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THEORY OF DIFFERENTIAL INEQUALITIES ASSOCIATED WITH nth ORDER NONLINEAR DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS TO THREE POINT BOUNDARY VALUE PROBLEMS

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Abstract. Differential inequalities are used as a tool to establish uniqueness of solutions to three point boundary value problems associated with n^{th} order nonlinear differential equations.

1. Introduction

We consider in this paper the uniqueness of solutions to three point boundary value problems associated with n^{th} order differential equation.

$$y^{(n)} = f\big(t, y, y' \dots y^{(n-1)}\big) \qquad n \geq 3 \tag{1.1}$$

$$y(\alpha) = y_1, y(\beta) = y_2, y'(\beta) = y_2', \dots y^{(n-3)}(\beta) = y_2^{(n-3)}, y(\gamma) = y_3 \tag{1.2}$$

where $\alpha \prec \beta \prec \gamma$ and $y_1, y_2, y_2', y_2'' \dots y_2^{(n-3)}$ and y_3 are all real

Existence and uniqueness of solutions to three-point boundary value problems have a long mathematical history going back to Dennis Barr and Tom Sherman [1]. Isolating the ideas involved in the discussion of such problems, we develop the theory of differential inequalities [6] which is of interest in itself and can be utilised to prove the existence and uniqueness of solutions to (1.1) satisfying (1.2). For an entire survey on this we refer [2], [3], [4] and [5].

Our assumptions are weaker than the known results and the proofs exploit the theory of differential inequalities developed. We firmly believe that our approach is new and will be useful in the investigation of other types of existence of such problems.

2. Differential Inequalities

We develop in this section the theory of differential inequalities which plays a crucial role in establishing existence and uniqueness of solutions to three-point boundary

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value problems associated with n^{th} order differential equations. For this purpose, it is convenient to define the following sets and classes of functions.
Let $y \in C^n[[x_1, x_3], R]$ and

$$\Omega_1 = \Big\{x \in [x_1, x_2)/y(x) \leq 0, y'(x) \succ 0, y''(x) \succ 0 \ldots y^{(n-2)}(x) \succ 0 \;\&\; y^{(n-1)}(x) = 0\Big\}$$

$$\Omega_2 = \Big\{x \in (x_2, x_3]/y(x) \geq 0, y'(x) \succ 0, y''(x) \succ 0 \ldots y^{(n-2)}(x) \succ 0 \;\&\; y^{(n-1)}(x) = 0\Big\}$$

$$G_1 = \Big\{g/g \in C\Big\{[x_1, x_3] \times R^n\Big\}g(x, 0, 0, \ldots 0) \geq 0 \forall x \in [x_1, x_2), g(x, u_1, u_2, \ldots u_n)$$
$$\text{is non increasing in } u_1 \text{ and strictly increasing in } u_2, u_3, \ldots u_{n-2}\}$$

and

$$G_2 = \Big\{g/g \in C\Big\{[x_1, x_3] \times R^n, R\Big\}g(x, 0, 0, \ldots 0) \geq 0 \forall x \in (x_2, x_3], g(x, u_1, u_2, \ldots u_n)$$
$$\text{is non decreasing in } u_1 \text{ and strictly increasing in } u_2, u_3, \ldots u_{n-2}\}$$

Lemma 2.1. *Assume that $y \in C^n[[x_1, x_2], R]$ with*

(i) $y(x_1) = 0, y(x_2) = 0, y'(x_2) = 0 \ldots y^{(n-3)}(x_2) = 0, y^{(n-2)}(x_2) \neq 0,\ y^{(n-1)}(x_2) = 0$
$y(x_1) = 0,\ y(x_2) = 0,\ y'(x_2) = 0 \ldots y^{(n-3)}(x_2) = 0,\ y^{(n-2)}(x_2) = 0,\ y^{(n-1)}(x_2) \neq 0$ *and*

(ii) $y^{(n)}(x) \geq g(x, y(x), y'(x) \ldots y^{(n-1)}(x)) \forall x \in \Omega_1$ *and* $g \in G_1$.

Then $y(x) \equiv 0$ *on* $[x_1, x_2]$.

