotent space like followings.

Definition 2.1. A space $X(\in T)$ is said to be a locally nilpotent space if

(1) $\pi_1(X)$ is a locally nilpotent group,
(2) the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$ [1].

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And we denote the category of locally nilpotent spaces and continuous maps as T_{LN}. We know that the category T_N is full subcategory of T_{LN}.

Generally, for a group G and a fixed $g \in G$, we denote by $[g, G]$ the subgroup of G generated by all commutators $[g, a]$ which means $g^{-1}a^{-1}ga$ where $a \in G$. Since $[g, a]^b = [g, b]^{-1}[g, ab]$ for each $a, b \in G$ (where $a^b = b^{-1}ab$), $[g, G]$ is a normal subgroup of G.

Definition 2.2. We say that a space $X(\in T)$ satisfies condition (T^*) if for all $g, t \in \pi_1(X)$ either $g[g, \pi_1(X)] = t[t, \pi_1(X)]$ or $g[g, \pi_1(X)] \cap t[t, \pi_1(X)] = \phi$.

Now we define the effective concept with respect to the locally nilpotent space.

Definition 2.3. For $X \in T$, we say that X satisfies the condition (T^{**}) if for all $g(\neq 1) \in \pi_1(X)$, then $g \notin [g, \pi_1(X)]$.

Since the $[g, \pi_1(X)]$ is a normal subgroup of $\pi_1(X)$, condition (T^{**}) is homotopy invariant property.

In fibration $F \to E \to B$, any path $\alpha : I \to B$ and singular q-complex $g : \Delta^q \to p^{-1}(\alpha(0))$ determine a map $G : \Delta^q \times I \to E$ over $\alpha \circ pr_2 : \Delta^q \times I \to I \to B$ and extending $G_0 = g : \Delta^q \times \{0\} \to E$. if α is a loop, then $G_1 : \Delta^q \times \{1\} \to E$ is a q-simplex in $p^{-1}(\alpha(1)) = p^{-1}(\alpha(0))$. Now do elements of $\pi_1(B)$ operate on $H_*(F)$ [9].

Definition 2.4. A fibration $F \to E \to B$ is said to be quasi-hilpotent if the action of $\pi_1(B)$ on $H_*(F)$ is nilpotent, $* \geq 0$.

Lemma 2.5. *For $X \in T_{LN}$, X satisfies the condition (T^{**}).*

Proof. (Step 1); first, we assert that X satisfies the condition (T^*). Since $\pi_1(X)$ is a locally nilpotent group, suppose $c \in a[a, \pi_1(X)] \cap b[b, \pi_1(X)]$ for some $a, b, c \in \pi_1(X)$. We only show that $a[a, \pi_1(X)] = b[b, \pi_1(X)]$. If $b \in a[a, \pi_1(x)]$ (for $a, b \in \pi_1(X)$) then $b[b, \pi_1(x)] \subset a[a, \pi_1(X)]$. We know that

$$c[c, \pi_1(X)] \subset a[a, \pi_1(X)] \cap b[b, \pi_1(X)] \qquad (*)$$

Clearly, $c = h^{-1}a$ for some $h = \prod_{i=1}^{m}[a, g_i]^{\epsilon_i} \in [a, \pi_1(X)](g_i \in \pi_1(X), \epsilon_i = \pm 1)$. Let $G_1 =< a, g_1, \ldots, g_m >$. Since $a = hc$, $h \equiv \prod_{i=1}^{m}[h, g_i]^{\epsilon_i}$ modulo $[c, G_1]$, that is, $h = \prod_{i=1}^{m}[h, g_i]^{\epsilon_i}$ in $\frac{g_1}{[c,G_1]}$. However, since the latter group is nilpotent, it follows that $h = 1$ in $\frac{G_1}{[c,G_1]}$ and $h \in [c, G_1]$. Therefore, $a = hc \in c[c, \pi_1(X)]$ and we get $a[a, \pi_1(X)] \subset c[c, \pi_1(X)]$ [2]. It follows from $(*)$ that $a[a, \pi_1(X)] = c[c, \pi_1(X)]$.

Similary, $b[b, \pi_1(X)] = c[c, \pi_1(X)]$ and consequently, $a[a, \pi_1(X)] = b[b, \pi_1(X)]$. Thus $X \in T_{LN}$, X satisfies the condition (T^*).

(Step 2); next, we assert that X above satisfies the condition (T^{**}). Assume that $g \in [g, \pi_1(X)]$ for some $(g \neq 1) \in \pi_1(X)$. Then $g^{-1} \in [g, \pi_1(X)]$ and $1 \in g[g, \pi_1(X)]$. Thus $g[g, \pi_1(X)] \cap 1[1, \pi_1(X)] \neq \phi$. Since X satisfies the condition (T^*) by (Step 1), $g[g, \pi_1(X)] = 1$. Since $g \in g[g, \pi_1(X)]$, we have a contradiction.