Proof. Let us first consider the case $y^{(n-2)}(x_2) \neq 0 \;\&\; y^{(n-1)}(x_2) = 0$. Suppose to the contrary $y(x) \neq 0$ on $[x_1, x_2]$ since $y(x_1) = 0$, $y(x_2) = 0$ there exists an $r \in (x_1, x_2)$ such that $y'(r) = 0$. By successive application of Rolle's theorem and using the conditions $y'(x_2) = 0$, $y''(x_2) = 0, \ldots y^{(n-3)}(x_2) = 0$, we get an $x_0 \in (r, x_2)$ such that $y^{(n-2)}(x_0) = 0$ and $y^{(n-2)}(x) \neq 0$ on $(x_0, x_2]$ without loss of generality we can suppose that $y^{(n-2)}(x) \succ 0$ on $(x_0, x_2]$. Since $y^{(n-1)}(x_2) = 0$, there exists a $p \in [x_0, x_2]$ such that $y^{(n-1)}(p) \succ 0$. Otherwise $y^{(n-1)}(x) \leq 0$ on $[x_0, x_2)$ gives $0 \geq \int_{x_0}^{x} y^{(n-1)}(x)dx = y^{(n-2)}(x) - y^{(n-2)}(x_0) = y^{(n-2)}(x)$. Thus $y^{(n-2)}(x) \leq 0$ which is a contradiction. Hence there exists a $q \in [p, x_2]$ such that $y^{(n-2)}(q) = 0 \;\&\; y^{(n-1)}(x) \succ 0$ on $[p, q]$. Also by successive integration of $y^{(n-1)}$ we get $0 \prec \int_q^{x_2} y^1(x)dx = y(x_2) - y(q) = -y(q)$. Thus $0 \prec -y(q)$ or $y(q) \prec 0$.

Thus $y(q) \prec 0$, $y'(q) \succ 0$, $y''(q) \succ 0 \ldots y^{(n-1)}(q) \succ 0$ and therefore $q \in \Omega_1$. By our assumption (ii) we get $y^{(n)}(q) \geq g\Big(q, y(q), y'(q) \ldots y^{(n-1)}(q)\Big)$. Thus $y^{(n)}(q) \succ 0$.
However $y^{(n)}(q) = \lim_{x \to q-} \frac{y^{(n-1)}(x) - y^{(n-1)}(q)}{x - q} \leq 0$. which is a contradiction.
Hence $y(x) \equiv 0$ on $[x_1, x_2]$.
Suppose $y^{(n-1)}(x_2) \neq 0$ and $y^{(n-2)}(x_2) = 0$. Then it is claimed that $y(x) \equiv 0$ on $[x_1, x_2]$.
To the contrary, suppose $y(x) \neq 0$, on $[x_1, x_2]$.
Since $y(x_1) = 0$, $y(x_2) = 0$, there exists an $r_1 \in (x_1, x_2)$ such that $y'(r_1) = 0$.

By successive application of Rolle's theorem and using the conditions.
$y'(x_2)=0,\ y''(x_2)=0\dots y^{(n-3)}(x_2)=0$, we get an $x_0 \in (r_1,x_2)$ such that $y^{(n-2)}(x_0)=0$ and $y^{(n-2)}(x) \neq 0$ on $(x_0,x_2]$ with out loss of generality we can suppose that $y^{(n-2)}(x) \succ 0$ on $(x_0,x_2]$. Since $y^{(n-2)}(x_2)=0$ there exists a point $p_1 \in (x_0,x_2)$ such that $y^{(n-1)}(p_1) \succ 0$. Otherwise $y^{(n-2)}(x) \leq 0$ on $[x_0,x_2]$.

Now $0 \geq \int_{x_0}^{x} y^{(n-1)}(x)dx = y^{(n-2)}(x) - y^{(n-2)}(x_0)$ ie $y^{(n-2)}(x) \leq 0$ on $x \in [x_0,x_2]$ which is a contradiction. Hence there exists a $q_1 \in [p_1,x_2]$ such that $y^{(n-1)}(q_1) = 0$, $y^{(n-2)}(x) \succ 0$ on $[p_1,q_1]$. Alos by successive integration of $y^{(n-1)}$, we get $\int_{q_1}^{x_2} y'(x)dx \succ 0$ ie $y(x_2)-y(q_1) \succ 0$ ie $-y(q_1) \succ 0$. Thus $y(q_1) \prec 0,\ y'(q_1) \succ 0,\ y''(q_1) \succ 0 \dots y^{(n-2)}(q_1) \succ 0,\ y^{(n-1)}(q_1) = 0$ therefore $q_1 \in \Omega_2$. By our assumption $y^{(n)}(q_1) \geq g(q_1 y(q_1), y'(q_1) \dots y^{(n-1)}(q_1))$ ie. $y^{(n)}(q_1) \succ g(q_1,0,0,\dots 0) \succ 0$.
However $y^{(n)}(q_1) = \lim_{x\to q_1-} \frac{y^{(n-1)}(x)-y^{(n-1)}(q_1)}{x-q_1} \prec 0$.
which is a contradiction. Hence $y(x) \equiv 0$ on $[x_1,x_2]$.