Remark. If $X(\in T)$ satisfies the condition (T^*), then X satisfies condition (T^{**}). But the converse statement need not be true in general [6].

Theorem 2.6. *For $X \in T_{LN}$, if $b \in [a, \pi_1(X)]$ then $a[a, \pi_1(X)] = b[b, \pi_1(x)]$, for $a, b \in \pi_1(X)$.*

Proof. If $b \in a[a, \pi_1(X)]$ (for $a, b \in \pi_1(X)$) then $b[b, \pi_1(X)] \subset a[a, \pi_1(X)]$. Since X satisfies the condition (T^*) by the (Step 1) of Lemma 2.5, thus our proof is completed by the following properties; when X satisfies the condition (T^*), for each $a, b \in \pi_1(X)$, if $a[a, \pi_1(X)] \subset b[b, \pi_1(X)]$ then $a[a, \pi_1(X)] = b[b, \pi_1(X)]$. Thus our proof is completed.

3. Euler Characteristic of the Space Satisfying Condition (T^{**})

In this section, we make results on the Euler characteristic number of the locally nilpotent spaces and spaces satisfying condition (T^{**}).

Furthermore, we study about the homotopy equivalent conditions of the nonnilpotent spaces.

Lemma 3.1.[3] *Fur finite X, if $\pi_1(X)$ contains a torsion free normal abelian subgroup $A \neq 1$ which acts nilpotently on $H_n(\tilde{X})$ where $n \geq 0$, then $\chi(X) = 0$.*

Theorem 3.2. *For finite X satisfying condition (T^{**}), if*

(1) *the action $\pi_1(X) \times H_n(\tilde{X}) \to H_n(\tilde{X})$ is nilpotent where $n \geq 0$,*
(2) *$\pi_1(X)(\neq 1)$ is finite,*

then $\chi(X) = 0$.

Proof. (Step 1); we check the nilpotent property of $\pi_1(X)$ under the above hypothesis. So assume that $\pi_1(X)$ is not nilpotent, then we don't have finite upper central series of $\pi_1(X)$. If $Z_n(\pi_1(X))$ denote the n-th center of $\pi_1(X)$, we can find an integer n such that $Z_{n+1}(\pi_1(X)) = Z_n(\pi_1(X)) \subsetneq \pi_1(X)$. It follows that if $x \notin Z_n(\pi_1(X))$, then $[x, \pi_1(X)] \not\subseteq Z_n(\pi_1(X))$. Choose any $x_1 \notin Z_n(\pi_1(X))$. we know $[x_1, \pi_1(X)] \not\subseteq Z_n(\pi_1(X))$ by above. If $x_1 \in [x_1, \pi_1(X)]$ then we have shown that the condition (T^{**}) does not hold, as required, so assume $x_1 \notin [x_1, \pi_1(X)]$. Then choose $x_2 \in [x_1, \pi_1(X)]$, $x_2 \notin Z_n(\pi_1(X))$. Since $[x_1, \pi_1(X)]$ is a normal subgroup of $\pi_1(X)$, $[x_2, \pi_1(X)] \subseteq [x_1, \pi_1(X)]$. If $x_2 \in [x_2, \pi_1(X)]$, we are done.

Otherwise, we have $[x_2, \pi_1(X)] \subsetneq [x_1, \pi_1(X)]$ but also we noted $[x_2, \pi_1(X)] \not\subseteq Z_n(\pi_1(X))$. So pick $x_3 \in [x_2, \pi_1(X)]$, $x_3 \notin Z_n(\pi_1(X))$ and continue. Since $\pi_1(X)$ is finite, this process must stop. Afer all we have α for which $x_\alpha(\neq 1) \in [x_\alpha, \pi_1(X)]$. This is a contradiction to the fact that X satisfies the condition (T^{**}). Thus we know that $\pi_1(X)$ is nilpotent group.

(Step 2); when $\pi_1(X)$ is finite, since X satisfies the condition (T^{**}) and by (step 1), $\pi_1(X)$ is nilpotent group. Thus $X \in T_N$. Since $\pi_1(X)$ is finite, $\chi(\tilde{X}) = \chi(X)$ and another property $\chi(\tilde{X}) = |\pi_1(X)|\chi(X)$ where $|\ |$ means the order of $\pi_1(X)$ and $\tilde{X}$ means the univeral convering space of X [11]. If $\pi_1(X) \neq 1$, $\chi(X) = 0$.

Theorem 3.3. *For finitely indexed set $\{X_\alpha, \alpha \in M :$ finite$\}$, $X_\alpha \in T_{LN}$ for each α with $\pi_1(X_\alpha)$ finite then $\chi(\prod_{\alpha \in M} X_\alpha) = 0$.*

Proof. (Step 1); we know that X_α satisfies the condition (T^{**}) by Lemma 2.5. Now let's check the finite product property of condition (T^{**}). For set $\{X_\alpha | \alpha \in M : \textit{finite}\}$, if X_α satisfies the condition (T^{**}) for each $\alpha \in M$ then $\prod_{\alpha\in M} X_\alpha$ satisfies the condition (T^{**}) from the following facts; let G be the group $\prod_{\alpha\in M}\pi_1(X_\alpha)$ and P_α be the projection of G on $\pi_1(X_\alpha)$ for any $\alpha \in M$. Suppose that $g \in [g, G]$ for $g \neq 1$ in G, there exists $\alpha(\in M)$ such that $P_\alpha(g) = g_\alpha$ is not identity in $\pi_1(X_\alpha)$. Then $g_\alpha \in [g_\alpha, \pi_1(X_\alpha)]$ this is a contradition to condition (T^{**}) (see Lemma 2.5). Thus $\prod_{\alpha\in M} X_\alpha$ satisfies the condition (T^{**}).

(Step 2); let's check the finite product property of the nilpotent actions. By the nilpotent property of $H_n(\tilde{X}_\alpha)$ under the action $\pi_1(X_\alpha)$, there is a lower central series of $H_n(\tilde{X}_\alpha)$ for each $\alpha \in M$. In this finite product space case, we only prove the arbitrary two product case of X_α, $X_\beta \in \{X_\alpha\}_{\alpha\in M}$. Put the lower central series of $H_n(\tilde{X}_\alpha)$ and $H_n(\tilde{X}_\beta)$ under the nilpotent action of $\pi_1(X_\alpha)$ and $\pi_1(X_\beta)$ respectively like followings; suppose that the nilpotent classes of X_α and X_β are n and m respectively. We get the following:

$$H_n(\tilde{X}_\alpha) \supset G_2 \supset G_3 \supset \cdots \supset G_j \supset \cdots \supset G_n = \{e\}$$
$$H_n(\tilde{X}_\beta) \supset E_2 \supset E_3 \supset \cdots \supset E_i \supset \cdots \supset E_m = \{e\}$$

Now we make the following sequence;
$H_n(\tilde{X}_\alpha) \times H_n(\tilde{X}_\beta) \supset H_n(\tilde{X}_\alpha) \times E_2 \supset G_2 \times E_2 \supset \cdots \supset G_{j-1} \times E_i \supset G_j \times E_i \supset G_j \times E_{i+1} \supset \cdots \supset G_n \times E_m = \{e\} \times \{e\} \cdots (*)$.

Then the above sequence $(*)$ is lower central series of $H_n(\tilde{X}_\alpha) \times H_n(\tilde{X}_\beta)$ under the action $\pi_1(X_\alpha) \times \pi_1(X_\beta)$ with the componentwise action. Futhermore, the nilpotent class of $X_\alpha \times X_\beta$ is less than $m \cdot n$. Thus there is a $\pi_1(\prod_{\alpha\in M} X_\alpha)$ nilpotent action on $H(\prod_{\alpha\in M} \tilde{X}_\alpha)$.

Since $\pi_1(\prod_{\alpha\in M} X_\alpha)$ is finite, our proof is completed by Theorem 3.2.

Remark. For set $\{X_\alpha | \alpha \in M : \textit{finite}\}$, $X_\alpha \in T_{LN}$ for each $\alpha \in M$ [5] if and only if $\prod_{\alpha\in M} X_\alpha \in T_{LN}$ [5].

Corollary 3.4. *For set $\{X_\alpha | \alpha \in M : \textit{finite}\}$, $X_\alpha(\in T_{LN})$ is finite oriented space without boundary then $\chi(\prod_\alpha \in M X_\alpha) = 0$ if $\pi_1(X_\alpha)(\neq 1)$ is finite for any $\alpha \in M$.*

Proof. Since $\chi(\prod_{\alpha\in M} X_\alpha) = \prod_{\alpha\in M}\chi(X_\alpha)$, by Theorem 3.2 and Theorem 3.3, our proof is completed.

We recall that a group G satisfies the maximal condition if it has no infinite strictly increasing chain of subgroups [12].

Theorem 3.5. *For finite $X(\in T_{LN})$, if*

(1) *$\pi_1(X)$ is infinite with the maximal condition on normal subgroups of $\pi_1(X)$ or*
(2) *$\pi_1(X)(\neq 1)$ is finite,*

then $\chi(X) = 0$.