Lemma 2.2. *Let $y \in C^n[[x_2,x_3],R]$ with*

i) $y(x_2) = 0,\ y'(x_2) = 0, \dots y^{(n-3)}(x_2) = 0,\ y^{(n-2)}(x_2) \neq 0$ & $y^{(n-1)}(x_2) = 0$ *and* $y(x_3) = 0\ y(x_2) = 0,\ y'(x_2) = 0 \dots y^{(n-2)}(x_2) = 0,\ y^{(n-1)}(x_2) \neq 0$ *and* $y(x_3) = 0$
ii) $y^{(n)} \geq g(x,y(x),y'(x),\dots y^{(n-1)})$ *for* $x \in \Omega_2$ *and* $g \in G_2$ *then* $y(x) \equiv 0$ *on* $[x_2,x_3]$.

Proof. Let us consider the case $y(x_2) = 0,\ y'(x_2) = 0 \dots y^{(n-3)}(x_2) = 0,\ y^{(n-2)}(x_2) \neq 0$, $y^{(n-1)}(x_2) = 0$ and $y(x_3) = 0$. Then it is claimed that $y(x) \equiv 0$ on $[x_2,x_3]$ to the contrary suppose that $y(x) \neq 0$ on $[x_2,x_3]$. Since $y(x_2) = 0,\ y(x_3) = 0$ there exists an $r \in (x_2,x_3)$ such that $y'(r) = 0$. By successive application of Rolle's theorem and using the given conditions $y'(x_2) = 0,\ y''(x_2) = 0, \dots y^{(n-3)}(x_2) = 0$, we get an $x_0 \in (x_2,r)$ such that $y^{(n-2)}(x_0) = 0$ and $y^{(n-2)}(x) \neq 0$ on $[x_2,x_0)$. With out loss of generality we can suppose that $y^{(n-2)}(x) \prec 0$ on (x_2,x_0) since $y^{(n-1)}(x_2) = 0$ there exists a point $p \in [x_2,x_0]$ such that $y^{(n-1)}(p) \prec 0$. Otherwise $y^{(n-1)}(x) \geq 0$ on (x_2,x_0) ie. $0 \leq \int_{x_0}^{x} y^{(n-1)}(x)dx = y^{(n-2)}(x) - y^{(n-2)}(x_0)\ 0 \leq y^{(n-2)}(x)$ which is a contradiction to (a). Hence there exists a $m \in [x_2,p]$ such that $y^{(n-2)}(m) = 0$ and $y^{(n-1)}(x) \prec 0$ on $[m,p]$. Also by successive integration of $y^{(n-1)}$ we arrive at $0 \succ \int_{m}^{x_2} y^1(x)dx = y(x_2)-y(m) = -y(m)$ thus $y(m) \succ 0$.
Thus $y(m) \succ 0,\ y'(m) \succ 0,\ y''(m) \succ 0 \dots y^{(n-1)}(m) \succ 0$ and $y^{(n-2)}(m) = 0$. Therefore $m \in \Omega_2$. By our assumption (ii), we get

$$y^{(n)}(m) \geq g(m,y(m),y'(m),\dots y^{(n-1)}(m)) \succ g(m.0\dots 0) \succ 0.$$

However $y^{(n)}(m) = \lim_{x\to m+} \frac{y^{(n-1)}(x)-y^{(n-1)}(m)}{x-m} \leq 0$ which is a contradiction. Hence $y(x) \equiv 0$ on $[x_2,x_3]$. Suppose $y(x_2) = 0,\ y'(x_2) = 0 \dots y^{(n-3)}(x_2) = 0,\ y^{(n-2)}(x_2) = 0$, $y^{(n-1)}(x_2) \neq 0$, and $y(x_3) = 0$. Then it is claimed that $y(x) \equiv 0$ on $[x_2,x_3]$. To the contrary suppose $y(x) \neq 0$ on $[x_2,x_3]$. Since $y(x_1) = 0,\ y(x_2) = 0$, there exists *a* $r_1 \in (x_2,x_3)$ such that $y^1(r_1) = 0$.

By successive application of Roll'e theorem and using the conditions $y'(x_2) = 0$, $y''(x_2) = 0 \dots y^{(n-3)}(x_2) = 0$, we get an $x_0 \in (x_2, r_1)$ such that $y^{(n-2)}(x_0) = 0$ and $y^{(n-2)}(x) \neq 0$ on (x_2, x_0). Without loss of generality we can suppose that $y^{(n-2)}(x) \prec 0$ on (x_2, x_0). Since $y^{(n-2)}(x_2) = 0$, there exists a point $p_1 \in (x_2, x_0)$ such that $y^{(n-1)}(p_1) \prec 0$. Other wise $y^{(n-1)}(x) \geq 0$ on $[x_2, x_0]$ now $0 \leq \int_{x_0}^{x} y^{(n-1)}(x)dx = y^{(n-2)}(x) - y^{(n-2)}(x_0) = y^{(n-2)}(x)$ thus $y^{(n-2)}(x) \geq 0$ which is a contradiction. Hence there exists a point $m_1 \in [x_2, p_1]$ such that $y^{(n-1)}(m_1) = 0$, $y^{(n-2)}(x) \prec 0$ on $[m_1, p_1]$. Also by successive integration of $y^{(n-1)}$, we get $0 \succ \int_{m_1}^{x_2} y(x)dx = y(x_2) - y(m_1)$ ie $0 \succ -y(m_1)$ or $y(m_1) \succ 0$.
Thus $y(m_1) \succ 0$, $y'(m_1) \succ 0$, $y''(m_1) \succ 0 \dots y^{(n-2)}(m_1) \succ 0$ and $y^{(n-1)}(m_1) = 0$. Therefore $m_1 \in \Omega_2$.
By our assumption (ii) $y^{(n)}(m_1) \geq g(m_1, y(m_1), y'(m_1) \dots y^{(n-1)}(m_1)) \succ g(m_1, 0, 0 \dots 0) \succ 0$.
However $y^{(n)}(m_1) = \lim_{x \to m_1+} \frac{y^{(n-1)}(x) - y^{(n-1)}(m_1)}{x - m_1} \prec 0$, which is a contradiction. Hence $y(x) \equiv 0$ on $[x_2, x_3]$.

Lemma 2.3. *Assume that $C^n[[x_1, x_3], R]$ satisfying*

(i) $y(x_1) = 0$, $y(x_2) = 0$, $y(x_3) = 0$, $y'(x_2) = 0$, $y''(x_2) = 0 \dots y^{(n-3)}(x_2) = 0$, $y^{(n-2)}(x_2) = 0$, $y^{(n-1)}(x_2) \neq 0$

(ii) $y^{(n)}(x) \geq g\Big(x, y(x), y'(x), y''(x), \dots y^{(n-1)}(x)\Big)$ *for* $x \in \Omega_1 \cup \Omega_2$ *and* $g \in G_1 \cup G_2$ *then* $y(x) \equiv 0$ *on* $[x_1, x_3]$