Proof. (Case 1); when $\pi_1(X)$ is finite, we know that X satisfies the condition (T^{**}) by the Theorem 2.5. By the similar method of (step 2) of the Theorem 3.2, we get $\chi(X) = 0$.

(Case 2); when $\pi_1(X)$ is infinite and $\pi_1(X)$ has maximal condition on normal subgroups then $\pi_1(X)$ is finitely generated nilpotent group. Thus $\pi_1(X)$ has the center group of $\pi_1(X)$ as the infinite normal abelian subgroup which acts nilpotently on $H_*(\tilde{X})$. Then by Lemma 3.1, we have $\chi(X) = 0$.

Corollary 3.6. *For finite X satisfying condition* (T^{**}) *with* $\pi_1(X)(\neq 1)$ *finite, suppose that*

(1) *the map* $f : \tilde{X} \to X$ *is a universal covering map,*
(2) *the action* $\pi_1(X) \times H_n(\tilde{X}) \to H_n(\tilde{X})$ *is nilpotent for all* $n \geq 0$

then $\chi(\tilde{X}) = 0$.

Proof. Since $\chi(\tilde{X}) = |\pi_1(X)|\chi(X)$ where $|\,|$ means the order of $\pi_1(X)$ and $\tilde{X}$ means the univeral convering space of X, and by the similar proof of the Theorem 3.5, our proof is completed.

In fibration $F_f \to E \xrightarrow{f} B$, if reduced homology group $\tilde{H}_*(F_f) = 0$, $* \geq 0$ we call that f is an acyclic map, where F_f is a homotopy fiber of f.

Corollary 3.7. *Let* $X(\in T_{LN})$ *be a finite aspherical polyhedron with* $\pi_1(X)$ *is infinite and has the maximal condition on normal subgroups of* $\pi_1(X)$ *then* $\chi(X) = 0$.

Proof. See Theorem of S. Rosset [13] and (Step 2) of Theorem 3.5.

Theorem 3.8. *For finite X satisfying condition* (T^{**})*, if*

(1) *the action* $\pi_1(X) \times H_n(\tilde{X}) \to H_n(\tilde{X})$ *is nilpotent for all* $n \geq 0$*,*
(2) $f : X \to Y$ *is an acyclic map with* $\pi_1(X)$ *finite,*

then $\chi(Y) = 0$.

Proof. By the (step 1) of the Theorem 3.2, $\pi_1(X)$ is a nilpotent group. Thus $X \in T_N$. From the fact that $f : X \to Y$ is an acyclic map and the classical homotopy exact sequence of fibration: $F_f \to X \xrightarrow{f} Y$, we know that $\pi_1(f)$ is an epimorphism, because $\pi_0(F_f) = 0$. Furthermore $H_1(F_f) \cong \frac{\pi_1(F_f)}{[\pi_1(F_f), \pi_1(F_f)]} = 0$ where [,] means the commutator subgroup. $\pi_1(F_f)$ is a perfect group and the homomorphic image of a perfect group is also a perfect group. Thus $\pi_1(X) \cong \frac{\pi_1(Y)}{P\pi_1(X)}$ where $P\pi_1(X)$ means a perfect normal subgroup of $\pi_1(X)$. Since $X \in T_N$, $\chi(X) = 0$ under the above condition and $P\pi_1(X)$ is trivial. Thus $\pi_1(f)$ is an isomorphism. By use of the Hurewicz Theorem inductively, $\pi_i(F_f) = 0$. Thus f is a weak homotopy equivalence. By the Whitehead Theorem [4], f is a homotopy equivalence. Therefore, our proof is completed.

Theorem 3.9. *For finite X satisfying condition* (T^{**})*, if*

(1) *$f : X \to Y$ is quasi-nilpotent homology equivalence with $\pi_1(X)$ finite,*
(2) *the action $\pi_1(X) \times H_n(\tilde{X}) \to H_n(\tilde{X})$ is nilpotent for all $n \geq 0$*

Then $\chi(Y) = 0$.

Proof. By the (step 1) of the Theorem 3.2, X is a nilpotent space. We know the homotopy fiber F_f of f is also nilpotent space. From the fact that f is quasi-nilpotent, Y is a nilpotent space. Thus we conclude that f is a nilpotent map. Since f is nilpotent map and homology equivalence, f is a homotopy equivalence. Thus our proof is completed.

Corollary 3.10. *For finite $X(\in T_{LN})$ if $f : X \to Y$ is quasi-nilpotent homology equivalence with $\pi_1(X)$ finite, then $\chi(Y) = 0$.*

Proof. By Lemma 2.5 and Theorem 3.9, our proof is completed.

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