Proof. Suppose that $y(x) \neq 0$ on $[x_1, x_3]$ with out loss of generality we can assume that $y^{(n-1)}(x_2) \succ 0$. If $y^{(n-1)}(x_2) \prec 0$, then a similar argument can be applied to $-y^{(n-1)}(x_2)$. Since $y(x_1) = 0$, $y(x_2) = 0$ and $y(x_3) = 0$ then there exists a $p_1 \in [x_1, x_2]$ and $p_2 \in [x_2, x_3]$ such that $y'(p_1) = 0$, $y'(p_2) = 0$. Since $y^{(n-1)}(x) \succ 0$ for all $x \in [x_1, x_3]$. By successive integration of $y^{(n-1)}(x) \succ 0$ between the limits x_2 to x, we get $0 \prec \int_{x_2}^{x} y''(x)dx = y'(x) - y'(x_2) = y'(x)$ thus $y'(x) \succ 0$ on $x \in [p_1, p_2]$. $y'(p_1) = 0$, $y'(p_2) = 0$ and $y'(x) \succ 0$ on $[p_1, p_2]$. Then it follows that there exists a $q_1 \in [p_1, x_2)$ and a $q_2 \in (x_2, p_2]$ such that $y''(q_1) = 0$, $y''(q_2) = 0$ and $y''(x)$ is of one sign in (q_1, q_2). For otherwise $0 \prec \int_{x}^{p_2} y''(x)dx = y'(p_2) - y'(x) = -y'(x)$ thus $y'(x) \prec 0$ on (p_1, p_2), which is a contradiction.
Since $y(x_2) = 0$ and $y'(x) \succ 0$ on (q_1, q_2) $0 \leq \int_{q_1}^{x_2} y^1(x)dx = y(x_2) - y(q_1) = -y(q_1)$ thus $0 \leq -y(q_1)$ or $y(q_1) \leq 0$ and $0 \leq \int_{x_2}^{q_2} y'(x)dx = y(q_2) - y(x_2) = y(q_2)$ thus $y(q_1) \geq 0$.
Then we have at $x = q_1$ $y(q_1) \leq 0$, $y'(q_1) \succ 0$, $y''(q_1) \succ 0 \dots y^{(n-2)}(q_1) \succ 0$ and $y^{(n-1)}(q_1) = 0$; at $x = q_2$ $y(q_2) \geq 0$, $y'(q_2) \succ 0$, $y''(q_2) \succ 0 \dots y^{(n-2)}(q_2) \succ 0$ and $y^{(n-1)}(q_1) = 0$.
Further more $y^{(n-1)}(x) \prec 0$ on (p_1, p_2) we have $y^{(n)}(q_1) = \lim_{x \to q_1+} \frac{y^{(n-1)}(x) - y^{(n-1)}(q_1)}{x - q_1} \leq 0$ and $y^{(n)}(q_2) = \lim_{x \to q_2+} \frac{y^{(n-1)}(x) - y^{(n-1)}(q_2)}{x - q_2} \leq 0$ using condition (ii) at q_1 as well as q_2 we arrive at a contradiction.
Hence $y(x) \equiv 0$ on $[x_1, x_3]$ and the proof is complete.

Lemma 2.4. *Assume that $y \in C^n[[x_1, x_2], R]$ with*

i) $y(x_1) = 0$, $y(x_2) = 0$, $y'(x_2) = 0 \cdots y^{(n-3)}(x_2) = 0$ *and either* $y^{(n-2)}(x_2) \prec 0$ *or* $y^{(n-1)}(x_2) \succ 0$

ii) $y^{(n)}(x) \geq g(x, y(x), y'(x), \ldots y^{(n-1)}(x))$ *for* $x \in \Omega_1$ *and* $g \in G_1$. *Then there exists a* $p \in [x_1, x_2)$ *such that* $y^{(n-2)}(x) \prec 0$ *for* $x \in [p_2, x_2)$ *and* $y(x) \geq 0$ *for* $x \in [p, x_2)$.

Proof. Consider first the case $y^{(n-2)}(x_2) \prec 0$.

Then there exists a $q \in [x_1, x_2)$ such that $y^{(n-2)}(x) \prec 0$ for $x \in (q, x_2)$ and $y^{(n-2)}(q) = 0$. Now to prove the required conclusion it is sufficient to prove that there exists a $p \in [q, x_2)$ such that $y^{(n-1)}(p) \leq 0$ suppose $y^{(n-1)}(x) \succ 0$ on $x \in [q, x_2)$ $0 \prec \int_q^x y^{(n-1)}(x)dx = y^{(n-2)}(x) - y^{(n-2)}(q) = y^{(n-2)}(x)$ thus $y^{(n-2)}(x) \succ 0$ which is a contradiction. Hence there exists a $p \in [q, x_2)$ such that $y^{(n-1)}(p) \leq 0$. Thus we have $y(q) \prec 0$, $y'(q) = 0$, $y''(q) = 0 \cdots y^{(n-1)}(q) \succ 0$.
Hence

$$\begin{aligned} y^{(n)}(q) &\geq g\Big(q, y(q), y'(q) \cdots y^{(n-1)}(q)\Big) \\ &\geq g(q, 0, 0 \ldots 0) \geq 0. \end{aligned}$$

However $y^{(n)}(q) = \lim_{x \to q^+} \frac{y^{(n-1)}(x) - y^{(n-1)}(q)}{x - q} \prec 0$.
Hence a contradiction and we are done. In case $y^{(n-1)}(x_2) \succ 0$ we arrive a similar contradiction. Hence the proof.

Lemma 2.5. *Assume that $y \in C^n[[x_2, x_3], R]$ satisfies*

i) $y(x_2) = 0$, $y(x_3) = 0$, $y'(x_2) = 0 \cdots y^{(n-3)}(x_2) = 0$ *and either* $y^{(n-2)}(x_2) \succ 0$ *or* $y^{(n-1)}(x_2) \prec 0$

ii) $y^{(n)}(x) \geq g\Big(x, y(x), y'(x), \ldots y^{(n-1)}(x)\Big)$ *for* $x \in \Omega_2$ *and* $g \in G_2$ *then there exists a* $p \in (x_2, x_3]$ *such that* $y^{(n-2)}(x) \prec 0$ *for* $x \in [x_2, p)$ *and* $y(x) \leq 0$ *for* $(x_2, p]$.

Proof. Proof is analogous to the previous Lemma and hence omitted.

3. Existence and Uniqueness

In this section we shall discuss a main problem of uniqueness of solution of (1.1) satisfying (1.2) before we proceed to do this we first establish that the following four boundary value problems (3.1_i), (3.2_i) $i = 1, 2$.

$$y^{(n)}(x) = f\Big(x, y, y', y'' \ldots y^{(n-1)}\Big)$$

$$y(x_1) = y_1, y(x_2) = y_2, y'(x_2) = y_2', \ldots y^{(n-3)}(x_2) = y_2^{(n-3)}, y^{(n-i)}(x_2) = m (i = 1, 2) \quad (3.1i)$$

or

$$y(x_2)=y_2, y'(x_2)=y_2' \cdots y^{(n-3)}(x_2)=y_2^{(n-3)}, y^{(n-i)}(x_2)=m(i=1,2), y(x_3)=y_3 \quad (3.2i)$$

have at most one solution. The next lemma establishes the existence of solutions to the boundary value problems (3.1_i), (3.2_i), $(i = 1, 2)$.

Lemma 3.1. *Assume that* $g \in G_1 \cup G_2$, $f \in C^n\Big[[x_1, x_3], R^n, R\Big]$ *and satisfies* $f(x, y_1, y_2, \ldots y_n) - f(x, z_1, z_2, \ldots z_n) \geq g(x, y_1 - z_1, y_2 - z_2, \ldots y_n - z_n)$ *whenever*

(a) $x \in (x_1, x_2]$, $y_1 \leq z_1$, $y_i \succ z_i (i = 2, 3, \ldots n-1)$ *and* $y_n = z_n$ *and*
(b) $x \in [x_2, x_3)$, $y_1 \geq z_1$, $y_i \succ z_i (i = 2, 3, \ldots n-1)$ *and* $y_n = z_n$.

Then the boundary value problem (1.1) satisfying (3.1i) and (3.2i) $(i = 1, 2)$ have at most one solution for each $m \in R$.

Proof. The proof of the uniqueness of solutions of the boundary value problem satisfying (3.1_2) will be given similar argument hold for other boundary value problems. Suppose there exist two solutions $y_1(x)$ and $y_2(x)$ on $[x_1, x_2]$. Then we write $y(x) = y_1(x) - y_2(x)$. Clearly $y(x_1) = 0$, $y^{(i)}(x_2) = 0 (i = 0, 1, 2, \ldots n-3)$ and $y^{(n-2)}(x_2) = 0$. If $y^{(n-1)}(x_2) = 0$, by uniqueness of solutions of initial problems of (1.1) we have $y(x) \equiv 0$ on $[x_1, x_2]$. We suppose that $y^{(n-1)}(x_2) \neq 0$. Let $x_0 \in \Omega_1$. Then $y(x_0) \leq 0$, $y'(x_0) \succ 0$, $y''(x_0) \succ 0 \cdots y^{(n-3)}(x_0) \succ 0$, $y^{(n-2)}(x_0) = 0$ and $y^{(n-1)}(x_0) \succ 0$ which implies that $y_1(x_0) \leq y_2(x_0)$, $y_1'(x_0) \succ y_2'(x_0)$, $y''(x_0) \succ y_2''(x_0) \cdots y_1^{(n-3)}(x_0) \succ y_1^{(n-3)}(x_0)$ $y_1^{(n-2)}(x_0) = y_2^{(n-2)}(x_0)$ and $y_1^{(n-1)}(x_0) \succ y_2^{(n-1)}(x_0)$. Hence by condition (a) $f\Big(x, y_1, y_1', y_1'' \ldots y_1^{(n-1)}\Big) - f\Big(x, y_2, y_2', y_2'' \ldots y_2^{(n-1)}\Big) \geq g\Big(x_0, y(x_0), y'(x_0) \cdots y^{(n-1)}\Big)$. Thus all the hypothesis of Lemma 2.1 are satisfied and hence $y(x) \equiv 0$ on $[x_1, x_2]$.

Thus uniqueness is established which implies that the boundary value problem (1.1) satisfying (3.1_2) has at most one solution. Similar arguments hold for other boundary value problems. Hence the proof of the lemma 3.1 is complete.

Theorem 3.1. *Assume that*

(a) *For each* $m \in R$ *there exist solutions of four boundary value problems* (1.1), (3.1_i), (3.2_i) $(i = 1, 2)$
(b) f, g *satisfy conditions in Lemma* 3.1. *Then there exists a unique solution of three point boundary value problem* (1.1) *satisfying* (1.2).

Proof. By lemma 3.1 it follows that the solutions to (1.1) satisfying (3.1i), (3.2i) $(i = 1, 2)$ when ever exist are unique. Let $m_2 \succ m_1$ and $y_1(x, m)$ be the unique solution of (1.1) satisfying (3.1_2). Set $V(x) = y_1(x, m_1) - y_1(x, m_2)$ then

$$V(x_1) = 0, V(x_2) = 0, V'(x_2) = 0, \ldots V^{(n-2)}(x_2) = 0 \text{ and } V^{(n-1)}(x_2) = m_1 - m_2 < 0$$

If $f \in G_1$ then $V(x) \leq 0$, $V'(x) \succ 0$, $V''(x) \succ 0 \cdots V^{(n-2)}(x) \succ 0$ and $V^{(n-1)}(x) = 0$.

Which implies $y_1(x,m_1) \leq y_1(x,m_2)$, $y_1'(x,m_1) \succ y_1'(x,m_2) \cdots y_1^{(n-2)}(x,m_1) \succ y_1^{(n-2)}$ (x,m_2) and $y_1^{(n-1)}(x,m_1) = y_1^{(n-1)}(x,m_2)$ and using (b) we get $V^{(n)}(x) \geq g\Big(x, V(x), V'(x), \ldots V^{(n-1)}(x)\Big)$ $x \in \Omega_1$ and $g \in G_1$ using Lemma 2.4 we have $V^{(n-2)}(x) \prec 0$ for all $x \in (p, x_2]$. In particular $V^{(n-2)}(x_2) \prec 0$.
Hence $y_1(x,m)$ is a strictly increasing function of m on $(x_1, x_2]$. Using Lemma 2.5, it can be proved that $y_2(x,m)$ is a strictly decreasing functin of m in the interval $[x_2, x_3)$. It follows from the Lemma 3.1 that the solutions of (1.1) satisfying (3.1_1), (3.1_2), (3.2_1) and (3.2_2) are unique. Further it follows that $y_1^{(n-2)}(x_2,m)$ and $y_2^{(n-2)}(x_2,m)$ are strictly monotone and range being set of reals, it follows that there exists an $m_0 \in R$ such that $y_1^{(n-2)}(x_2,m_0) = y_2^{(n-2)}(x_2,m_0)$. Thus $y(x)$ is defined by

$$y(x) = \begin{cases} y_1(x,m_0), x \in [x_1, x_2) \\ y_2(x,m_0), x \in (x_2, x_3] \end{cases}$$

is a solution of the three point boundary value problem. Now to prove uniqueness, suppose there are two solutions $y_1(x)$ and $y_2(x)$ for the boundary value problems (1.1) and (1.2). Then from lemma 3.1, it follows that $y_1^{(n-2)}(x_2) \neq y_2^{(n-2)}(x_2)$ and $y_1^{(n-1)}(x_2) \neq y_2^{(n-1)}(x_2)$.
Setting $V(x) = y_1(x) - y_2(x)$. Implies it is easy to see that as before all assumptions are satisfied and therefore $y(x) \equiv 0$ on $[x_1, x_3]$. Thus uniqueness is established.

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